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ERROR ASSESSMENT OF UPWIND INTEGRAL TRANSFORMS: NON-LINEAR ONE-DIMENSIONAL CONVECTIVE HEAT TRANSFER

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Abstract. *This paper provides an error assessment of the Upwind Generalized Integral Transform Technique (UDS-GITT) for solving the nonlinear one-dimensional Burgers' equation. In this technique, the advective terms are approximated using an upwind discretization scheme (UDS) and the transformation of the problem is carried-out using the Generalized Integral Transform Technique (GITT). An average square error is proposed and a table showing the errors for many cases is displayed.*

Keywords: *Upwind Generalized Integral Transform Technique, Mixed Formulation, Advective-Diffusion Problems, Nonlinear Burgers' Equation*

1. NOMENCLATURE

ξ	spatial coordinate
τ	time
Θ	temperature
θ	filtered temperature
$\bar{\theta}$	transformed filtered temperature
Pe	Péclet number
U	velocity
α, β	velocity parameters
F	filter function
N	norm

Greek Symbols

Λ	discrete derivative
μ	eigenvalues
ψ	eigenfunctions
ϵ	mean square error
δ	discrete step-size
Λ	discrete derivative

Subscripts

i, j, k	GITT indexes

2. INTRODUCTION

The majority of numerical routines are based on discretizing the physical domain and approximating governing equation within the smaller sub-regions created by the discretization. A more recent methodology that combined ideas from numerical methods and analytical approaches, has been gaining popularity over the last few decades. This analytical-numerical methodology is the so-called Generalized Integral Transform Technique (GITT) (Cotta, 1990). This method has been demonstrated to serve as an effective instrument for solving advection-diffusion problems. Some of the most recent applications of the Generalized Integral Transform Technique include convective heat transfer in flows within microchannels (Sphaier, 2012; Castellões *et al.*, 2010), and the solution of general convection-diffusion problems using a unified integral transformation scheme (Sphaier *et al.*, 2011). Although both dicretization-based methodologies and eigenfunction-expansion solutions have been effectively applied to a variety of advection-diffusion problems, combined solution algorithms that utilize both types of methodologies have been used sparingly. GITT solutions have notable success rate when applied to diffusion problems. The method has also been applied to convective problems, but when advection is dominant a worse performance is generally seen. Since its common to use upwind approximation schemes (UDS) for handling advection transport terms, the idea of employing a hybrid methodology becomes particularly interesting for solving advective-dominant problems. This idea has been employed in recent works (Chalhub *et al.*, 2012b,a); nevertheless, an objective criteria for assessing the error of the combined UDS-GITT solutions has not been addressed. Under this scenario, this work proposes a criteria for assessing the global error of a UDS-GITT solution of a one-dimensional nonlinear convective heat transfer problem.

3. PROBLEM FORMULATION AND SOLUTION METHODOLOGY

The problem considered in this study is based on a one-dimensional non-linear formulation of Burgers' equation for convective heat transfer, given by the following equations:

$$\frac{\partial \Theta(\xi,\tau)}{\partial \tau} + U(\Theta)\,\frac{\partial \Theta(\xi,\tau)}{\partial \xi} = \frac{1}{\mathrm{Pe}}\,\frac{\partial^2 \Theta(\xi,\tau)}{\partial \xi^2}, \tag{1}$$

$$\Theta(0,\tau) = 1, \tag{2}$$

$$\left(\frac{\partial \Theta}{\partial \xi}\right)_{\xi=1} = 0, \tag{3}$$

$$\Theta(\xi,0) = 0, \tag{4}$$

where the velocity depends on the temperature field through:

$$U(\Theta) = \alpha + \beta\,\Theta \tag{5}$$

As usual in integral transform solutions, filtering is employed for removing non-homogeneous boundary terms. The solution separation $\Theta(\xi,\tau) = \theta(\xi,\tau) + F(\xi)$ is proposed, where the filter problem is chosen from a linearized steady version of the problem with similar boundary conditions:

$$\frac{\mathrm{d}F}{\mathrm{d}\xi} = \frac{1}{\mathrm{Pe}}\,\frac{\mathrm{d}^2 F}{\mathrm{d}\xi^2} \tag{6}$$

$$F(0) = 1, \tag{7}$$

$$F'(1) = 0 \tag{8}$$

which yields a constant solution $F(\xi) = 1$. With the solution separation, the following filtered system is obtained:

$$\frac{\partial \theta}{\partial \tau} + \big(\alpha + \beta\,(\theta + F)\big)\,\frac{\partial \theta}{\partial \xi} = \frac{1}{\mathrm{Pe}}\,\frac{\partial^2 \theta}{\partial \xi^2}, \tag{9}$$

$$\theta(\xi,0) = -F(\xi), \tag{10}$$

$$\theta(0,\tau) = 0, \tag{11}$$

$$\left(\frac{\partial \theta}{\partial \xi}\right)_{\xi=1} = 0, \tag{12}$$

The filtered problem is the transformed using and orthogonal eigenfunctions basis. Due to the nature of the selected problem, the simple one-dimensional Helmholtz problem is employed for providing the eigenfunctions:

$$\Psi''(\xi) + \mu^2\,\Psi(\xi) = 0, \tag{13}$$

$$\Psi(0) = 0, \tag{14}$$

$$\Psi'(1) + \mathrm{Bi}\,\Psi(1) = 0, \tag{15}$$

This eigenproblem, has the simple solution:

$$\Psi_i(\xi) \,=\, \sin(\mu_i\,\xi), \tag{16}$$

$$\mu_i \,=\, (i-1/2)\,\pi, \tag{17}$$

$$N_i \,=\, \int_0^1 \Psi_i^2(\xi)\,\mathrm{d}\xi \,=\, \frac{1}{2}, \tag{18}$$

for $i = 1, 2, \ldots$. Once the eigenfunctions have been calculated, the problem is transformed according to the following integral transformation pair:

$$\text{Inversion} \quad \rightarrow \quad \theta \,=\, \sum_{i=1}^{\infty} \frac{\bar{\theta}_i(\tau)\,\Psi_i(\xi)}{N_i}, \tag{19}$$

$$\text{Transform} \quad \rightarrow \quad \bar{\theta}_i(\tau) \,=\, \int_0^1 \theta(\xi,\tau)\,\Psi_i(\xi)\,\mathrm{d}\xi, \tag{20}$$

3.1 Traditional Solution Approach

Using the solution arising from the previous eigenproblem, the transformation of the original problem is carried-out by operating equation (9), and the associated initial condition, with the integral operator $\int_0^1 \bullet\,\Psi_i(\xi)\,\mathrm{d}\xi$, employing the associated boundary conditions, eigenproblem information, and the inversion formula. This leads to the following transformed system:

$$\frac{\mathrm{d}\bar{\theta}_i}{\mathrm{d}\tau} + \sum_{j=1}^{\infty} A_{i,j}\,\bar{\theta}_j + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} B_{i,j,k}\,\bar{\theta}_j\,\bar{\theta}_k \,=\, -\frac{\mu_i^2}{\mathrm{Pe}}\,\bar{\theta}_i, \tag{21}$$

$$\bar{\theta}_i(0) \,=\, \bar{f}_i^*, \tag{22}$$

for $i = 1, 2, \ldots, \infty$. The involved integral coefficients are given by:

$$A_{i,j} = \frac{1}{N_j}\int_0^1 (\alpha + \beta\,F(\xi))\,\Psi_i(\xi)\,\Psi_j'(\xi)\,\mathrm{d}\xi, \tag{23a}$$

$$B_{i,j,k} \,=\, \frac{\beta}{N_j\,N_k}\int_0^1 \Psi_i(\xi)\,\Psi_j(\xi)\,\Psi_k'(\xi)\,\mathrm{d}\xi, \tag{23b}$$

$$\bar{f}_i^* \,=\, -\int_0^1 F(\xi)\,\Psi_i(\xi)\,\mathrm{d}\xi. \tag{23c}$$

where, as previously mentioned, the double summation involving the coefficient $B_{i,j,k}$ arises from an additional substitution of the inversion formula into the velocity profile due to the non-linearity of the problem. The solution of the transformed potentials is then obtained by truncating the infinite system representation (21) to a finite order $i_{\max}$ and employing a commercially or publicly available dedicated ODE solver. In this study, *Mathematica*'s function NDSolve was employed for this purpose.

3.2 Proposed Solution Scheme

The proposed solution scheme involves employing an upwind approximation formula to the advective term:

$$U(\Theta)\,\frac{\partial\theta}{\partial\xi} \,\approx\, U(\Theta)\,\Lambda(\theta,\delta) \,=\, U(\Theta)\,\frac{f(\xi)-f(\xi-\delta)}{\delta}. \tag{24}$$

Based on the above approximation, the modified transformed system is given by the following equation:

$$\frac{\mathrm{d}\bar{\theta}_i}{\mathrm{d}\tau} + \sum_{j=1}^{\infty} A^{+}_{i,j}\,\bar{\theta}_j + \sum_{k=1}^{\infty}\sum_{k=1}^{\infty} B^{+}_{i,j,k}\,\bar{\theta}_j\,\bar{\theta}_k = -\frac{\mu_i^2}{\mathrm{Pe}}\,\bar{\theta}_i + \bar{p}_i^{+} \tag{25}$$

and the initial condition remains unchanged. The modified integral coefficients are given by:

$$A^{+}_{i,j} = \frac{1}{N_j}\left(\int_0^1 (1+\beta\,F(\xi))\,\Psi_i(\xi)\,\Lambda(\Psi_j,\delta)\,\mathrm{d}\xi + \beta\int_0^1 \Lambda(F,\delta)\,\Psi_i(\xi)\,\Psi_j(\xi)\,\mathrm{d}\xi\right), \tag{26a}$$

$$B^{+}_{i,j,k} = \frac{\beta}{N_j\,N_k}\int_0^1 \Psi_i(\xi)\,\Psi_j(\xi)\,\Lambda(\Psi_k,\delta)\,\mathrm{d}\xi, \tag{26b}$$

$$\bar{p}_i^{+} = \beta\int_0^1 F(\xi)\,\Lambda(F,\delta)\,\Psi_i(\xi)\,\mathrm{d}\xi. \tag{26c}$$

Finally, the infinite system representation given by (25) is truncated to a finite order $i_{\max}$ and numerically solved using a dedicated ODE solver.

4. RESULTS AND DISCUSSION

In order to assess the error of the different solutions an average square error, calculated based on a converged solution is defined:

$$\epsilon = \sqrt{\int_0^1 (\Theta_{conv}(\xi,\tau) - \Theta(\xi,\tau,n_{\max},\delta))^2\,\mathrm{d}\xi} \tag{27}$$

where $\Theta_{conv}(\xi,\tau)$ represents a converged solutions, whereas $\Theta(\xi,\tau,n_{\max},\delta)$ represents a solution calculated with a given truncation order and an upwind approximation parameter. The equivalent purely-GITT solution does not depend on the δ-parameter and can be simply given by $\Theta(\xi,\tau,n_{\max})$.

The following table present the average errors for many two nonlinear cases ($\alpha = 0.9$, $\beta = 0.1$ and $\alpha = 0.1$, $\beta = 0.9$) for $\mathrm{Bi} = 0$ and $\mathrm{Pe} = 1000$ and times $\tau = 0.5$ and $\tau = 1$. It is highlighted in bold the minimum error for each truncation order. It can be clearly seen that the optimum size of δ reduces with the increase of the truncation order. Also, the rise of the nonlinearity (β) requires of a higher δ in order to achieve the best error value. For the case where the front wave velocity is higher $\alpha = 0.9$ the increase in time requires a higher δ value.

5. CONCLUSIONS

This paper provides an error assessment of the Upwind Generalized Integral Transform Technique (UDS-GITT) for solving the nonlinear one-dimensional Burgers' equation. The results showed a good performance if a proper step-size (δ) is chosen. It can be clearly seen that the δ parameter must depend on the truncation order and the time for each case. As a result, an important suggestion for further research involves the determination of the optimum value of δ. One can even propose a variable d, which is function of x, in order to adapt the approach locally inside the domain. This proposition is similar to the one used with the flux limiters introduced by TVD (Total Variation Diminishing) schemes.

6. REFERENCES

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Table 1. Variation of the average square error with different nonlinearity cases, times and truncation orders for $\mathrm{Bi} = 0$ and $\mathrm{Pe} = 1000$.

		$n_{\max}$								
	δ	5	10	15	20	30	35	40	45	50
$\tau = 0.5$	GITT	0.1076	0.0510	**0.0244**	**0.0122**	**0.0033**	**0.0019**	**0.0013**	**0.0009**	**0.0007**
$\alpha = 0.9$	0.0001	**0.108**	**0.051**	0.025	0.013	0.004	0.003	0.003	0.003	0.002
$\beta = 0.1$	0.001	0.109	0.053	0.029	0.021	0.017	0.016	0.016	0.016	0.015
	0.005	0.112	0.066	0.054	0.052	0.051	0.051	0.050	0.050	0.050
	0.01	0.118	0.082	0.077	0.076	0.075	0.075	0.075	0.075	0.075
	0.05	0.158	0.151	0.151	0.151	0.151	0.151	0.151	0.151	0.151
	0.1	0.194	0.193	0.193	0.193	0.193	0.193	0.193	0.193	0.193
$\tau = 1$	GITT	0.1283	0.0522	0.0204	0.0085	**0.0022**	**0.0020**	**0.0017**	**0.0013**	**0.0011**
$\alpha = 0.9$	0.0001	0.128	0.052	**0.020**	**0.008**	0.003	0.003	0.003	0.003	0.002
$\beta = 0.1$	0.001	0.127	**0.050**	0.022	0.017	0.018	0.018	0.018	0.018	0.018
	0.005	0.124	0.061	0.052	0.053	0.054	0.054	0.054	0.054	0.054
	0.01	**0.123**	0.077	0.074	0.075	0.076	0.076	0.076	0.076	0.076
	0.05	0.142	0.132	0.132	0.132	0.132	0.132	0.132	0.132	0.132
	0.1	0.162	0.157	0.156	0.156	0.156	0.156	0.156	0.156	0.156
$\tau = 0.5$	GITT	0.163	0.188	0.198	0.173	0.103	0.085	0.061	0.053	0.045
$\alpha = 0.1$	0.0001	0.163	0.188	0.181	0.153	0.095	**0.081**	**0.060**	**0.051**	**0.045**
$\beta = 0.9$	0.001	0.163	0.183	0.169	**0.122**	**0.087**	0.083	0.084	0.081	0.083
	0.005	0.161	0.167	**0.131**	0.133	0.158	0.166	0.170	0.173	0.175
	0.01	0.160	**0.158**	0.148	0.171	0.199	0.203	0.205	0.206	0.207
	0.05	**0.157**	0.191	0.225	0.232	0.229	0.225	0.221	0.219	0.218
	0.1	0.160	0.220	0.224	0.207	0.206	0.207	0.206	0.205	0.205
$\tau = 1$	GITT	0.291	0.349	0.240	0.176	0.102	0.078	0.064	0.053	**0.043**
$\alpha = 0.1$	0.0001	0.290	0.343	0.232	0.167	0.097	**0.075**	**0.062**	**0.053**	0.045
$\beta = 0.9$	0.001	0.286	0.296	**0.185**	**0.141**	**0.127**	0.131	0.135	0.138	0.141
	0.005	0.271	**0.205**	0.209	0.232	0.247	0.252	0.258	0.261	0.263
	0.01	0.257	0.215	0.266	0.284	0.301	0.306	0.308	0.309	0.309
	0.05	**0.250**	0.342	0.353	0.354	0.351	0.348	0.345	0.343	0.343
	0.1	0.286	0.348	0.343	0.328	0.326	0.326	0.325	0.324	0.324

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