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INTERNAL MIXED CONVECTION: - CRITERIA FOR TRANSITION FROM NATURAL TO FORCED REGIME (PRESCRIBED WALL TEMPERATURE)

Jacques PADET
Université de Reims Champagne Ardenne, Faculté des Sciences
Laboratoire de Thermomécanique, B.P. 1039, 51687 Reims, France
Email: jacques.padet@univ-reims.fr

Renato M. COTTA
Engenharia Mecânica – COPPE-POLI/UFRJ
Universidade Federal do Rio de Janeiro, Brasil
Cx. Postal 68503 – Cidade Universitária
Rio de Janeiro, RJ, 21945-970
Email: cotta@serv.com.ufrj.br

Nelu - Cristian CHERECHES
Technical University "Gh. Asachi" of Iasi
Str. Lascar Catargi, nr. 38, cod 700107, Iasi, Romania
Email: nc.chereches@univ-reims.fr

Nadim EL - WAKIL
Université de Reims Champagne Ardenne, Faculté des Sciences
Laboratoire de Thermomécanique, B.P. 1039, 51687 Reims, France
Email: nadim.elwakil@univ-reims.fr

Abstract. *Fully developed mixed convection between parallel plates for steady-state laminar flow is analyzed by making use of the Mathematica system symbolic computation capabilities. The expressions obtained for the fully developed velocity, pressure and temperature distributions, are employed to examine different criteria for the definition of the relative importance of the natural and forced convection effects, always in terms of the Reynolds and Richardson numbers. Initially, three previously studied criteria are considered, namely, the ratio of wall shear stresses, the ratio of the quadratic means of the buoyancy and viscous forces terms, and the ratio of the quadratic means of the buoyancy and pressure forces terms. A new criterion based on the ratio of kinetic energy generated in the flow due to natural convection and that generated in total, due to both natural and forced effects, is also proposed. A closer examination of this criterion is then performed, in the attempt to establish recommendations for practical use. An application dealing with convection in water flow is presented for illustration.*

Keywords. Internal Convection, Channel Flow, Transition, Symbolic Computation, Mixed Convection.

1. Introduction

The study of mixed convection within channels and ducts has historically received less attention than the analysis of pure forced or natural convective heat transfer. Nevertheless, these two extreme situations are in fact special cases of the more general formulation accounting for the combination of the buoyancy and imposed pressure gradient effects. In addition, the literature lacks a general criterion for the selection of the range of validity of these specific situations, in terms of the governing parameters or relevant dimensionless groups. Fully developed flow conditions were considered by Aung and Worku (1986) for laminar mixed convection inside parallel plates at different prescribed temperatures, and later employed by Padet (1997) to investigate the conditions of predominance of either the natural or forced effects in this fairly general situation. Padet (1997) established a criterion based on the ratio of wall shear stresses, in terms of the product of the Reynolds and Richardson numbers. Additional studies about fully developed mixed convection between two parallel plates at uniform wall temperature, uniform temperature on a wall and a uniform wall heat flux on the opposite wall or uniform wall heat fluxes on both walls, have also been performed by Cheng et all. (1990), A. Barletta and E. Zanchini (1998), Hamadah and Wirtz (1991).

The developed *Mathematica* notebook here reported performs the symbolic computation of fully developed mixed convection between parallel plates, for steady-state laminar flow. The expressions obtained for the fully developed velocity, pressure and temperature distributions, are employed to examine different criteria for the definition of the relative importance of the natural and forced convection effects, always in terms of the Reynolds and Richardson numbers. Initially, three previously studied criteria are considered, namely, the ratio of wall shear stresses, the ratio of the quadratic means of the buoyancy and viscous forces terms, and the ratio of the quadratic means of the buoyancy and pressure forces terms. A new criterion based on the ratio of kinetic energy generated in the flow due to natural convection and that generated in total, due to both natural and forced effects, is also proposed. A closer examination of this criterion is then performed, in the attempt to establish recommendations for practical use. The complete notebook is readily available to interested readers upon request.

2. Problem Formulation

We start by writing the flow and energy equations for mixed convection between two vertical parallel plates subjected to different wall temperatures, T_1 and T_2, respectively at y=0 and y=e, where e is the distance between the two plates, Fig. (1). The flow is considered to be two-dimensional, laminar, incompressible, and ascendant with an average velocity V_d, with negligible viscous dissipation. The main flow occurs thus along the longitudinal direction x, and the Boussinesq approximation is recalled to deal with the buoyancy term, while all the other physical properties are taken as constant:

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0 \tag{1}$$

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=g\,\beta\left(T-\frac{T_1+T_2}{2}\right)-\frac{1}{\rho}\frac{\partial p^*}{\partial x}+\nu\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right) \tag{2}$$

$$U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}=-\frac{1}{\rho}\frac{\partial p^*}{\partial y}+\nu\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right) \tag{3}$$

$$U\frac{\partial T}{\partial x}+V\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \tag{4}$$

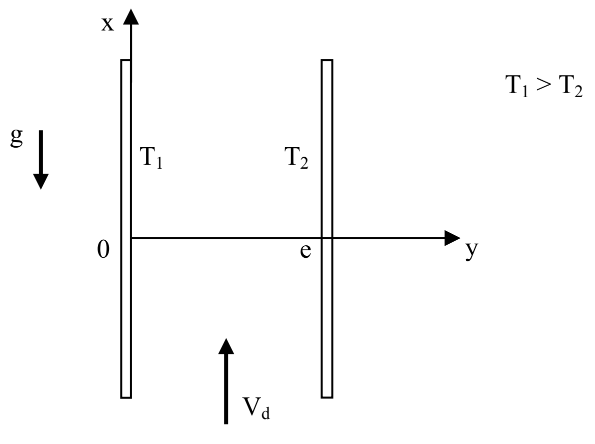

Figure 1. Geometry and coordinates system for mixed convection analysis

Now, if we seek the fully developed solution for the above system, we must replace the dependent variables in the equations shown above, as a function of the transversal coordinate only, starting with the continuity equation. From continuity, the result is that the derivative with respect to y of the fully developed transversal velocity component is zero. If we merge this information with the non-penetration boundary conditions, we obtain the known result that the transversal component is zero for fully developed flow:

$$V = cte = 0, \quad x \longrightarrow \infty \tag{5}$$

The momentum equations are also simplified through the same path to yield:

$$0 = g\,\beta\left(T - \frac{T_1 + T_2}{2}\right) - \frac{1}{\rho}\frac{\partial p^*}{\partial x} + \nu\frac{d^2 U}{d\,y^2}, \quad x \longrightarrow \infty \tag{6}$$

Finally, the energy equation is simplified as:

$$\frac{d^2 T}{d\,y^2} = 0, \quad x \longrightarrow \infty \tag{7}$$

Eq. (7) may now be solved in order to obtain the temperature field after prescribing the related boundary conditions $T(0) = T_1$ and $T(e) = T_2$:

$$T = \frac{T_2 - T_1}{e}y + T_1, \qquad x \longrightarrow \infty \tag{8}$$

3. Fully developed flow

The temperature field obtained in Eq. (8) is a heat condution result. By introducing this equation into Eq. (6) and integrating it once, we obtain :

$$\frac{dU}{dy} = -\frac{g\beta}{\nu e}\left(T_2 - T_1\right)\frac{y^2}{2} - \frac{g\beta}{\nu}\frac{T_1 - T_2}{2}y + \frac{1}{\mu}\frac{dp^*}{dx}y + C_1 \tag{9}$$

By integrating twice we can obtain the velocity field corresponding to fully developed flow in terms of the prescribed temperature boundary conditions:

$$U = -\frac{g\beta}{6\nu e}\left(T_2 - T_1\right)y^3 - \frac{g\beta}{4\nu}\left(T_1 - T_2\right)y^2 + \frac{1}{2\mu}\frac{dp^*}{dx}y^2 + C_1 y + C_2 \tag{10}$$

The integral constants C_1 and C_2 can be obtained by taking into account the no-slip velocity boundary conditions , and the final result for the fully developed velocity field is written below:

$$U = \frac{g\beta}{6\nu e}\left(T_1 - T_2\right)y^3 + \left(-\frac{g\beta}{4\nu}\left(T_1 - T_2\right) + \frac{1}{2\mu}\frac{dp^*}{dx}\right)y^2 + \left(\frac{g\beta}{12\nu}\left(T_1 - T_2\right) - \frac{1}{2\mu}\frac{dp^*}{dx}\right)ey \tag{11}$$

We may also take from this general expression the specific contributions of the natural and the forced convection effects to the fully developed flow:

$$U_{natural} = \frac{g\beta\left(T_1 - T_2\right)\left(e - 2y\right)\left(e - y\right)y}{12\nu e} \tag{12}$$

$$U_{forced} = -\frac{\left(e - y\right)y}{2\mu}\frac{dp^*}{dx} \tag{13}$$

Some of the most important flow parameters can now be readily determined for future evaluations in the analysis of different transition criteria. The volumetric flow rate is obtained from integration of the velocity field over the cross-sectional area, and can be simplified to yield the final expression below:

$$q_v = \int_0^e U\,dy \tag{14}$$

$$q_v = -\frac{e^3}{12\mu}\frac{dp^*}{dx} \tag{15}$$

As expected, the flow rate is due essentially to the forced flow component, and the average flow velocity can be evaluated, providing the pressure gradient:

$$V_d = -\frac{e^2}{12\mu}\frac{dp^*}{dx}\ ;\ \frac{dp^*}{dx} = -\frac{12\mu}{e^2}V_d \tag{16, 17}$$

Now, Eq.(11) can be rewritten in terms of the average velocity V_d:

$$U = \frac{g\beta}{6\nu e}(T_1 - T_2)y^3 - \left(\frac{g\beta}{4\nu}(T_1 - T_2) + \frac{6V_d}{e^2}\right)y^2 + \left(\frac{g\beta}{12\nu}(T_1 - T_2) + \frac{6V_d}{e^2}\right)ey \tag{18}$$

The velocity gradients at the two walls can be also readily determined as:

- at the warm wall:

$$\left(\frac{dU}{dy}\right)_{y=0} = \frac{g\beta}{12\nu}(T_1 - T_2)e + 6\frac{V_d}{e} \tag{19}$$

- at the cold wall:

$$\left(\frac{dU}{dy}\right)_{y=e} = \frac{g\beta}{12\nu}(T_1 - T_2)e - 6\frac{V_d}{e} \tag{20}$$

Also, we can obtain the minimum value of V_d when the gradient at the wall y = e will be zero, and recirculation will start, assuming $T_1 > T_2$:

$$\frac{g\beta}{12\nu}(T_1 - T_2)e - 6\frac{V_d}{e} = 0\ ;\ V_d = \frac{g\beta(T_1 - T_2)e^2}{72\nu} \tag{21}$$

The dimensionless numbers product (Richardson and Reynolds) relevant to mixed convection is then written below:

$$Ri.Re = \frac{g\beta(T_1 - T_2)4e^2}{V_d\nu} \tag{22}$$

If we replace the mean velocity from Eq. (21) into Eq. (22), it results:

$$Ri.Re = 288 \tag{23}$$

This number corresponds to the limit between upward flow ($Ri.Re < 288$) and downward flow ($Ri.Re > 288$).

4. Transition Criteria

We now examine different possible criteria for the consideration of the mixed convection phenomena.

4.1. 1st Criterion - Relative shear stresses

The first possibility here considered is the establishment of a criterion for the dominance of natural or forced convection, based on the relative magnitudes of the shear stresses (or velocity gradients) at the channel walls. First, for the dominance of *natural convection*:

$$\left(\frac{dU}{dy}\right)_{y=0} - \left(\frac{dU}{dy}\right)_{y=e} < \frac{1}{10}\left(\frac{dU}{dy}\right)_{y=0} \tag{24}$$

This equation can be solved considering Eqs. (19) and (20) to result:

$$Ri.Re \geq 5470 \tag{25}$$

The same analysis can be employed to come up with a criterion for the relative importance of *forced convection*:

$$\left(\frac{dU}{dy}\right)_{y=0} + \left(\frac{dU}{dy}\right)_{y=e} < \frac{1}{10}\left(\frac{dU}{dy}\right)_{y=0} \tag{26}$$

which provides:

$$Ri.Re \leq 15.2 \tag{27}$$

4.2. 2nd Criterion - Comparison of gravitational and viscous forces

The second criterion here analyzed concerns the utilization of the ratio between the buoyancy and the viscous forces, as obtained from the above formulation for fully developed flow. Equation (6) is repeated below, explicitly showing the two terms to be compared:

$$0 = \underbrace{g\,\beta\left(T - \frac{T_1+T_2}{2}\right)}_{a} \underbrace{- \frac{1}{\rho}\frac{\partial p^*}{\partial x}}_{b} + \underbrace{\nu \frac{d^2U}{d\,y^2}}_{c} \tag{6}$$

The gravitational term:

$$a = g\,\beta\left(T - \frac{T_1+T_2}{2}\right) \tag{28}$$

and the viscous term:

$$c = \nu \frac{d^2U}{d\,y^2} \tag{29}$$

If we take into account Eqs. (8), (9) and (17), these two terms can be rewritten below:

$$a = g\,\beta\left(T_1 - T_2\right)\left(\frac{1}{2} - \frac{y}{e}\right) \tag{30}$$

$$c = g\beta(T_1 - T_2)\frac{y}{e} - \frac{g\beta}{2}(T_1 - T_2) - \frac{12\nu V_d}{e^2} \quad (31)$$

The comparison is now performed by taking the quadratic average of these two functions over the solution domain $(0,e)$:

$$\overline{a^2} = \frac{1}{12} g^2 \beta^2 \Delta T^2 \quad (32)$$

$$\overline{c^2} = \frac{1}{12} g^2 \beta^2 \Delta T^2 + 144 \frac{\nu^2 V_d^2}{e^4} \quad (33)$$

where $\Delta T = T_1 - T_2$.

The ratio of the quadratic average terms can be written below:

$$P^2 = \frac{\overline{a^2}}{\overline{c^2}} = \frac{(Ri.Re)^2}{(Ri.Re)^2 + 27648} \quad (34)$$

which finally yields:

$$Ri.Re = \frac{166.28\,P}{\sqrt{1 - P^2}} \quad (35)$$

The two extreme situations provide the forced convection case (P=0) and the natural convection case (P→1):

$P = 0 \rightarrow Ri.Re = 0$ – forced convection

$P \rightarrow 1 \rightarrow Ri.Re \rightarrow \infty$ – natural convection

We can also assign different values for the ratio P, so as to establish a desired criterion, for instance, P=0.95 (the buoyancy quadratic term is 95% of the overall), and P=0.05 (buoyancy quadratic term is only 5% of the global term):

$P > 0.95 \rightarrow \mathrm{Ri.Re} > 505.89$ – natural convection

$P < 0.05 \rightarrow Ri.Re < 8.32$ – forced convection

4.3. 3rd Criterion - Comparison of gravitational and pressure forces

The third criterion here analyzed concerns the utilization of the ratio between the buoyancy (a) and the pressure forces (b), from Eq. (6). If we consider Eq. (15) the pressure term becomes:

$$b = \frac{12\nu}{e^2} V_d$$

and the quadratic average is:

$$\overline{b^2} = \frac{144\nu^2}{e^4} V_d^2 \quad (36)$$

The comparison is now performed by taking the ratio of these two quadratic terms:

$$\Gamma^2 = \frac{\overline{a^2}}{\overline{b^2}} = \frac{(Ri.Re)^2}{27648} \tag{37}$$

$$Ri.Re = 166.28\ \Gamma \tag{38}$$

The two extreme situations provide the forced convection case (Γ=0) and the natural forced convection case (Γ=∞):

$\Gamma = 0.05 \rightarrow Ri.Re = 8.31$ – forced convection

$\Gamma = 0.95 \rightarrow Ri.Re = 157.97$ – natural convection

4.4. 4th Criterion - Comparison of the kinetic energy produced by natural and total effects

The fourth criterion here analyzed concerns the utilization of the ratio between the kinetic energy produced by the natural convection effects and by the total flow, as obtained from the formulation above for the fully developed flow. The two flow components, Eqs. (12) and (13), have been previously obtained as:

$$U_{natural} = \frac{g\beta\left(T_1 - T_2\right)\left(e-2y\right)\left(e-y\right)y}{12\nu e}$$

$$U_{forced} = -\frac{\left(e-y\right)y}{2\mu}\frac{dp^*}{dx} = \frac{6V_d\left(e-y\right)y}{e^2}$$

Therefore, the total kinetic energy produced by each flow component per unit volume is computed from:

$$K_{e\ natural} = \frac{\rho}{2e}\int_0^e U_{nat}^2 dy\ ; \quad K_{e\ forced} = \frac{\rho}{2e}\int_0^e U_{forced}^2 dy$$

and the ratio of kinetic energy generated by natural convection effects and the total kinetic energy becomes:

$$K_e = \frac{K_{e\ natural}}{K_{e\ forced} + K_{e\ natural}} \tag{39}$$

Finally, we obtain:

$$Ri.Re = \frac{761.98\ K_e}{\sqrt{1-K_e^2}} \tag{40}$$

The two extreme situations provide the forced convection case (K_e = 0) and the natural convection case (K_e = 1), when the energy generated by the forced flow is considered negligible. This criterion is very similar to the previous one analyzed (2nd criterion), but offers another physical point of view.

We can also assign different values for the ratio K_e, so as to establish a desired criterion, for instance, K_e = 0.95 (the kinetic energy generated by natural convection is 95% of the overall), and K_e = 0.05 (the kinetic energy generated by natural convection is only 5% of the global energy in the flow):

$K_e = 0.95 \rightarrow Ri.Re = 2318.26$

$K_e = 0.05 \rightarrow Ri.Re = 38.15$

In Fig. (2) we represent the variation of the dimensionless product $Ri.Re$ as a function of K_e, from Eq. (40):

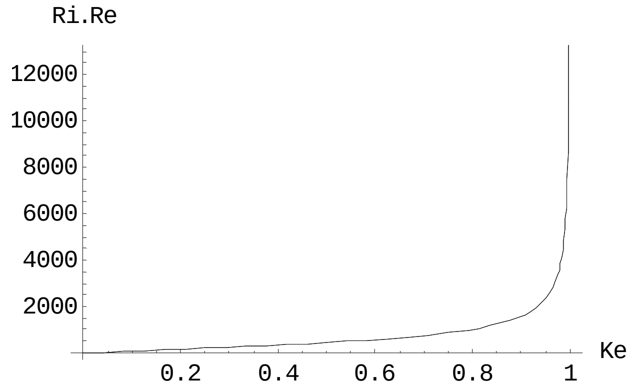

Figure 2. Product *Ri.Re* in terms of kinetic energy ration, K_e

We may also present this function in the inverse order, with the kinetic energy ratio in terms of the product of *Ri* and *Re:*

$$K_e = \frac{RiRe}{\sqrt{580608 + (RiRe)^2}} \tag{41}$$

Finally, we can see in the Fig.3 below, the region that covers the 5% to 95% level of relative importance of the kinetic energy generation by the buoyancy effects:

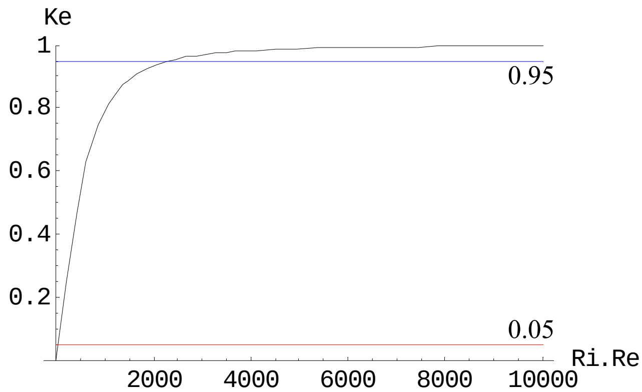

Figure 3. Kinetic energy ration, K_e, as a function of product Ri.Re, showing the 95% and 5% natural convection levels.

5. Application

We now consider a specific case previously consided (Padet, 1997) for mixed convection with water, provided by the following pertinent data: $e = 2\cdot10^{-2}$ m, $V_d = 3\cdot10^{-2}$ m/s, $g = 9.81$ m/s^2.

The thermophysical properties of water at 30°C are: $\beta \approx 4\cdot10^{-4}$ K^{-1} ; $\nu \approx 8{,}5\cdot10^{-7}$ m^2/s. First, we raise some basic information about the problem:

$$Re = \frac{V_d\ 2e}{\nu} = 1411.76$$

We can observe that the flow is laminar. The Richardson number is obtained as:

$$Ri = \frac{g\,\beta\,\Delta T\,2\,e}{V_d^2} = 0.174\ \ \Delta T$$

and the dimensionless numbers product becomes:

$$Ri.Re = 245.65\ \ \Delta T \qquad (42)$$

We then form the equation that relates the temperature difference between the two walls, Eq. (42, and the ratio of kinetic energy generation, Eq. (40):

$$245.65\ \ \Delta T = \frac{761.98\ K_e}{\sqrt{1-K_e^2}}\ ; \quad \Delta T = 3.10\frac{K_e}{\sqrt{1-K_e^2}}$$

The temperature differences that characterize the 95% and 5% levels of natural convection influence in the total kinetic energy are given below:

$K_e = 0.95 \rightarrow\ \Delta T = 9.43\ ^{o}C$

$K_e = 0.05 \rightarrow\ \Delta T = 0.15\ ^{o}C$

The temperature difference required for the onset of recirculation at the wall y = e is also obtained below, by taking into account Eqs. (23) and (42):

$288 = 245.65\ \Delta T\ , \quad \Delta T = 1.17\ ^{o}C$

6. Conclusions

The constructed solutions and the symbolic implementation offer an analysis tool for various mixed convection heat transfer problems. Not only various possibilities of heating/cooling phenomena can be simulated, but several extensions to the present notebook may be undertaken, such as considering turbulent flow, different boundary conditions, other geometries, etc. Besides the potential as a teaching tool as well, this implementation can be utilized in the verification of other physical situations and/or different criteria for the predominance of natural/forced convection.

7. Nomenclature

a, b, c	buoyancy, pressure and viscous terms
C_1, C_2	integration constants
e	distance between walls, m
g	gravity acceleration, m·s^{-2}
K_e	kinetic energy ratio
q_v	volumetric flow rate, m^3·s^{-1}
p^*	pressure, Pa
Ri	Richardson number
Re	Reynolds number
$Ri.Re$	dimensionless numbers product
T	temperature, ^{o}C
T_1	hot wall temperature, ^{o}C
T_2	cold wall temperature, ^{o}C
U	fluid velocity component in x direction, m·s^{-1}
V	fluid velocity component in y direction, m·s^{-1}
V_d	average fluid velocity, m·s^{-1}
x	vertical coordinate, m
y	transversal coordinate, m
α	thermal diffusivity, m^2·s^{-1}

β isobaric coefficient of thermal expansion of fluid, K^{-1}
μ dynamic viscosity of fluid, $kg{\cdot}m^{-1}{\cdot}s^{-1}$
ν kinematic viscosity of fluid, $m^2{\cdot}s^{-1}$
ρ fluid density, $kg{\cdot}m^{-3}$

8. References

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