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USE OF THE GROBNER BASIS IN THE STUDY OF MANIPULATORS TOPOLOGY

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Abstract. *In the classic cases used in industry, manipulators need to pass through singularities of the joint space to change their posture: the end-effector must bump into the frontier of the workspace. A 3-DOF manipulator can execute a non singular change of posture if and only if there is at least one point in its workspace which has exactly three coincident solutions of the inverse kinematic model. Since it is difficult to express this condition directly from the kinematic model, it is proposed to eliminate two joint variables from the system in order to obtain a condition that depends only on the last joint variable. For this purpose, a powerful algebraic tool is used: the Grobner basis. With this approach, it is possible to obtain analytical expressions of the surfaces of the parameters space that separate the different types of manipulators. The determinant of Jacobian matrix of the direct kinematic model is considered equal to zero to obtain the other surfaces that separate the various regions for different topologies. In this paper, it will be presented the process of obtaining the surfaces of separation, as well as the corresponding curves due changes in the parameters of 3R orthogonal manipulators. Detailed knowledge of the various regions of the space of parameters is important for the optimal design of manipulators which obeys a topology specified by the designe*r.

Keywords*: robotics, Grobner basis, manipulators topology*

1. INTRODUCTION

In the study of manipulator robots is essential to know the topology of the singularity surfaces in the workspace. These singularities are defined as places where the determinant of the Jacobian matrix of direct kinematic model (DKM) is annulled, resulting in the equations of surfaces which divide the workspace in several regions that have manipulators with same properties (binary or quaternary, regions with the same numbers of cusps and node points). These regions are called domains. Since it is difficult to express them directly from the kinematic model, it is proposed to eliminate two joint variables from the system in order to obtain a condition that depends only on the last joint variable. For this purpose, a powerful algebraic tool is used: the Grobner basis. With this approach, it is possible to obtain analytical expressions of the surfaces of the parameters space that separate the different types of manipulators. The annulment of the determinant of Jacobian matrix of the inverse kinematic model (IKM) enables to obtain the other surfaces that separate the various regions for different topologies.

Wenger and El Omri (1996a; 1996b) showed that for some choices of the parameters, manipulators with three rotational joints (3R) may be able to change posture without meeting a singularity in the joint space. They succeed in characterizing such manipulators (Wenger, 1998), but they needed general conditions on the design parameters. Corvez (2002) found important results about this issue. In 2004, Baili realized researches on the proprieties of 3R manipulators with orthogonal axes and made a classification in the parameters space.

This article aims to show the achievement of the separation surfaces in parameters space of manipulators with three rotational joints with orthogonal axes as described in Fig. 1. It is described the process of obtaining the surfaces, as well as the corresponding curves due to changes in the parameters of 3R orthogonal manipulators. In addition, a brief classification of 3R manipulators is presented; this classification is based on two criteria to define a topology of the workspace: the number of cusp points and the number of nodes (intersection of two singularity branches).

The study of this type of manipulator is done according to the Denavit-Hartenberg parameters: d_2, d_3, d_4, r_2, and r_3. To reduce the number of parameters, will be considered $d_2 = 1$ and $r_3 = 0$. The joint variables are θ_1, θ_2 and θ_3 which represent the input angles of the actuators.

For this type of manipulator, the direct kinematic model, obtained in Saramago et al. (2007), is given by:

$$x=\left[1+\left(d_3+d_4c_3\right)c_2\right]c_1-\left(r_2+d_4s_3\right)s_1$$
$$y=\left[1+\left(d_3+d_4c_3\right)c_2\right]s_1-\left(r_2+d_4s_3\right)c_1 \tag{1}$$

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$$z = -(d_3 + d_4 c_3) s_2$$

in which $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$, for i = 1, 2, 3.

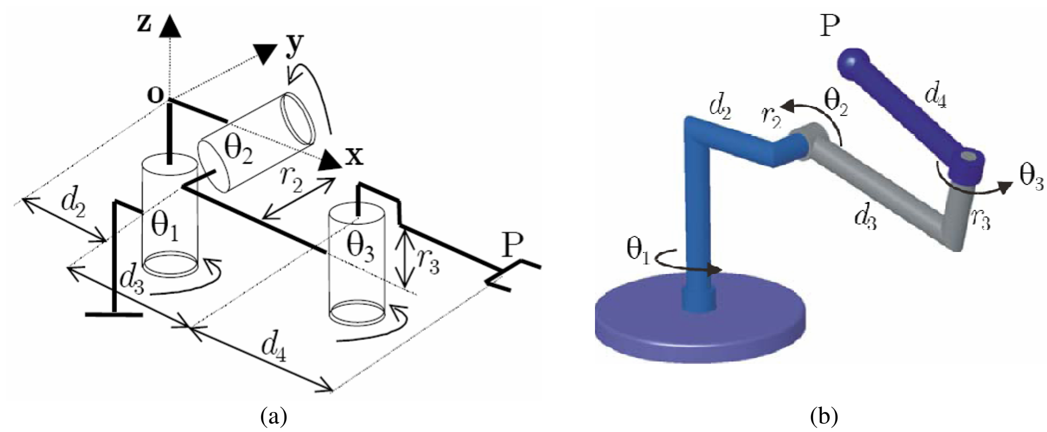

Figure 1. Manipulator with three rotational joints (3R) with orthogonal axes

2. SEPARATION SURFACES FORMULATION

The existence of at least a point of singularity in the workspace with exactly three solutions coincident in IKM is difficult to be shown directly from DKM. The idea is to eliminate the variables of joint, θ_1 and θ_2 to obtain a condition on the last variable, θ_3. For this, the algebraic relations $c_i^2 + s_i^2 = 1$, i = 1, 2, 3, are added to Eqs. (1), resulting in a system of six algebraic equations:

$$\begin{aligned} x &= \left[1 + (d_3 + d_4 c_3) c_2\right] c_1 - (r_2 + d_4 s_3) s_1 \\ y &= \left[1 + (d_3 + d_4 c_3) c_2\right] s_1 + (r_2 + d_4 s_3) c_1 \\ z &= -(d_3 + d_4 c_3) s_2 \\ c_1^2 + s_1^2 &= 1 \\ c_2^2 + s_2^2 &= 1 \\ c_3^2 + s_3^2 &= 1 \end{aligned} \tag{2}$$

which can be written in equivalent form:

$$\begin{aligned} f_1 &= x - \left[1 + (d_3 + d_4 c_3) c_2\right] c_1 + (r_2 + d_4 s_3) s_1 \\ f_2 &= y - \left[1 + (d_3 + d_4 c_3) c_2\right] s_1 - (r_2 + d_4 s_3) c_1 \\ f_3 &= z + (d_3 + d_4 c_3) s_2 \\ f_4 &= c_1^2 + s_1^2 - 1 \\ f_5 &= c_2^2 + s_2^2 - 1 \\ f_6 &= c_3^2 + s_3^2 - 1 \end{aligned} \tag{3}$$

To solve these equations, which means to eliminate θ_1 and θ_2, it is calculated a Grobner basis of the ideal generated by the polynomials $f_1, f_2, f_3\, f_4, f_5$ and f_6, according to Cox et al. (2005).

To achieve this goal, the function ***gbasis*** of Maple9® is used. The polynomials given by Eq. (3) are informed, besides them, a command that indicates the order of the monomial, to the elimination of variables θ_1 and θ_2, is added. This command is described below:

G := ***gbasis***([f1,f2,f3,f4,f5,f6], ***lexdeg***([c1,s1,c2,s2],[c3,s3,r2,d3,d4,x,y,z]));

Thus, the reduced Grobner basis, for a 3R manipulator, is composed by the polynomials:

$$p_1(\theta_3) = c_3^2 + s_3^2 - 1 \tag{4}$$

and

$$p_2(\theta_3) = m_5 c_3^2 + m_4 s_3^2 + m_3 c_3 s_3 + m_2 c_3 + m_1 s_3 + m_0 \tag{5}$$

where,

$$\begin{cases} m_0 = -x^2 - y^2 + r_2^2 + \dfrac{(R+1-L)^2}{4} \\ m_1 = 2r_2 d_4 + (L-R-1)d_4 r_2 \\ m_2 = (L-R-1)d_3 d_4 \\ m_3 = 2r_2 d_3 d_4^2 \\ m_4 = d_4^2(r_2^2+1) \\ m_5 = d_3^2 d_4^2 \end{cases} \tag{6}$$

$$R = x^2 + y^2 + z^2 \quad \text{and} \quad L = d_4^2 + d_3^2 + r_2^2 \tag{7}$$

Considering a new variable t = tan (θ_3 / 2) for which the following relations can be adopted:

$$\cos\theta_3 = \frac{1-t^2}{1+t^2} \quad \text{and} \quad sin\theta_3 = \frac{2t}{1+t^2} \tag{8}$$

and replacing the Eq. (8) in (5), the polynomial $p_2(\theta_3)$ becomes:

$$p_2(t) = \frac{m_5\left(1-2t^2+t^4\right)}{\left(1+t^2\right)^2} + \frac{4m_4 t^2}{\left(1+t^2\right)^2} + \frac{\left(m_3 - m_3 t^2\right)2t}{\left(1+t^2\right)^2} + \frac{\left(m_2 - m_2 t^2\right)}{\left(1+t^2\right)} + \frac{2m_1 t}{\left(1+t^2\right)} + m_0 \tag{9}$$

After further algebraic manipulations, it is possible to obtain the polynomial:

$$P(t) = at^4 + bt^3 + ct^2 + dt + e \tag{10}$$

where,

$$\begin{cases} a = m_5 - m_2 + m_0 \\ b = -2m_3 + 2m_1 \\ c = -2m_5 + 4m_4 + 2m_0 \\ d = 2m_3 + 2m_1 \\ e = m_5 + m_2 + m_0 \end{cases} \tag{11}$$

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A manipulator is called cuspidal if the polynomial *P(t)*, which has degree 4 and coefficients that depend only on x, y, z, d_4, d_3 and r_2 , admits triple real root. This is equivalent to solve the following system:

$$S\left(t,a,b,c,d,e\right)=\begin{cases} P(t,Z,R,d_3,d_4,r_2)=0 \\ \dfrac{\partial P\left(t,Z,R,d_3,d_4,r_2\right)}{\partial t}=0 \\ \dfrac{\partial^2 P\left(t,Z,R,d_3,d_4,r_2\right)}{\partial t^2}=0 \end{cases} \tag{12}$$

In polynomial system S, the variables of the problem are t, R and Z, while the parameters of the strictly positive problem are d_3, d_4 and r_2 (note that, if one of these parameters is canceled it results in a non-cuspidal manipulator).

To solve the system (12) is necessary to divide the first quadrant of the space of parameters (d_3, d_4 and r_2) in several regions where the number of the problem solutions is constant. In each region, the number of solutions of the problem and a set of parameters of manipulator representing this region are obtained.

Under the hypotheses $Z = z^2 > 0$ and $R - Z > 0$, where R is given in Eq. (7), is possible to eliminate the variables t, Z and R. After several changes of variables and using the Grobner basis, Corvez (2002), applying the Cylindrical Algebraic Decomposition algorithm - CAD, obtained the five polynomials below:

$$g_1 : r_2^2 + d_3^2 - d_4^2 = 0 \tag{13}$$

$$g_2 : d_4^2\left[\left(1 + r_2^2 + d_3^2\right)^2 - 4d_3^2\right]\left(r_2^2 + d_3^2 - d_4^2\right) - r_2^2 d_3^2 = 0 \tag{14}$$

$$g_3 : r_2^2 d_4 - d_3 + d_4 = 0 \tag{15}$$

$$g_4 : \left(d_3 - 1\right)^2\left(d_3^2 - d_4^2\right) + r_2^2 d_3^2 = 0 \tag{16}$$

$$g_5 : \left(d_3 + 1\right)^2\left(d_3^2 - d_4^2\right) + r_2^2 d_3^2 = 0 \tag{17}$$

To continue the process of obtaining the singularity surfaces in the workspace, the polynomial given by Eq. (10), is rewritten as

$$P\left(t\right) = at^4 + 4bt^3 + 6ct^2 + 4dt + e = 0 \tag{18}$$

where the coefficients are calculated by:

$$a = d_3^2 d_4^2 - d_3 d_4 V - R_{11} + \frac{V^2}{4} + r_2^2 \tag{19}$$

$$b = \frac{d_4 r_2\left(-2d_3 d_4 + V + 2\right)}{2} \tag{20}$$

$$c = -\frac{d_3^2 d_4^2}{3} + \frac{2d_4^2 r_2^2}{3} + \frac{2d_4^2}{3} - \frac{R_{11}}{3} + \frac{V^2}{12} + \frac{r_2^2}{3} \tag{21}$$

$$d = \frac{d_4 r_2\left(2d_3 d_4 + V + 2\right)}{2} \tag{22}$$

$$e = d_3^2 d_4^2 + d_3 d_4 V - R_{11} + \frac{V^2}{4} + r_2^2 \tag{23}$$

considering $V = -x^2 - y^2 - z^2 - 1 + d_3^2 + r_2^2 + d_4^2$ and $R_{11} = x^2 + y^2$ (24)

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To determine another equation of the surface of separation, the condition of existence of quadruple roots of polynomial equation must be assumed. Consequently, for $P(t)$ admits quadruple roots, the three following equations must be verified:

$$ae - bd = 0 \tag{25}$$

$$ad - bc = 0 \tag{26}$$

$$ac - b^2 = 0 \tag{27}$$

In order to write a condition that depends only on the DH parameters, the cartesian coordinates x, y and z must be removed. That is, the variables V and R_{11} should be eliminated. This elimination can be done applying the algebraic tools developed in Buchberger (1982). These tools are available in some symbolic calculation software. Initially, the variable R_{11} is eliminated from Eqs. (25) and (27), resulting a polynomial of degree 4 in V. Then, R_{11} is eliminated from Eqs. (26) and (27), obtaining a polynomial of degree 3 in V. Finally, using the two polynomials found previously the variable V is canceled, resulting in the following equation:

$$d_3^{12} d_4^2 r_2^4 Q_1 Q_2 Q_3 = 0 \tag{28}$$

where Q_1, Q_2 and Q_3 are three polynomials depending on d_3, d_4 and r_2:

$$\begin{aligned} Q_1 = {} & d_3^6 d_4^2 - 2d_3^4 d_4^2 - d_3^4 d_4^4 + 3r_2^2 d_3^4 d_4^2 + d_3^2 d_4^2 + 2d_3^2 d_4^4 - 2r_2^2 d_3^2 d_4^4 + 3r_2^4 d_3^2 d_4^2 - d_4^4 + r_2^2 d_4^2 \\ & - 2r_2^2 d_4^4 + 2r_2^4 d_4^2 - r_2^4 d_4^4 + r_2^6 d_4^2 - r_2^2 d_3^2 \end{aligned} \tag{29}$$

$$\begin{aligned} Q_2 = {} & -543 d_4^5 d_3^2 r_2^2 + 648 d_4^5 d_3^4 - 81 d_4^5 d_3^2 - 32 d_4^4 r_2^4 d_3 - 8 d_4^4 r_2^2 d_3 + 1110 d_3^3 r_2^2 d_4^4 - 25 r_2^2 d_3^5 - 47 d_3^2 d_4^3 r_2^2 \\ & -141 d_3^2 d_4^3 r_2^4 - 47 d_3^2 d_4^3 r_2^8 - 210 d_3^6 d_4 r_2^2 + 486 d_3^{10} d_4^3 + 972 d_3^6 d_4^3 - 1215 d_3^8 d_4^3 - 1458 d_3^5 d_4^4 \\ & -8 d_3 d_4^4 r_2^{10} - 48 d_3 d_4^4 r_2^6 - 32 d_3 d_4^4 r_2^8 - 141 d_3^2 d_4^3 r_2^6 + 243 d_3^3 d_4^4 + 81 d_3^5 d_4^2 - 162 d_3^7 d_4^2 + 81 d_3^9 d_4^2 \\ & -243 d_3^4 d_4^3 + 1224 d_3^3 d_4^4 r_2^6 + 300 d_3^3 d_4^4 r_2^8 + 1791 d_3^3 d_4^4 r_2^4 - 801 d_3^4 d_4^3 r_2^2 + 35 d_3^4 d_4 r_2^4 + 29 d_3^3 d_4^2 r_2^6 \\ & +444 d_3^5 d_4^2 r_2^2 + 58 d_3^3 d_4^2 r_2^4 + 35 d_3^4 d_4 r_2^2 + 16 d_4^5 r_2^{12} + 80 d_4^5 r_2^{10} + 160 d_4^5 r_2^8 + 160 d_4^5 r_2^6 + 80 d_4^5 r_2^4 \\ & +16 d_4^5 r_2^2 - 177 d_3^5 d_4^2 r_2^4 - 813 d_3^7 d_4^2 r_2^2 - 2340 d_3^8 d_4^5 r_2^2 + 2052 d_3^4 d_4^5 r_2^4 - 2025 d_3^6 d_4^5 + 3078 d_3^8 d_4^5 \\ & -3096 d_3^7 d_4^4 r_2^4 + 1440 d_3^5 d_4^4 r_2^6 - 459 d_3^6 d_4^3 r_2^4 + 72 d_3^9 d_4^4 r_2^2 + 2268 d_3^{10} d_4^5 r_2^2 + 1872 d_3^8 d_4^5 r_2^4 \\ & +648 d_3^{12} d_4^5 + 972 d_3^{11} d_4^4 - 2268 d_3^{10} d_4^5 + 3159 d_3^7 d_4^4 - 1188 d_3^5 d_4^4 r_2^4 + 2601 d_3^6 d_4^3 r_2^2 - 72 d_3^6 d_4^5 r_2^6 \\ & +180 d_3^4 d_4^5 r_2^6 + 2340 d_3^6 d_4^5 r_2^4 + 2352 d_3^4 d_4^5 r_2^2 - 1773 d_3^8 d_4^3 r_2^2 - 2682 d_3^5 d_4^4 r_2^2 - 1557 d_3^6 d_4^5 r_2^2 \\ & +1872 d_3^7 d_4^4 r_2^2 - 2916 d_3^3 d_4^4 + 29 d_3^3 d_4^2 r_2^2 - 168 d_3^4 d_4^5 r_2^8 - 552 d_3^2 d_4^5 r_2^8 - 1227 d_3^2 d_4^5 r_2^4 \\ & -1233 d_3^2 d_4^5 r_2^6 - 84 d_3^2 d_4^5 r_2^{10} - 1845 d_3^4 d_4^3 r_2^4 - 1287 d_3^4 d_4^3 r_2^6 \end{aligned} \tag{30}$$

$$Q_3 = -d_3 + d_4 r_2^2 + d_4 \tag{31}$$

Since the elimination of variables from one or several equations can add complementary solutions (Buchberger, 1982), the solutions of Eq. (28) may include other surfaces not suitable, in addition the surface that separates the binary and quaternary manipulators in space of parameters. But, it is also known that the separating surface of the binary and quaternary manipulators can be given by Eqs. (13) to (17). Thus, these sets of equations are compared.

Observing the Eqs. (28) to (31) and Eqs. (13) to (17) is possible to verify that Eq. (29) (considering $Q_1 = 0$) is equal to Eq. (14). Furthermore, it appears that all other equations are different. It deducted that the only appropriate solution is $Q1 = 0$ (Baili, 2004). Note that this equation is of degree 2 in d_4 and its resolution allows writing the equations of two surfaces C_{1a} and C_{1b} as:

$$C_{1a} : d_4 = \sqrt{\frac{1}{2}\left(d_3^2 + r_2^2 - \frac{\left(d_3^2 + r_2^2\right)^2 - d_3^2 + r_2^2}{AB} \right)} \tag{32}$$

$$C_{1b} : d_4 = \sqrt{\frac{1}{2}\left(d_3^2 + r_2^2 + \frac{\left(d_3^2 + r_2^2\right)^2 - d_3^2 + r_2^2}{AB} \right)} \tag{33}$$

where,

$$A = \sqrt{(d_3 + 1)^2 + r_2^2} \tag{34}$$

$$B = \sqrt{(d_3 - 1)^2 + r_2^2} \tag{35}$$

The Eq. (32) is the surface of separation between the manipulators of domain 1 and domain 2. The manipulators that belong to domain 1 are binary, have a toroidal cavity in its workspace and do not have cusps and nodes points. The domain 2 represents the manipulators that have 4 points of cusp, but do not have the same number of nodes.

The other surfaces are obtained from the annulment of determinant of Jacobian matrix J of the direct kinematic model given by Eqs. (1), using the Maple software, as follows:

$$\det(J) = d_4\left(d_3 + d_4 c_3\right)\left[d_2 s_3 + \left(d_3 s_3 + \left(d_3 s_3 - r_2 c_3\right) c_2\right)\right] \tag{36}$$

Considering $det\ (J) = 0$, this results in two possibilities:

$$d_3 + d_4 c_3 = 0 \tag{37}$$

or

$$s_3 + \left(d_3 s_3 - r_2 c_3\right) c_2 = 0 \tag{38}$$

From Equation (37), one can obtain $c_3 = -d_3/d_4$ and applying in fundamental equation of trigonometry results that $s_3 = \varepsilon\sqrt{1-\left(d_3/d_4\right)^2}$ with $d_3 \le d_4$ and $\varepsilon = \pm 1$.

In order to obtain the separation surface between the other domains, the values of c_3 and s_3 are replaced in Eq. (38) and the variable c_2 calculated as

$$c_2 = -\frac{\varepsilon\sqrt{1-(d_3/d_4)^2}}{\left(\varepsilon\sqrt{1-\left(d_3/d_4\right)^2}\right) d_3 + \left(d_3/d_4\right) r_2} \tag{39}$$

Remembering that $c_2 = \cos\theta_2$, it has:

$$-1 \le c_2 \le 1 \tag{40}$$

Manipulating the right side of the inequality (40), it is possible to obtain the surface C_2, which separates the domains 2 and 3:

$$c_2 \le 1 \Rightarrow \frac{1}{d_3}\left(\sqrt{1-\left(d_3/d_4\right)^2}\right) \le -\left(\sqrt{1-\left(d_3/d_4\right)^2}\right) + \left(r_2/d_4\right) \Rightarrow \left(\frac{1+d_3}{d_3}\right)\left(\sqrt{1-\left(d_3/d_4\right)^2}\right) \le \left(r_2/d_4\right)$$

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$$\Rightarrow\left(\frac{1+d_3}{d_3}\right)^2\left(1-\left(d_3/d_4\right)^2\right)\leq\left(r_2/d_4\right)^2 \Rightarrow\left(\frac{1}{d_4}\right)^2\left(r_2^2+(1+d_3)^2\right)\geq\left(\frac{1+d_3}{d_3}\right)^2 \Rightarrow d_4\leq\left(\frac{d_3}{1+d_3}\right)\sqrt{r_2^2+(1+d_3)^2}$$

Thus, the surface of separation C_2 between the domains 2 and 3 is defined by

$$C_2 : d_4 = \frac{d_3}{1+d_3}.\sqrt{\left(d_3+1\right)^2+r_2^2} \tag{41}$$

The domain 3 is composed by manipulators which present 2 cusps points on internal envelopment. In the case of domain 4, the manipulators have 4 points of cusp and 4 nodes. The surface C_3, which separates the manipulators of the domains 3 and 4, is obtained by using the left side of the inequality (40), considering $\varepsilon=\pm1$:

$$c_2\geq-1 \Rightarrow -\frac{1}{d_3}\sqrt{1-\left(\frac{d_3}{d_4}\right)^2}\geq-\left(\sqrt{1-\left(\frac{d_3}{d_4}\right)^2}\right)+\frac{r_2}{d_4} \Rightarrow \frac{1}{d_3}\sqrt{1-\left(\frac{d_3}{d_4}\right)^2}\leq\left(\sqrt{1-\left(\frac{d_3}{d_4}\right)^2}\right)-\frac{r_2}{d_4} \Rightarrow$$

$$\Rightarrow\left(\frac{1-d_3}{d_3}\right)\left(\sqrt{1-\left(\frac{d_3}{d_4}\right)^2}\right)\leq-\frac{r_2}{d_4} \Rightarrow \left(\frac{d_3-1}{d_3}\right)\left(\sqrt{1-\left(\frac{d_3}{d_4}\right)^2}\right)\geq\left(\frac{r_2}{d_4}\right) \Rightarrow \left(\frac{d_3-1}{d_3}\right)^2\left(1-\left(\frac{d_3}{d_4}\right)^2\right)\geq\left(\frac{r_2}{d_4}\right)^2$$

$$\Rightarrow\left(\frac{1}{d_4}\right)^2\left(r_2^2+(d_3-1)^2\right)\leq\left(\frac{d_3-1}{d_3}\right)^2 \Rightarrow d_4\geq\frac{d_3}{|1-d_3|}\sqrt{r_2^2+(d_3-1)^2} \text{ with } d_3>1$$

Therefore, the Equation (42) that defines the surface C_3 is given by

$$C_3 : d_4 = \frac{d_3}{d_3-1}.\sqrt{\left(d_3-1\right)^2+r_2^2} \tag{42}$$

Finally, the domain 5 corresponds to manipulators that have no cusp points. Unlike of manipulators of the type 1, the internal envelope is not defined by a toroidal cavity, but by a region with 4 solutions in IKM. To obtain the surface of separation C_4 between the domains 3 and 5, are used again to inequality $|c_2|\leq1$ considering $\varepsilon=\pm1$:

$$|c_2|\leq1 \Rightarrow C_2\geq-1 \Rightarrow -\frac{1}{d_3}\sqrt{1-\left(\frac{d_3}{d_4}\right)^2}\geq-\left(\sqrt{1-\left(\frac{d_3}{d_4}\right)^2}\right)-\frac{r_2}{d_4} \Rightarrow \left(\frac{1-d_3}{d_3}\right)\left(\sqrt{1-\left(\frac{d_3}{d_4}\right)^2}\right)\geq\left(\frac{r_2}{d_4}\right)\Rightarrow$$

$$\Rightarrow\left(\frac{1}{d_4}\right)^2\left(r_2^2+(1-d_3)^2\right)\leq\left(\frac{1-d_3}{d_3}\right)^2 \Rightarrow d_4\geq\frac{d_3}{|1-d_3|}\sqrt{r_2^2+(d_3-1)^2} \text{ with } \mathrm{d}_3\leq1$$

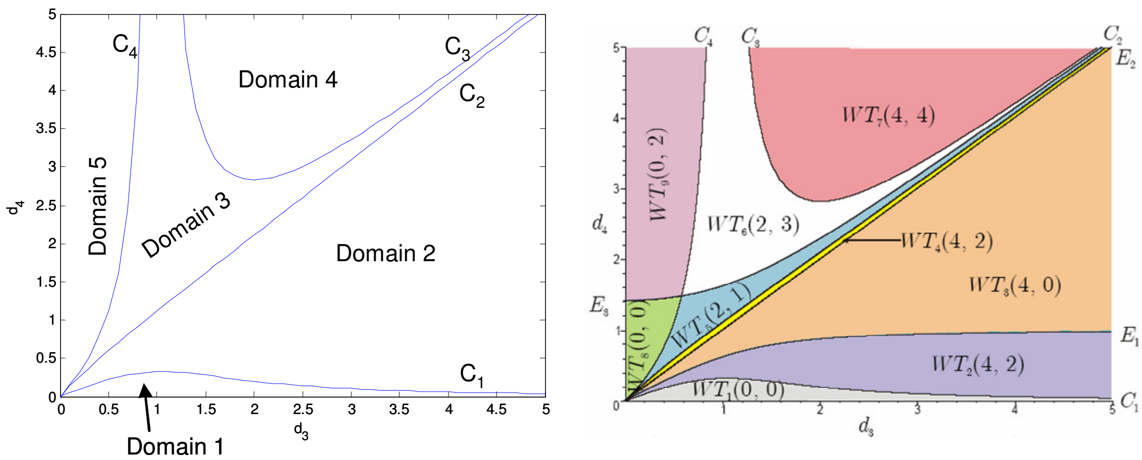
In this way, the equation (43) defines the surface of separation C_4:

$$C_4 : d_4 = \frac{d_3}{1-d_3}.\sqrt{\left(d_3-1\right)^2+r_2^2} \tag{43}$$

Summarizing, the space of parameters (d_3, d_4 and r_2) of a 3R orthogonal manipulator was divided into 5 domains separated by surfaces C_1, C_2, C_3 and C_4, defined by Eqs. (32), (41), (42) and (43), respectively.

The Figure 2a shows the curves of separation in a plane section (d_3, d_4) of the space of parameters, resulting in 5 domains, adopting a fixed value for $r_2=1$.

The Figure 2b shows the space of parameters divided according to the number of cusps points and nodes points. The domains according to the number of cusps points are divided into sub-domains that contain the same number of nodes. Each sub-domain defines a topology of the workspace denoted WT_i (α, β), where α represents the number of cusp points and β the number of nodes points.



(a) (b)

Figure 2. Division of parameters space considering $r_2 = 1$:
(a) Plane sections of the 4 surfaces; (b) according to the number of points of cusps and nodes

As explained above, the manipulators of the domain 1 have a toroidal cavity and do not have cusps and nodes points. The manipulator represented in Fig. 3a characterize the first type of manipulator, whose topology is known as $WT_1(0, 0)$.

The manipulators that belong to domain 2 have 4 points of cusp. This region can be subdivided into 3 sub-domains through the surfaces E_1 and E_2. The topology of the workspace $WT_2(4, 2)$, represented by Fig. 3b, has 4 cusp points, 2 nodes, a toroidal cavity, two regions with 4 solutions and a region with 2 solutions in IKM. The topology $WT_3(4, 0)$ contains manipulators with 4 cusp points, zero node, without toroidal cavity, a region with 4 solutions and other with 2 solutions in IKM, as illustrated in Fig. 3c. The transition between the topologies WT_2 and WT_3 is the boundary between the manipulators containing a toroidal cavity in its workspace and those that do not contain. According to Baili (2004), the surface of separation between these topologies is given by the expression:

$$E_1 : d_4 = \frac{1}{2}(A - B) \tag{44}$$

where A and B are given by Eq. (34) and (35), respectively.

In domain 2 is still possible to characterize the topology represented in the Fig. 4d, denoted by $WT_4(4, 2)$, containing 4 points of cusp and 2 nodes. These nodes are different from nodes of WT_2 since not delimit a toroidal cavity but a region of 4 solutions in IKM. In this case, the surface of separation E_2, between topology WT_3 and WT_4 is defined by:

$$E_2 : d_4 = d_3 \tag{45}$$

The domain 3 is composed by manipulators which have 2 cusp points and can be divided into 2 sub-domains through the surface E_3. The manipulators described by $WT_5(2, 1)$ have 2 cusps points on internal envelope, a node point and has the shape of a fish, as shown Fig. 3e. Moreover, the Fig. 5f presents a manipulator that belongs to the workspace $WT_6(2, 3)$, which has 2 points of cusp and 3 nodes. The Eq. (46) defines the separation surface between the topology WT_5 and WT_6. Besides, this surface also separates the topology of the workspace WT_8 and WT_9 that are contained in the domain 5. This last surface can be obtained as

$$E_3 : d_4 = \frac{1}{2}(A + B) \tag{46}$$

In domain 4, the manipulators are of type 4, represented by $WT_7(4, 4)$, have 4 cusp points and 4 nodes, as can be seen in Fig. 3g. The 4 points of cusp are shared between the internal and external singularity surfaces.

Finally, the domain 5 corresponds to manipulators that have no cusp points. Unlike of manipulators of the type 1, the internal envelopment is not defined by a toroidal cavity, but by a region with 4 solutions in IKM.

The domain 5 corresponds to manipulators of type 5 and do not have cusp points. This region is divided into 2 sub-domains through the surface E_3. In the Fig. 3h, the topology represented by WT_8 (0, 0) does not have cusp points and

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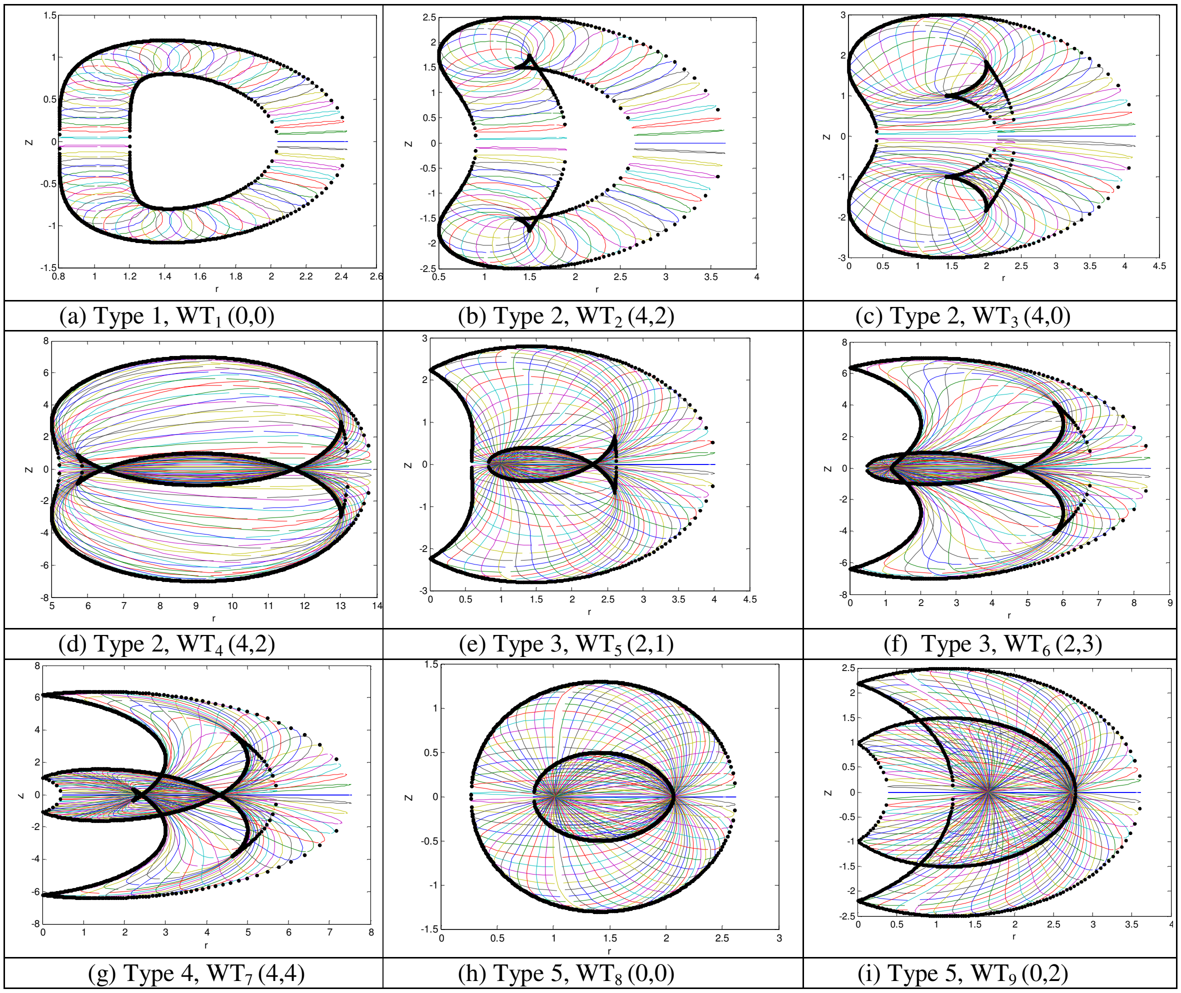
nor nodes. As mentioned earlier, its internal envelope is not defined by a toroidal cavity, but by a region with 4 solutions in IKM. Finally, Fig. 3i features a manipulator which belongs to the topology WT_9 (0, 2), with 0 cusp points and 2 nodes points obtained by the intersection of internal and external envelopment.



(a) Type 1, WT_1 (0,0)
(b) Type 2, WT_2 (4,2)
(c) Type 2, WT_3 (4,0)
(d) Type 2, WT_4 (4,2)
(e) Type 3, WT_5 (2,1)
(f) Type 3, WT_6 (2,3)
(g) Type 4, WT_7 (4,4)
(h) Type 5, WT_8 (0,0)
(i) Type 5, WT_9 (0,2)

Figure 3. Radial section for 3R orthogonal manipulator, showing the 5 types of manipulators

A first application of the surfaces obtained in this paper can be seen in Oliveira et al. (2008). In this work, an optimization problem is formulated with the objective of obtaining the optimal geometric parameters of the manipulator so as to maximize the workspace for a pre-determined topology.

3. CONCLUSIONS

This work showed the formulation of the separation surfaces in parameters space of manipulators with three rotational joints with orthogonal axes. It was described the process of obtaining the surfaces, as well as the corresponding curves due to changes in the parameters of 3R orthogonal manipulators. A brief classification of 3R manipulators was presented; this classification was based on two criteria to define a topology of the workspace (the number of cusp points and the number of nodes).

Detailed knowledge of the various regions of the space of parameters is important for the optimal design of manipulators which obeys a topology specified by the designer. In the next step of this research, the optimal design of manipulators will be formulated in order to obtain the maximum volume of the workspace, the maximum stiffness of the mechanism and the optimization of dexterity. In addition, the pre-determined topology of the workspace will be considered by using appropriate constraints, written according to the separation surfaces of different domains.

The objective of the developed formulation is obtained a useful methodology for the user, allowing the optimum design of manipulations considering the most appropriate topology for the tasks.

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