AGGLOMERATION SCHEMES STUDIES IN THE ADDITIVE CORRECTION MULTIGRID METHOD APPLIED TO NUMERICAL SOLUTION OF NAVIER-STOKES EQUATIONS WITH UNSTRUCTURED GRIDS

Susie C. Keller, susie.keller@joinville.ufsc.br
Mobility Engineering Center - Federal University of Santa Catarina
Joinville, SC, Brazil

Clovis R. Maliska, maliska@sinmec.ufsc.br
SINMEC – Laboratory of Numerical Simulation in Fluid Dynamics and Computational Heat Transfer, Dept. of Mechanical Engineering, Federal University of Santa Catarina
Florianopolis, SC, Brazil

Abstract. *Multigrid methods efficiently solve large linear equations systems resulting from the discretization of partial differential equations. These methods improve the convergence rate of iterative methods by reducing all error frequency modes iteratively solving the problem in several meshes. The Additive Correction Multigrid Method (ACM) has several advantages compared to the traditional multigrid methods, normally the geometrical ones, in which the agglomeration is done based on the geometry of the grid and interpolation is required for passing the solution from one grid to the coarse one. In the ACM the discretization is done only in the original (fine) grid with the approximate equations in the coarse grid obtained using coefficients from the previous grid. This paper presents results obtained with ACM parameters previously tested, and special attention is given to the agglomeration process. The agglomeration scheme is of utmost importance for the method and, in order to improve this procedure, different ways to perform cells agglomerations were tested here. For instance, current literature only mentions the usage of pressure coefficient-based schemes for the mass conservation equation (Raw, 1996), but this work shows that other schemes also produce good results.*

Keywords: *Additive Correction Multigrid, Agglomeration Schemes, Navier-Stokes Equations, Unstructured Grids*

1. INTRODUCTION

The main objective of this paper is to analyse some agglomeration schemes suitable for using with the Additive Correction Multigrid Method (Hutchinson and Raithby, 1986). This method enables the use of adaptive agglomeration of the cells based on the strength of the coefficients taking into account, therefore, the anisotropy causes on the numerical method performance. As the coefficient contains information of the physics as well as of the geometry of the problem, if the agglomeration is performed efficiently, the resulting matrix will be coefficient-balanced, improving considerably the rate of convergence.

In all the cases here analyzed the ACM method will be applied to solve fluid flow problems with the differential equations been discretized by the EbFVM-Element based Finite Volume Method (Raw, 1985; Maliska, 2004). This paper deals with 2D problems, having, therefore, nine coefficients forming the numerical stencil.

2. ADDITIVE CORRECTION MULTIGRID METHOD (ACM)

The Additive Correction Multigrid method (ACM) distinguishes from conventional multigrid methods for generating coarse grid equations without the use of fixed stencils. Classical multigrid methods form coarse-grid equations by discretizing the governing equations on each grid and interpoling the fine-grid residuals to coarser-grid equations. Obviously, this is not suitable for conservative methods like EbFVM, since conservation is lost when one passes from one grid to another. And conservation at discrete level is highly desirable for robustness of the scheme. ACM forms coarse-grid equations by asserting integral conservation over blocks of control volumes. It determines a constant correction in each coarse grid cell by forcing the sum of residuals to be zero after the correction is applied.

In the ACM method, discretization is made only in the finest grid, which makes the computational complexity and cost smaller than other multigrid methods, eliminating the possibility of inconsistent approximations between the grids. Coarse-grid equations are formed by adding up fine-grid equations, which requires integral form of the conservation equation to be satisfied over the block. The solution of equations in a coarse grid and its subsequent adjustment into a fine grid produces a conservative solution on each block. This is a physically desirable property, as pointed above.

To exemplify the ACM, an unstructured mesh is depicted in Fig. 1. Even though the blocks may be formed in any convenient way, in this example the volumes are joined in blocks with four cells on average. We can see that the new coarse-grid blocks (B1, B2, …, B9) are formed by joining fine-grid control volumes in different directions, resulting in an agglomeration with highly unstructured forms.

Initially, we can considerate the system of equations to be solved written in the common form as

$$A_p^i \phi_i = \sum_{nb} A_{nb}^i \phi_{nb} + b_i \,, \tag{1}$$

where A_p^i is the central coefficient of the control volume i considered in the discrete equation for ϕ, A_{nb}^i are the coefficients connecting the control volume i to the neighbors control volumes, b_i is the source coefficient and ϕ is the solution. The coefficients of Eq. (1) can be obtained by applying Finite Volume Method (FVM) or Element based Finite Volume Method (EbFVM) to either structured or unstructured grids (Maliska, 2004).

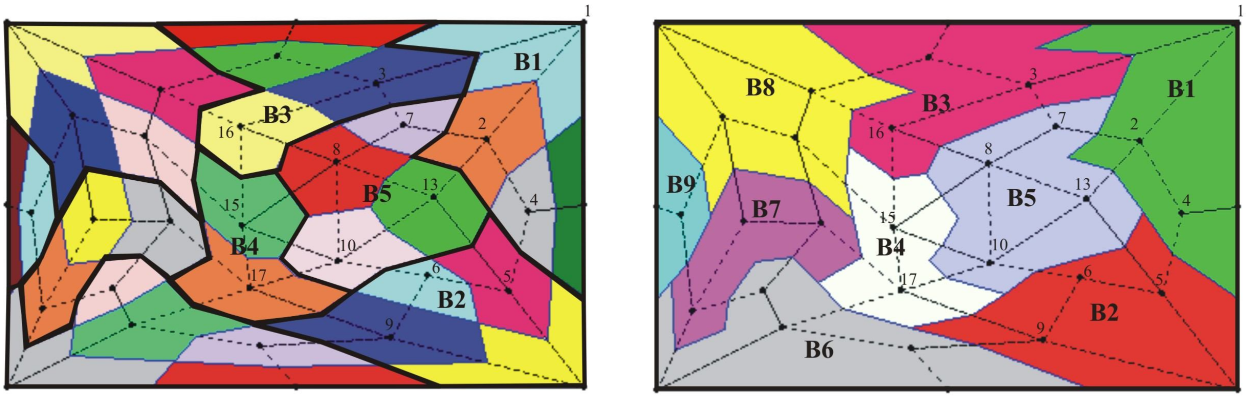


Figure 1. Unstructured mesh: (a) fine mesh divided in blocks, (b) coarse mesh.

The core of ACM is the definition of a correction equation that has the role of adding coarse-grid corrections (ϕ^*) to the best estimate of ϕ on the fine grid, as

$$\tilde{\phi}_i == \phi_i + \phi_{I,i}^* \,, \tag{2}$$

where $\tilde{\phi}_i$ is the improved solution in each cell and $\phi_{I,i}^*$ is the correction related to all volumes that lie within the I block.

Requiring that the residual be zero for the corrected solution, one obtains the linear system for $\phi_{I,i}^*$ (Hutchinson and Raithby, 1986)

$$A_P^* \phi_P^* = \sum_{nb} A_{NB}^* \phi_{NB}^* + b_P^* \,, \tag{3}$$

where

$$b_P^* = \sum_{i,I} r_i \,, \tag{4}$$

$$A_P^* = \sum_{i,I} A_p^i - \sum A_{nb}^i \,, \tag{5}$$

$$A_{NB}^* = \sum_{i,I} A_{nb}^i \,. \tag{6}$$

In these equations $\sum A_{nb}^i$ denotes the connection among the control volumes inside of one same block and $\sum_{i,I} A_{nb}^i$ represents the connection among volumes of neighbor blocks.

Equation (3) must be solved in order to obtain the correction $\phi_{I,i}^*$. This correction is added to each ϕ value of control volumes that lie within the I-block, Eq. (2). Thus, the improved estimate $\tilde{\phi}_i$ is obtained, as already stated. Additional details of the ACM method can be found elsewhere (Hutchinson and Raithby, 1986; Maliska, 2004; Keller, 2007).

2.1 The Additive Correction Multigrid Method applied to the Coupled Solution of Linear Equation Systems

The ACM strategy can be applied to solve a coupled system of partial differential equations. In our case, the coupling is between pressure and velocity. In this case some observations must be done. The two-dimensional fluid flow problems herein discussed are described by three fundamental equations (Navier-Stokes in x and y directions and mass conservation equation). In a coupled form, these equations represent the velocities in x and y-directions and the pressure. These strategies are used so that all three dependent variables are active in the three equations (Trottenberg et al., 2001). Thus, the coefficients matrix has nine coefficients for each control volume. The linear system has $3xN$ equations with $3xN$ unknowns, where N is the number of control volumes of discretized domain. The resulting matrix is a blocked matrix with 3x3 coefficients matrix in each block, as represented by Eqs. (7) and (8),

$$\begin{bmatrix} [A] & \cdots & & [A] & & \\ & [A] & \cdots & & & [A] \\ & & \cdot & & & \\ [A] & \cdots & \cdot & \cdot & & \\ & & \cdot & & \cdot & \\ & & [A] & \cdots & & [A] \end{bmatrix} \begin{bmatrix} [\phi] \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} [B] \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \tag{7}$$

with the block matrices given by

$$[A] = \begin{bmatrix} [A^{uu}] & [A^{uv}] & [A^{uP}] \\ [A^{vu}] & [A^{vv}] & [A^{vP}] \\ [A^{Pu}] & [A^{Pv}] & [A^{PP}] \end{bmatrix} \qquad [\phi] = \begin{bmatrix} u \\ v \\ P \end{bmatrix} \qquad [\mathrm{B}] = \begin{bmatrix} B^u \\ B^v \\ B^P \end{bmatrix} \tag{8}$$

where the first line of the matrix represents Navier-Stokes in x-direction: A^{uu} is velocity u (x-direction), A^{uv} is velocity v (y-direction) and A^{uP} is the pressure. The second line of the matrix represents Navier-Stokes in y-direction: A^{vu} is velocity u (x-direction), A^{vv} is velocity v (y-direction) and A^{vP} is the pressure. The third line of the matrix represents mass conservation equation where A^{Pu} is velocity u (x-direction), A^{Pv} is velocity v (y-direction) and A^{PP} is the pressure.

This blocked matrix is to be solved by an iterative scheme using the ACM multigrid method to accelerate the convergence rate of the process.

3. ADAPTIVE AGGLOMERATION SCHEME

The process begins with the choice of a fine grid for solving the problem. The coefficients are generated, the solution in the finer grid determined up to a level in which the explicit method is still efficient. When the convergence is poor on that grid, another grid is obtained based on the agglomeration algorithm. Therefore, through a set of rules, it is determined which of the neighbors control volumes will form the new control volume, generating the coarser grid.

Following the nomenclature of a typical family tree, as proposed by Elias et al (1997), we call the fine grid cell, which neighbors are being examined, as the parent, and its neighbors included in the same coarse cell as its children. The current parent's parent is known as the grandparent.

There are two main rules used to decide which fine grid cells are added to the new coarse grid cells considered (Elias, 1993). The first agglomeration rule states that a neighbor of a parent cell can be a child if the transport timescale between the parent and the neighbor is of the same order or smaller than the timescale between the parent and the grandparent. Representing the parent by the index i, the neighbor by j and the grandparent by h, we have that j is a possible child of i if:

$$\max\left(a_{i,j}, a_{j,i}\right) \geq \max\left(a_{i,h}, a_{h,i}\right)/2 \, . \tag{9}$$

The second rule states that a cell is excluded if the interface timescale is very large. The concept of what would be "large" is defined relative to all other timescales which affect the cell in question. A possible child can be agglomerated with the parent if the timescale between the parent and the possible child is of the same order or smaller than the timescale between the child and its other neighbors. Thus, taking the coefficients again and representing the parent by the index i, the possible child by the index j and the possible child's neighbors by the index h, j is agglomerated with i if:

$$\max\left(a_{i,j}, a_{j,i}\right) \geq \max\left(a_{j,h}, a_{h,j}\right)/2 \, . \tag{10}$$

In order to improve the efficiency of this agglomeration scheme some details must be also considered. It can be seen in more details in (Keller, 2007).

3.1 Agglomeration Schemes for the Coupled Solution of Linear Equations System

The task is to choose from the available coefficients, which ones will be used to perform the agglomeration: the nine coefficients of matrix *A* (Eq. (2)) or between any possible combinations of those. Since the system under consideration is a coupled one, it is not clear which equation coefficients should be used, since more than one equation, with different physics would require different coarse grids. In this paper the nine coefficients, some matrix norms and a matrix determinant, shown below, were tested.

3.1.1 Frobenius norm

The Frobenius norm of an *m x n* matrix *A* is defined as the square root of the sum of the absolute square of its elements:

$$\|A\|_F = \left(\sum_{j=1}^{m} \sum_{i=1}^{n} |a_{ij}|^2 \right)^{1/2} \tag{11}$$

3.1.2 Sum norm (maximum absolute colum sum norm):

The Sum norm matrix is the maximum colum sum of the absolute values of matrix *A* elements:

$$\|A\|_1 = \max_{j=1,\dots,m} \sum_{i=1}^{n} |a_{ij}| \tag{12}$$

3.1.3 Maximum norm (maximum absolute row sum norm):

The Maximum norm matrix is the maximum row sum of the absolute values of matrix *A* elements:

$$\|A\|_\infty = \max_{i=1,\dots,n} \sum_{j=1}^{m} |a_{ij}| \tag{13}$$

3.1.4 Euclidian norm:

This norm is defined by:

$$\|A\|_2 = \left[\rho\left(A^H A\right)\right]^{1/2} = \left[\rho\left(A A^H\right)\right]^{1/2} \tag{14}$$

where $\rho(A)$ is the spectral ratio of matrix *A*.

The spectral ratio is defined as $\rho(A) = \max_{i=1,\dots,n} |\lambda_i|$, with $\rho_1, \dots, \rho_n$ being the eigenvalues of matrix *A*.

3.1.5 Trace of matrix

The trace of *n x n* matrix *A* is:

$$tr(A) = \sum_{i=1}^{n} a_{ii} \tag{15}$$

3.1.6 Matrix determinant

A matrix determinant of third order is one of the options tested. The results obtained by applying the above schemes and coefficients of matrix (Eq. (8)) are presented in next Section.

4. RESULTS

The problem chosen to perform the tests was the lid driven cavity with Reynolds number equal to 1000. The parameters used were: W cycle multigrid, ILU solver as the base solver, direct solver with maximum of 60 cells, 5 fixed iterations for the iterative solver, maximum of 50 cycles, agglomeration on the beginning of the process only, and maximum solution residue equal to 10^{-5}. The agglomeration schemes used are presented in Tab. 1.

Table 1. Agglomeration parameters.

Agglomeration Scheme	Kind of Agglomeration	Agglomeration Scheme	Kind of Agglomeration	Agglomeration Scheme	Kind of Agglomeration
1	Coef. A^{uu}	6	Coef A^{vP}	11	Sum Norm
2	Coef. A^{uv}	7	Coef. A^{Pu}	12	Maximum Norm
3	Coef. A^{uP}	8	Coef. A^{Pv}	13	Euclidian Norm
4	Coef. A^{vu}	9	Coef. A^{PP}	14	Trace of matrix
5	Coef. A^{vv}	10	Frobenius Norm	15	Matrix Determinant

As can be seen in Tab. 2, in the finest grid, 62883 volumes, the 9th scheme showed a gain in computational time compared to other schemes. This scheme uses the pressure coefficient in continuity equation. However, one can observe that in unstructured grids formed by elements, the order the elements are arranged is defined by the generation of the grid. Thus, small variations of computational time do not determine the best agglomeration scheme.

Table 2. Computational times for the agglomeration schemes in the lid driven cavity problem in grid with 62883 volumes.

Agglomeration Scheme	Processing time (s)
1	514,391
2	535,438
3	566,422
4	535,859
5	534,625
6	600,5
7	563,906
8	601,047
9	499,813
10	513,922
11	513,078
12	513,547
13	518,391
14	510,735
15	510,734

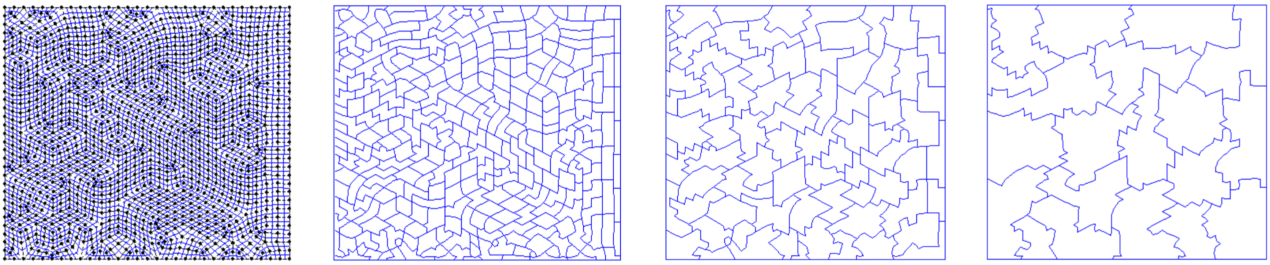
Schemes 1, 9, 10, 11, 12, 13, 14 and 15 presented the better results in CPU time; therefore they were tested in other grid sizes (Tab. 3). It should be noted that each mesh here defined is the original (fine) mesh and as many coarse meshes as necessary can be obtained from that mesh until you obtain the coarsest mesh having 60 volumes at most.

As can be seen in Tab. 3, using other grids (25681, 10167, 2627 and 1089 volumes), the 9th scheme had a worse time that schemes 1, 10, 11, 12 and 13. In general, we can say that the 9th scheme is the best, but other schemes are good alternatives too.

Table 3. Computational times for the agglomeration schemes in the lid driven cavity problem.

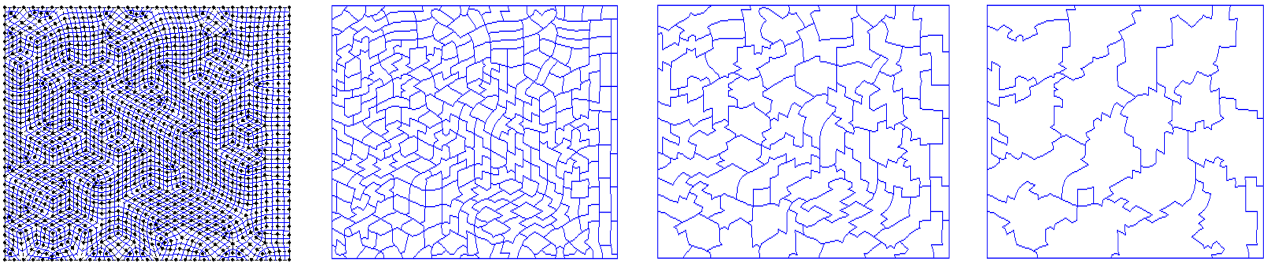
Agglomeration Scheme	Processing time (s)				
	Grid with 1089 volumes	Grid with 2627 volumes	Grid with 10167 volumes	Grid with 25681 volumes	Grid with 62883 volumes
1	5,75	15,25	66,875	179,031	514,391
9	5,734	15,703	65,453	180,984	499,813
10	5,688	15,328	65,25	178,64	513,922
11	5,75	15,235	66,953	178,375	513,078
12	5,718	15,266	65,125	179,593	513,547
13	5,781	15,453	67,485	179,578	518,391
14	5,797	15,734	65,5	184,266	510,735
15	6,375	16,719	70,032	186,562	510,734

Figure 2 shows the original grid and three levels of agglomeration applied to a grid with 1089 volumes using the 9th scheme. It can be noted that, on the right and left wall, volumes are more elongated following the direction of flow (the effect of advective terms). Scheme 14 also showed good results and the meshes resulting from agglomeration are presented in Fig 3. It can be observed that the meshes shown in Fig. 3 are similar to meshes of Fig. 2.



(a) (b) (c) (d)

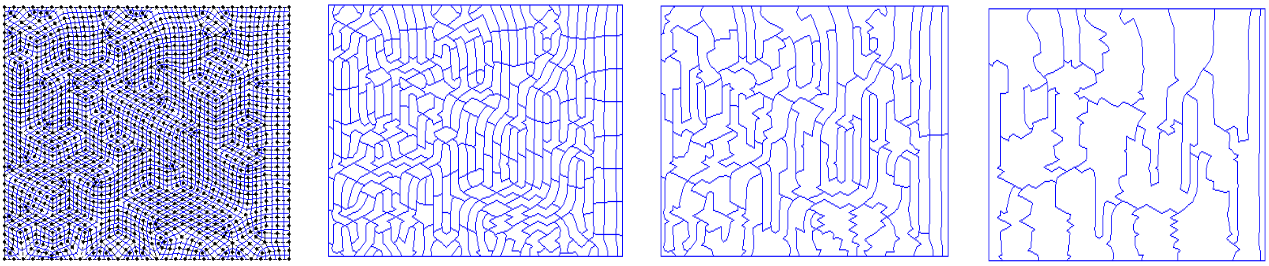
Figure 2. Grids resulting from the adaptive agglomeration of the 9th scheme: (a) original grid (1089 volumes), (b) 281 volumes, (c) 75 volumes and (d) 21 volumes.



(a) (b) (c) (d)

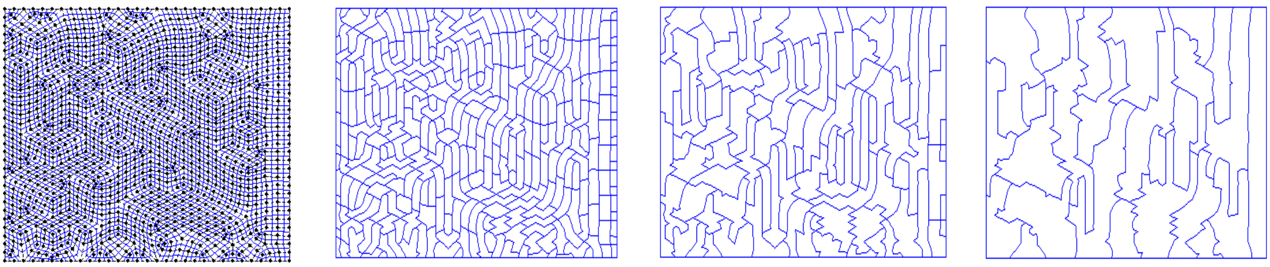
Figure 3. Grids resulting from the adaptive agglomeration of the 14th scheme: (a) original grid (1089 volumes), (b) 282 volumes, (c) 77 volumes and (d) 21 volumes.

Figures 4 and 5 show the meshes resulting from the use of schemas 8 (velocity *v* coefficient in mass conservation equation) and 6 (pressure coefficient in Navier –Stokes y-direction equation) that had the worst results. It can be noted that the flow direction is vertical because it involves velocity v (in scheme 8) and the equation for velocity v (in scheme 6). These meshes do not represent the physical phenomenon properly and therefore does not present good results.



(a) (b) (c) (d)

Figure 4. Grids resulting from the adaptive agglomeration of the 8th scheme: (a) original grid (1089 volumes), (b) 290 volumes, (c) 81 volumes and (d) 23 volumes.



(a) (b) (c) (d)

Figure 5. Grids resulting from the adaptive agglomeration of the 6th scheme: (a) original grid (1089 volumes), (b) 299 volumes, (c) 84 volumes and (d) 25 volumes.

Results obtained performing the agglomeration process every time the linear system is modified are presented below. In this case, meshes generated are consistent with the physical phenomenon, once fluid flow main directions are considered. On the other hand, this process produces high computational costs due to the agglomeration process being performed several times.

Comparing Tab. 4 with Tab. 2, it can be seen that performing the agglomeration process every time the linear system is modified increases the computational time considerably. Thus, this procedure does not provide enough benefits to justify its implementation.

Table 4. Computational times for the agglomeration schemes in the lid driven cavity problem in the grid with 62883 volumes, agglomerating every time the linear system is modified.

Agglomeration Scheme	Processing Time (s)
1	620,938
2	650,391
3	707,672
4	654,61
5	638,703
6	758,781
7	709,718
8	736,234
9	630,687
10	627,687
11	615,64
12	633,641
13	742,656
14	653,578
15	652,704

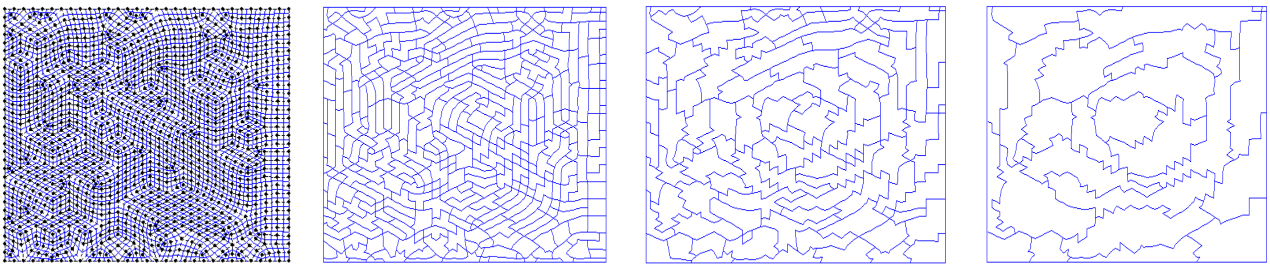
Still, after testing all agglomeration schemes in the grid with 62883 volumes (Tab. 4), the most promising agglomeration schemes are analyzed in Tab. 5. These are schemes number 1, 5, 9, 10, 11 and 12. In general, schemes 1 (velocity *u* coefficient in Navier –Stokes *x*-direction equation) and 11 (sum norm) presented the best results.

It can be noted that these results differ from results obtained previously, where agglomeration was performed only once. In that case the best result was obtained using scheme 9.

Table 5. Computational times for the agglomeration schemes in the lid driven cavity problem, agglomerating every time the linear system is modified.

Agglomeration Scheme	Processing time (s)				
	Grid with 1089 volumes	Grid with 2627 volumes	Grid with 10167 volumes	Grid with 25681 volumes	Grid with 62883 volumes
1	6,484	17,453	74,313	202,125	620,938
5	9,657	17,437	74,437	211,328	638,703
9	9,438	17,141	74,016	209,969	630,687
10	9,562	17,359	74,453	205,859	627,687
11	9,422	17,219	74,438	208,266	615,64
12	9,422	17,312	74,328	210,438	633,641

Figure 6 shows the original grid and three levels of agglomerations using scheme 1. This grid shows the effects of convective terms in x-direction, mainly in the last grid. Scheme 11 generates grids shown in Fig. 7. In this case, there is a regular distribution of cells.



(a) (b) (c) (d)

Figure 6. Grids resulting from the adaptive agglomeration of the 1th scheme: (a) original grid (1089 volumes), (b) 281 volumes, (c) 75 volumes and (d) 21 volumes.

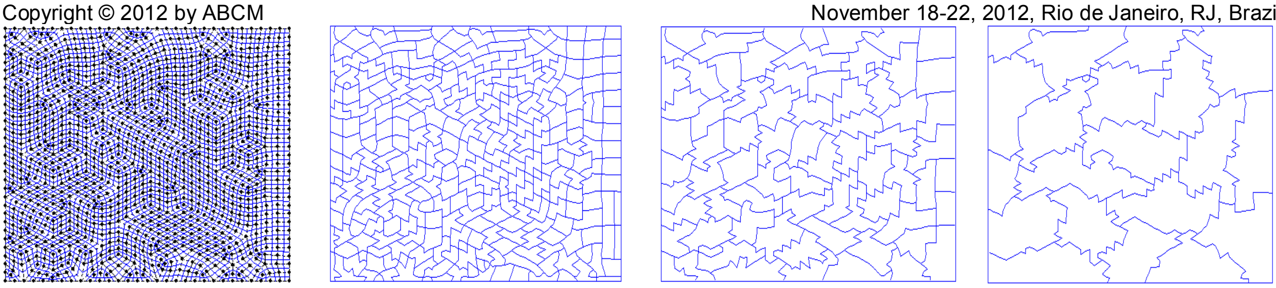


Figure 7. Grids resulting from the adaptive agglomeration of the 11th scheme: (a) original grid (1089 volumes), (b) 281 volumes, (c) 75 volumes and (d) 21 volumes.

5. CONCLUSIONS

The main purpose of this study was the implementation of the Additive Correction Multigrid method with several different agglomeration schemes, since the literature only mentions the pressure variable in the mass conservation equation (Raw, 1996). Fifteen schemes were created to perform the agglomerations. Nine of them are based on the coefficients of the control volume matrix which arise when a three equations system (Navier-Stokes in x and y directions plus the mass conservation equation) is estimated in order to yield a coupled set of equations. The other six schemes were devised by aiming at finding a characteristic value that represents the matrix. Different types of norms were chosen, such as the determinant and the trace of the matrix in order to obtain this characteristic value. Many of these schemes have shown good results, mainly schemes 1, 9, 10, 11, 12, 13, 14 and 15.

An interesting aspect of this study, which should be developed in future works, is the fact that the agglomeration procedure performed only at the beginning of the process showed better results than when it is performed each time the linear system is modified. This indicates that while the agglomeration that is performed multiple times creates the most suitable linear system to be solved by the ACM internal solvers, this procedure takes excessive computing time. Setting up a procedure to verify if the linear system has suffered significant changes may be an interesting option. Thus, the agglomeration would be performed only a few times during the solution process. However the implementation of a procedure to verify changes in linear system incurs in the increase of computation time for solving the problem. Depending on how the procedure is done, it can also become unviable. Another alternative would be to define the execution of agglomerations every *n* number of times the linear system is modified.

The different parameters that can be exhaustively tested are some of the interesting aspects of studying the ACM method. That's because by knowing all the features of the method one can promote changes to make it a more efficient and robust solver. More tests should be performed in different physical problems as well in order to improve this Multigrid method.

6. BIBLIOGRAFY

Elias, S. R., 1993 *Enhancements to Additive Correction Multigrid*, Master thesis, Waterloo, Ontario, Canada.

Elias, S.R., Stubley, G.D., Raithby, G.D., 1997 "An Adaptative Agglomeration Scheme for Additive Correction Multigrid", *International Journal of Numerical Methods in Engineering*, Vol. 4 (5), pp. 887-903.

Hutchinson, B. R., Raithby, 1986 "A Multigrid Method based on the Additive Correction Strategy", *Numerical Heat Transfer*, Vol. 9, pp. 511-537.

Keller, S. C., 2007, *O Método Multigrid de Correções Aditivas para a Solução Numérica Acoplada das Equações de Navier-Stokes com Malhas Não-Estruturadas*, tese de doutorado, Universidade Federal de Santa Catarina, Florianópolis, Brasil.

Maliska, C. R., 2004, *Transferência de Calor e Mecânica dos Fluidos Computacional*, Ed. LTC, Rio de Janeiro, Brazil.

Raw, M. J., 1985, *A New Control-Volume-Based Finite Element Procedure for the Numerical Solution of the Fluid Flow and Scalar Transport Equations*, Doctor thesis, Waterloo, Ontario, Canada.

Raw, M. J., 1996, "Robustness of Coupled Algebraic Multigrid for the Navier-Stokes Equations", *AIAA 96-0297, 34th Aerospace and Sciences Meeting & Exhibit*, 15-18 de Janeiro, Reno, NV.

Trottenberg, U., Oosterlee, C., Schüller, A., 2001,*Multigrid,* Academic Press, Londres, Inglaterra.

7. RESPONSIBILITY NOTICE

The authors are the only responsible for the printed material included in this paper.