STREAMLINE DESIGN OF STABILITY COEFFICIENTS FOR THE STANDARD FAMILY OF STABILIZED METHODS

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Abstract. *The classical Galerkin finite element method performs poorly in the computation of convection-dominated transport phenomena. This deficiency may be alleviated by stabilization. A family of stabilized methods has evolved over the last two decades, including Galerkin/least-squares, SUPG (also known as streamline diffusion), and the unusual stabilized finite element method. These three methods share the approach of appending to the Galerkin equation terms containing residual-based operators multiplied by stabilization coefficients. The residual-based operators naturally account for the direction of the flow. The stability coefficient is typically designed on the basis of model problems or bounds from error analyses. Heretofore the flow direction has been ignored or regarded on an ad hoc basis. In this work we analyze the spurious anisotropy inherent in the Galerkin method, i.e., the dependence of the solution on the orientation of the mesh with respect to the flow direction. On the basis of this analysis we propose definitions of the stability parameter that rationally incorporate the flow direction. Numerical tests compare the performance of the proposed methods with established techniques.*

Keywords: *Advection-diffusion, Stabilized method, Stability coefficient*

1. INTRODUCTION

The Galerkin finite element method with low-order piecewise polynomials performs poorly for advection-dominated equations. Adding terms to the variational formulation is well-accepted practice, leading to stabilized methods.

Stabilized finite elements have been around for more than 20 years. These methods have the desirable properties of improving the numerical stability of the Galerkin method

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and of preserving good accuracy properties. The streamline upwind/Petrov-Galerkin (SUPG, or streamline diffusion) method was introduced by Hughes and Brooks (1979) and (Brooks and Hughes, 1982). Variations of this idea considered for advective-diffusive equations are: the Galerkin/least-squares (GLS) version, introduced by Hughes, Franca and Hulbert (1989), and a few years later, the version termed unusual stabilized finite element method (USFEM) (Franca, Frey and Hughes, 1992; Franca and Farhat, 1995).

The additional terms are residual-based and contain stabilization coefficients. The residual-based operators in these terms translate into a streamline diffusion effect. The degree of stabilization in this direction depends on the stabilization coefficients. These were originally conceived based on comparisons to exact solutions of one-dimensional test problems on uniform meshes (Brooks and Hughes, 1982). They were improved to take into account general polynomial discretizations using error estimates (Franca et al., 1992).

The design of the stability parameter in previous work ignores the flow direction, or accounts for it in ad hoc fashion, see, e.g., (Brooks and Hughes, 1982). Here we analyze the spurious *anisotropy* inherent in the Galerkin method, i.e., the dependence of the solution on the orientation of the mesh with respect to the flow direction. We propose definitions of the stability parameter that rationally incorporate the flow direction. Numerical tests compare the performance of the proposed methods with established techniques.

2. STABILIZED METHODS FOR ADVECTION-DIFFUSION

Let $\Omega \subset \mathbb{R}^d$ be a d-dimensional, open, bounded region with smooth boundary Γ. We partition Ω into nonoverlapping regions (element domains) in the usual way, denoting the union of element interiors $\widetilde{\Omega}$, such that $\overline{\Omega} = \overline{\widetilde{\Omega}}$.

2.1. Boundary-value problem

Consider the (homogeneous Dirichlet) advective-diffusive problem of finding a scalar field $u(\boldsymbol{x})$, such that

$$\mathcal{L}u = f \qquad \text{in } \Omega \tag{1}$$

$$u = 0 \qquad \text{on } \Gamma \tag{2}$$

where $\mathcal{L}u = -\nabla \cdot (\kappa \nabla u) + \boldsymbol{a} \cdot \nabla u$, the diffusivity $\kappa(\boldsymbol{x}) > 0$ is known, $\boldsymbol{a}(\boldsymbol{x})$ is the given flow velocity, and $f(\boldsymbol{x})$ is the prescribed source distribution. Generalization of the results presented herein to problems with other types of boundary conditions is straightforward.

2.2. Galerkin approximation

Galerkin approximation is stated in terms of the set of functions $\mathcal{V}^h \subset H_0^1(\Omega)$. The standard finite element method is: find $u^h \in \mathcal{V}^h$ such that

$$a(v^h, u^h) = (v^h, f), \quad \forall v^h \in \mathcal{V}^h \tag{3}$$

where $(\cdot, \cdot)$ is the $L_2(\Omega)$ inner product. (The form of the right-hand side assumes sufficiently smooth f.) The bilinear operator is

$$a(v, u) = (\nabla v, \kappa \nabla u) + (v, \boldsymbol{a} \cdot \nabla u) \tag{4}$$

2.3. Stabilized methods

The standard family of stabilized methods is obtained by appending to the Galerkin equation (3) terms containing residual-based operators multiplied by stabilization coefficients τ, namely

$$a(v^h, u^h) + (\bar{\mathcal{L}} v^h, \tau \mathcal{L} u^h)_{\widetilde{\Omega}} = (v^h, f) + (\bar{\mathcal{L}} v^h, \tau f)_{\widetilde{\Omega}} \tag{5}$$

Subscripts on inner products denote domains of integration other than Ω. Different stabilized methods are obtained via definitions of the differential operator

$$\bar{\mathcal{L}} v = \begin{cases} \mathcal{L} v, & \text{GLS (Hughes et al., 1989)} \\ \mathcal{L}_{\mathrm{adv}} v = \boldsymbol{a} \cdot \nabla v, & \text{SUPG (Brooks and Hughes, 1982)} \\ -\mathcal{L}^* v = \nabla \cdot (\kappa \nabla v) + \boldsymbol{a} \cdot \nabla v, & \text{USFEM (Franca et al., 1992)} \end{cases} \tag{6}$$

The methods differ in the treatment of $\nabla \cdot (\kappa \nabla v^h)$ in the added terms.

Definition of the stability parameter τ is discussed in the following. We restrict the discussion to linear elements with constant diffusivity within each element. In this case $\nabla \cdot \nabla v^h = 0$ in $\widetilde{\Omega}$ and the *three methods coincide*.

3. ONE-DIMENSIONAL ANALYSIS AND DESIGN

For completeness we review the analysis of the Galerkin method in one dimension (representing the case of a uniform d-dimensional mesh aligned with a constant velocity) and the design of stability coefficients based on this analysis (Brooks and Hughes, 1982). The presentation in the following analysis is different from the one in (Brooks and Hughes, 1982), but the results and conclusions are identical.

In addition to the constant, an exact, free-space solution to the advection-diffusion equation (1) in one dimension, with constant coefficients and in the absence of sources, is of the form

$$u = \exp\left(|\boldsymbol{a}| x / \kappa\right) \tag{7}$$

3.1. Spurious oscillations in the Galerkin method

We consider a uniform mesh of linear elements of size h, with nodes at $x_A = Ah$. Nodal values of the exact solution (7) are

$$u(x_A) = \left(\exp\left(2\,\alpha\right)\right)^A \tag{8}$$

where $\alpha = \dfrac{|\boldsymbol{a}| h}{2\kappa}$ is the element Péclet number. Similarly, we assume that corresponding nodal values of finite element solutions are

$$u_A = \left(\exp\left(2\,\alpha^h\right)\right)^A \tag{9}$$

where $u_A = u^h(x_A)$. The dependence of α^h on α is determined by the analysis of a three-node stencil. (This presentation includes the constant for $\alpha^h = 0$.)

The Galerkin method (3) yields the following equation at interior node A

$$\sinh(\alpha^h) \left(\alpha \cosh(\alpha^h) - \sinh(\alpha^h)\right) = 0 \tag{10}$$

Solutions to this equation are the trivial solution $\alpha^h = 0$ (i.e., the constant is represented exactly) and

$$\alpha^h = \operatorname{arctanh}\, \alpha \tag{11}$$

which indicates that α^h approximates α accurately for $\alpha \ll 1$. This presentation may be reconciled with familiar analyses such as (Brooks and Hughes, 1982) by noting that

$$\operatorname{arctanh}\, \alpha = \frac{1}{2} \log \frac{1+\alpha}{1-\alpha} \tag{12}$$

so that

$$\exp\left(2 \operatorname{arctanh}\, \alpha\right) = \frac{1+\alpha}{1-\alpha} \tag{13}$$

According to Eq. (11), α^h is real valued for $\alpha < 1$, approximating α with increasing accuracy as $\alpha \to 0$ (Fig. 1). There is significant degradation in accuracy even prior to the onset of spurious oscillations at $\alpha = 1$. For $\alpha > 1$, spurious oscillations are marked by α^h being complex valued, with $\operatorname{Im} \alpha^h = \pi/2$, so that

$$u_A = \left(-\exp\left(2 \operatorname{Re} \alpha^h\right)\right)^A \tag{14}$$

The real part of α^h is shown in Fig. 2.

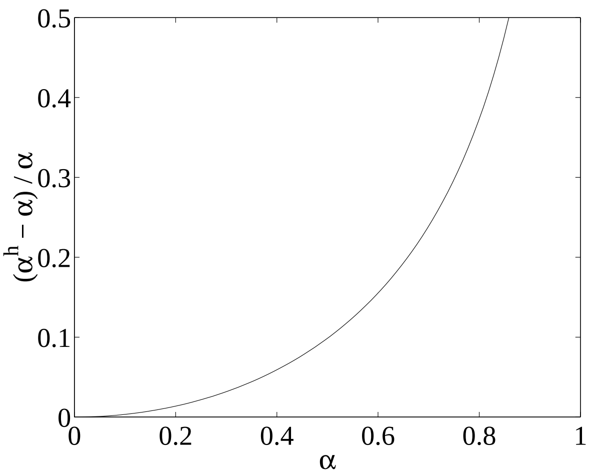

Figure 1: Error in α^h in the range $\alpha < 1$.

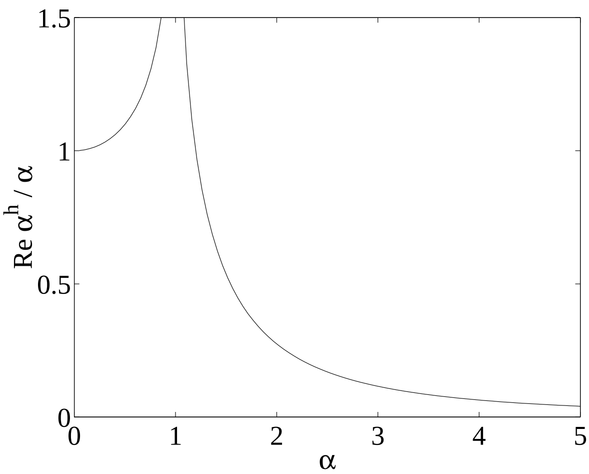

Figure 2: $\operatorname{Re} \alpha^h$ for Galerkin.

3.2. Stability parameter

Repeating the preceding analysis for the stabilized methods (5) (all coincide for linear elements) shows that defining the stability parameter as $\tau = \dfrac{h}{2|\boldsymbol{a}|}\, \xi_0$, where

$$\xi_0 = \frac{1}{\tanh \alpha} - \frac{1}{\alpha} \tag{15}$$

leads to $\alpha^h = \alpha$.

A different approach to designing the stability parameter is based on bounds from error estimates (Franca et al., 1992). For linear elements this results in $\tau = \frac{h}{2|\boldsymbol{a}|}\,\xi_{\mathrm{FFH}}$, where

$$\xi_{\mathrm{FFH}} = \begin{cases} \alpha/3, & 0 \le \alpha < 3 \\ 1, & 3 \le \alpha \end{cases} \tag{16}$$

Brooks and Hughes (1982) refer to this as a doubly asymptotic approximation (see Fig. 3). Franca et al. (1992) defined the parameter in terms of the p-norm of $\boldsymbol{a}$. Here we employ the 2-norm. In the following numerical results we refer to this as **FFH**.

4. SPURIOUS ANISOTROPY AND STREAMLINE DESIGN

In addition to the constant, an exact, free-space solution to the multi-dimensional advection-diffusion equation (1), with constant coefficients and in the absence of sources, is of the form

$$u = \exp\left(\boldsymbol{a} \cdot \boldsymbol{x}/\kappa\right) \tag{17}$$

4.1. Spurious anisotropy in the Galerkin method

In contrast to exact solutions, Galerkin solutions are anisotropic in the sense that they depend on the orientation of the mesh with respect to the given velocity. This phenomenon is demonstrated in the following analysis.

We consider a uniform, two-dimensional mesh of bilinear elements of size h, aligned with the global axes, with nodes at $\boldsymbol{x}_A = (mh, nh)$. Since $\boldsymbol{a}^T = |\boldsymbol{a}|\langle\cos\theta, \sin\theta\rangle$, nodal values of the exact solution (17) are

$$u(\boldsymbol{x}_A) = \left(\exp\left(2\,\alpha c\right)\right)^m \left(\exp\left(2\,\alpha s\right)\right)^n \tag{18}$$

where $c = \cos\theta$ and $s = \sin\theta$. Similarly, we assume that corresponding nodal values of finite element solutions are

$$u_A = \left(\exp\left(2\,\alpha^h c\right)\right)^m \left(\exp\left(2\,\alpha^h s\right)\right)^n \tag{19}$$

where $u_A = u^h(\boldsymbol{x}_A)$. The dependence of α^h on the element Péclet number α and the orientation of the mesh with respect to the streamline direction is determined by the analysis of a nine-node patch (Fig. 4). (This presentation includes the constant for $\alpha^h = 0$.)

The Galerkin method (3) yields the following equation at interior node A

$$\begin{aligned} &\sinh(\alpha^h c)\left(\alpha c\cosh(\alpha^h c) - \sinh(\alpha^h c)\right)\left(3 + 2\,\sinh^2(\alpha^h s)\right) + \\ &\quad \sinh(\alpha^h s)\left(\alpha s\cosh(\alpha^h s) - \sinh(\alpha^h s)\right)\left(3 + 2\,\sinh^2(\alpha^h c)\right) \;=\; 0 \end{aligned} \tag{20}$$

The trivial solution $\alpha^h = 0$ satisfies this equation (i.e., the constant is represented exactly). There is an additional solution, corresponding to Eq. (11) when the mesh is aligned with the flow. The variation of this solution with the orientation of the mesh with respect to the streamline direction (θ) is shown in Fig. 5. For brevity, cases in which α^h is complex valued ($\alpha > 1$) are omitted. Note that the best performance is attained when the flow is along element diagonals ($\theta = \pi/4$).

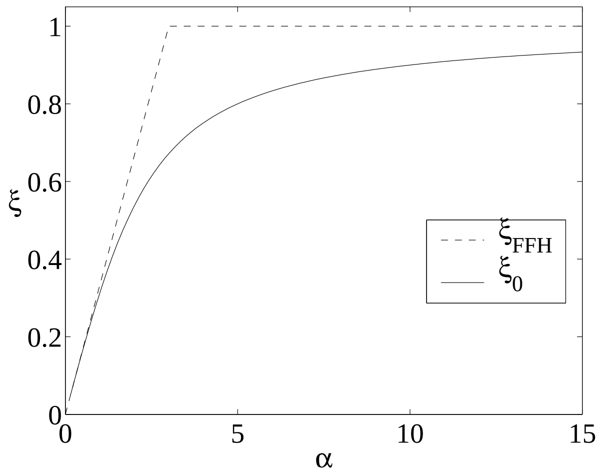

Figure 3: Terms in the stability parameter.

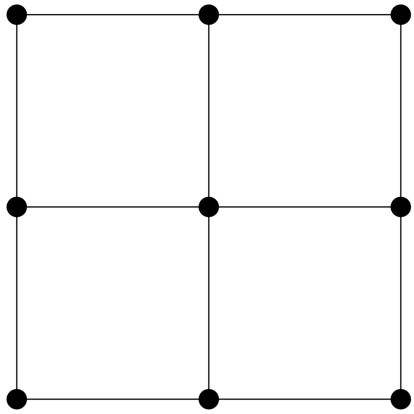

Figure 4: Nine-node patch.

4.2. Streamline design of the stability parameter

Repeating the preceding analysis for the stabilized methods (5) (all coincide for the mesh considered) provides a definition of the stability parameter that leads to $\alpha^h = \alpha$, which is omitted for brevity. In the limits, this parameter may be expressed simply as $\tau = \dfrac{h}{2|\boldsymbol{a}|} D(\theta)\, \xi_0(\alpha)$, where

$$D = \begin{cases} c^4 + s^4, & \alpha = 0 \\ \dfrac{c+s}{1+3\,cs} \quad (0 \leq \theta \leq \pi/2), & \alpha \to \infty \end{cases} \tag{21}$$

The least amount of stabilization is applied when the flow is along element diagonals ($\theta = \pi/4$, Fig. 6), i.e. when the performance of Galerkin is at its best (Fig. 5).

The difference between the two cases of D is not large. This suggests a definition of the parameter that may be employed in practice. Since the advective-dominated case ($\alpha \gg 1$) is the challenging regime, we propose

$$\tau = \frac{h}{2|\boldsymbol{a}|} \, \frac{\cos\theta + \sin\theta}{1 + 3\,\cos\theta\,\sin\theta} \left(\frac{1}{\tanh\alpha} - \frac{1}{\alpha} \right) \tag{22}$$

Note that the orientation should be regarded so that $0 \leq \theta \leq \pi/2$. This presents no practical limitation. In the following numerical results we refer to the parameter that leads to $\alpha^h = \alpha$ as the streamline parameter (**STR**), and the one defined by Eq. (22) is called the estimated parameter (**EST**).

5. NUMERICAL RESULTS

In this section we compare the numerical performance of stabilized finite element methods with the proposed parameters to established techniques. We consider the following methods:

STR Stabilized finite elements with the streamline parameter that leads to $\alpha^h = \alpha$ in the analysis in Sec. 4.2.

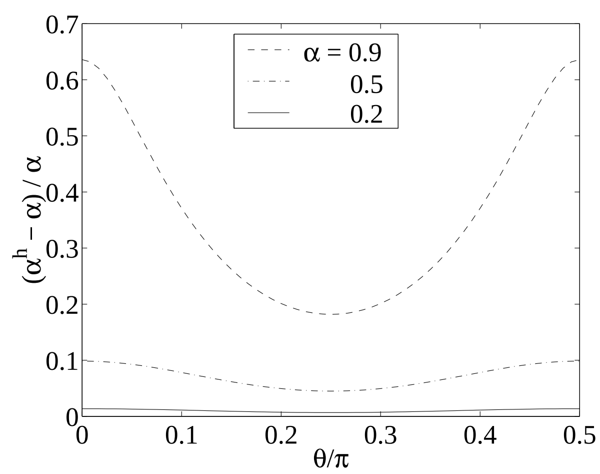

Figure 5: Anisotropy in Galerkin method.

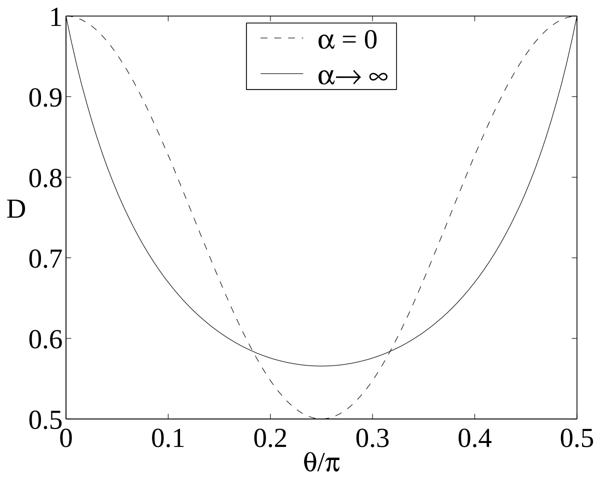

Figure 6: Directivity in both limits.

EST Stabilized finite elements with the estimated streamline parameter (22).

FFH Stabilized finite elements with the FFH parameter (Franca et al., 1992), see (16).

RFB The method of residual-free bubbles, with the bubble derived for the advective limit (Brezzi, Franca and Russo, 1998).

We use bilinear elements in all tests.

5.1. Smooth boundary layer

Consider a constant-coefficient advective-diffusive problem in the unit square, without distributed sources ($f = 0$) and with inhomogeneous Dirichlet boundary conditions selected so that the solution is of the form (17). We use a uniform mesh with 20×20 elements. Table 1 shows the relative error, measured in the L_2 norm. The error relative to the exact solution at $\theta = 0$ is consistently larger since the boundary layer spreads along an entire side of the domain, whereas in other cases it is concentrated in a corner. In all cases, the interpolation error dominates. For **STR** and **EST** the approximation error is negligible. The **EST** results are comparable to **STR**, so from here on we show only **EST** results.

5.2. Advection skew to the mesh

We modify Problem 5.1. so that there is a discontinuity in the inflow Dirichlet data which is propagated into the domain creating an internal layer, with homogeneous Neumann outflow conditions. Here $\alpha = 2.5{\times}10^4$. A piecewise constant reference solution (based on the advective limit) is set equal to the inhomogeneous Dirichlet value to the left of the discontinuity, and zero to the right. **EST** provides some improvement over **FFH**, yet **RFB** exhibits the best performance for these problems with discontinuities, particularly when the flow is along element diagonals (Fig. 7).

5.3. Advection skew to the mesh with outflow boundary layers

The outflow conditions of Problem 5.2. are changed to homogeneous Dirichlet conditions, leading to outflow boundary layers (Brezzi et al., 1998; Franca et al., 1992).

Table 1: L_2 relative errors [%], Problem 5.1.

		rel. to exact sol'n			rel. to nodal interpolant		
α	θ	**STR**	**EST**	**FFH**	**STR**	**EST**	**FFH**
2.5	0	7.62	7.62	8.59	4.51×10^{-14}	5.25×10^{-14}	1.81
	30	1.14	1.15	1.25	5.04×10^{-14}	3.28×10^{-2}	0.337
	45	1.14	1.15	1.26	5.58×10^{-14}	4.74×10^{-2}	0.362
250	0	12.8	12.8	12.9	1.13×10^{-13}	9.75×10^{-14}	3.65×10^{-2}
	30	1.67	1.67	1.75	5.22×10^{-14}	1.27×10^{-3}	0.361
	45	1.67	1.67	1.77	6.17×10^{-14}	1.20×10^{-3}	0.411
2.5×10^4	0	12.9	12.9	12.9	9.87×10^{-14}	1.00×10^{-13}	3.66×10^{-4}
	30	1.67	1.67	1.75	7.31×10^{-14}	1.28×10^{-5}	0.361
	45	1.67	1.67	1.77	6.12×10^{-14}	1.21×10^{-5}	0.410

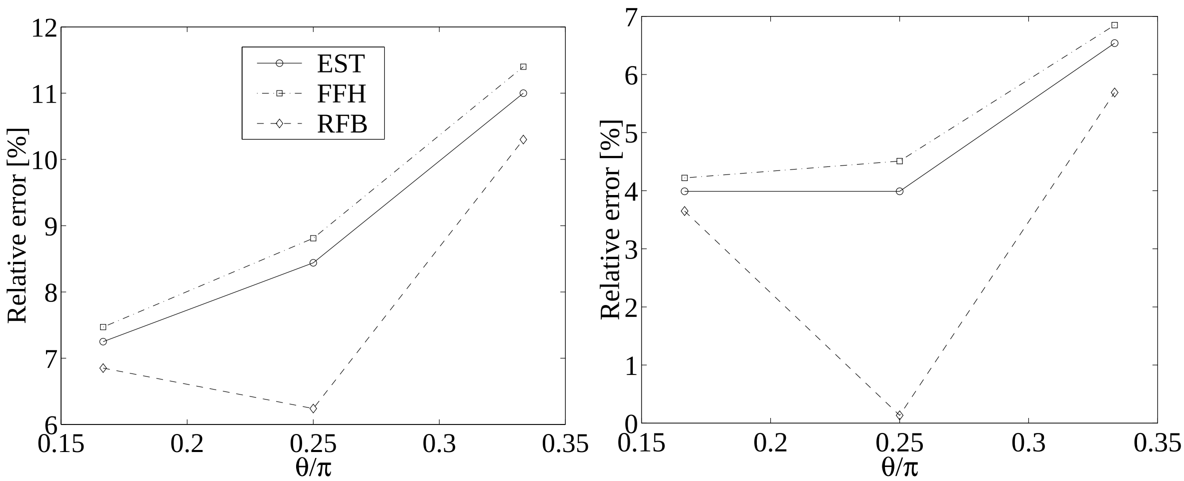

Figure 7: L_2 error [%] in Problem 5.2. relative to reference solution (left) and nodal interpolant (right).

Solutions at $\theta = 60°$ are shown in Fig. 8. The outflow boundary layers are numerically challenging, but may not represent typical physical configurations. The **EST** parameter is designed to reduce stabilization based on the streamline direction, see Fig. 6, which is inappropriate for the outflow boundary layers in this problem, leading to the relative deterioration in the **EST** results (Fig. 9).

5.4. Advection in a rotating flow field

Consider a homogeneous Dirichlet advective-diffusive problem (Franca et al., 1992; Hughes and Brooks, 1979) in the unit square (centered at the origin), without distributed sources ($f = 0$), and with $\kappa = 10^{-6}$ and a rotating velocity field $\boldsymbol{a}^T = \langle -y, x \rangle$. There is an internal boundary along the negative y-axis, with the boundary condition

$$u(0, y) = \frac{1}{2}[\cos(4\pi y + \pi) + 1] \quad , \quad -0.5 \le y \le 0 \tag{23}$$

An **FFH** solution on a uniform mesh of 200×200 elements is set as the reference solution, and the tests are performed on a uniform mesh of 40×40 elements. Stability parameters

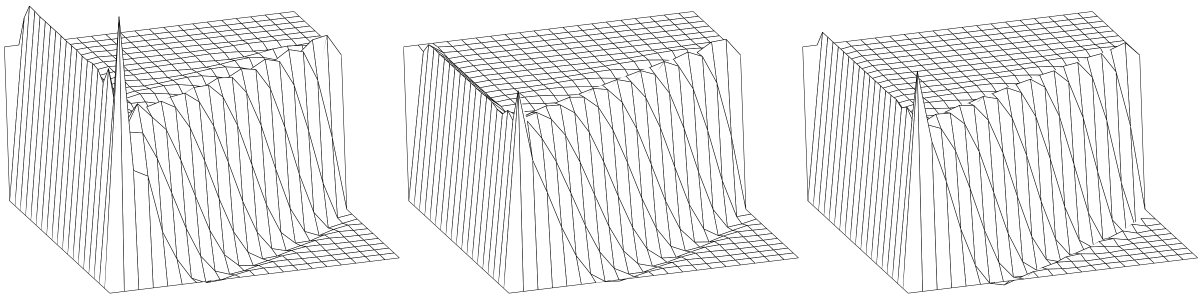

Figure 8: Solutions of Problem 5.3. at $\theta = 60°$: **EST** (left), **FFH** (center), and **RFB**.

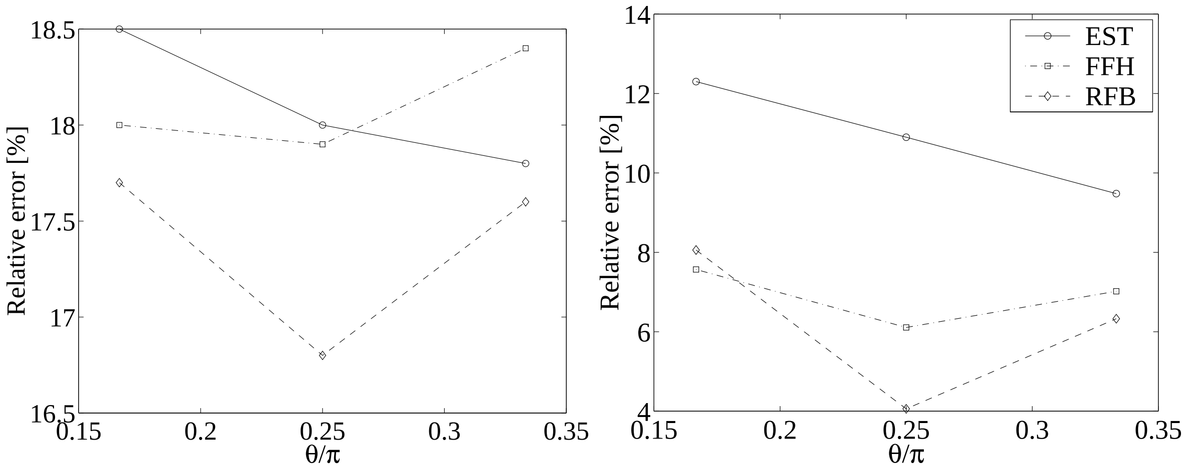

Figure 9: L_2 error [%] in Problem 5.3. relative to reference solution (left) and nodal interpolant (right).

are evaluated in terms of velocity at element centers. Solutions are shown in Fig. 10. Table 2 shows the relative error, measured in the L_2 norm. **EST** exhibits the best performance on this smooth problem. We note that the version of **RFB** implemented herein is designed for the advective limit, while this problem contains diffusion-dominated regions.

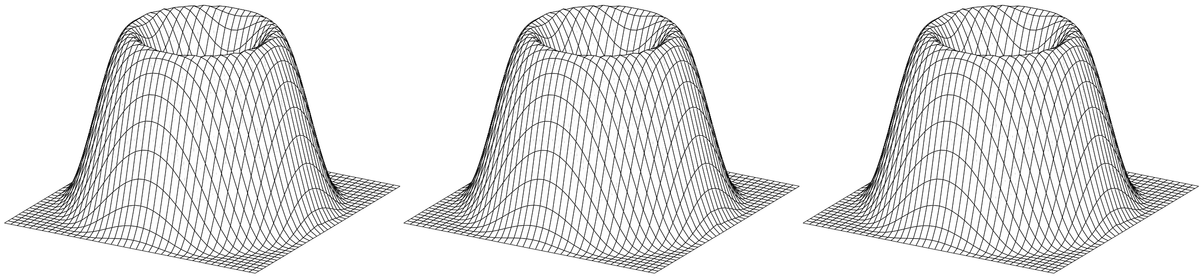

Figure 10: Solutions of Problem 5.4.: **EST** (left), **FFH** (center), and **RFB**.

6. CONCLUSIONS

In this work we analyze the dependence of numerical solutions on the orientation of the mesh with respect to the flow direction. We propose definitions of the stability

Table 2: L_2 relative errors [%], Problem 5.4.

	rel. to ref. sol'n	rel. to nodal interp.
EST	0.779	0.344
FFH	0.904	0.484
RFB	0.809	0.353

parameter that rationally incorporate the flow direction. Numerical tests compare the performance of the proposed methods with established techniques.

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