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Contributions to the theory of longitudinal vibrations and wave propagation in rods and tubes: I. A mathematical model for linear elastic and hereditary elastic materials

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Abstract: After a brief historical survey of some work done on the linear theory of longitudinal vibrations and wave propagation in rods and tubes of uniform cross-section, a simple mathematical model for rods and tubes of linear elastic materials is proposed. Three suitably selected propagation modes (one extensional and two shear modes) with dispersion relations corresponding to mixed boundary conditions are coupled in order to approximately comply with zero-stress boundary conditions. The coupling gives a set of partial differential equations in the mode amplitudes, with time and a single space coordinate (along the axis of symmetry of the rod or tube) as independent variables. Then, the model is generalized to a set of partial integral-differential equations in order to be able to describe vibrations and wave propagation in rods and tubes made of linear hereditary-elastic solids. In this first part of the work, the focus is in either very low frequency or very high frequency phenomena using a simple model with only two coupled modes. The model allows a fairly elegant and comparatively powerful analytical approach to longitudinal vibrations and to longitudinal pulse propagation in solid waveguides. Analytical formulae for group velocities are derived, as well as asymptotic expressions for the propagation of mode amplitudes. The limitations and pitfalls of the model are assessed, and new experiments and digital simulations are suggested to test some of its predictions.

Keywords: mechanical vibrations; wave propagation; elastic and hereditary-elastic materials; propagation modes in rods and tubes.

INTRODUCTION

Longitudinal vibrations and wave propagation in rods and tubes remains an important subject even today (Benaroya, 2005; Da Silva, 2005). By themselves, or coupled with torsional or flexural wave motion, these oscillations appear in the classical fields of civil, mechanical, electrical and aerospace engineering, of non destructive testing, and in the new fields of mechatronics and micro-electromechanical systems.

The mathematical study of longitudinal vibrations began almost three centuries ago, with the derivation of the classical one dimensional wave equation for extensional oscillations in rods or tubes, assuming that the material behaves as linear elastic and the cross-sections remain plane during the passage of the mechanical waves. This approach neglects both the effects of internal friction and the lateral expansions and contractions related with Poisson's ratio. The equations of the dynamic theory of linear elasticity, (derived in the XIX century), allows to take into account the resultant distortion of the transverse sections when the wavelength is of the same order of magnitude as the rod's radius. The effect of internal friction on longitudinal oscillations has been considered, from a phenomenological point of view, since the last quarter of the XIX century. In 1876 Boltzmann suggested two important things: (a) the relation between stress and strain in a given solid is a function of its entire past history, and (b) a superposition principle can be applied to study these hereditary materials. The equations of dynamic linear elasticity theory were soon generalized to take into account viscoelastic effects. Their theoretical study continued during the whole XX century, including now some mechanisms that explain internal friction at the microscopic level.

For an infinite cylindrical and linear elastic waveguide, with zero stress boundary conditions and assuming a purely sinusoidal time dependence, the abovementioned equations of the dynamic theory of elasticity were posed in cylindrical coordinates and solved by separation of variables by Pochhammer as early as in 1876 (Kolsky, 1963, pp.54-59). The boundary conditions then give a transcendental equation that relates the angular frequency ω with the wave number k along the axis of the rod: the so called dispersion relation, in terms of Bessel functions. Assuming a suitable spatial symmetry (to eliminate flexural and torsional vibrations) a numerable infinity of propagation modes is obtained from the aforementioned dispersion relation. Each mode is characterized by a definite connection between ω and k, as well as by certain typical space patterns of strain and stress. In principle, by superposition of the contributions of the different modes and the different frequencies that belong to an elastic pulse or wave pattern, we can follow the propagation and distortion of the pattern along the axis of a bar. In the case of finite cylinders, realistic boundary conditions at the end plane cross sections introduced seemingly insurmountable additional complications to obtain a solution by separation of variables in the partial differential equations (Nadeau, 1964, pp.268-271). This analytical approach could be extended to cylindrical tubes, but always with increasing difficulties. However, some work along these lines was done at the end of the fifties and during the sixties, including some extensions to wave propagation and vibrations of hereditary materials in rods. It was soon realized that it may be actually easier to make an ab-initio numerical analysis without the introduction of the mode concept and without spectral decomposition of the wave pattern. This is the contemporary trend, fostered by the increase in computer power and the development of high quality numerical algorithms (Conca and Gatica, 1997, De Silva, 2005). But even with the present computing facilities, the problem of longitudinal vibrations and wave propagation can be fairly difficult to tackle in certain cases of practical interest, even if internal friction is neglected. One disadvantage of digital simulations is that, sometimes, they don't show clearly the effects of changing the physical parameters in the problem.

So, a simplified analytical approach may be of interest, both as a guide for the design of ab-initio digital simulations of wave propagation or vibrations, and for the interpretation of experimental results. The main purpose of this work is to develop a simple mathematical model for longitudinal vibration and wave propagation in linear elastic bars or tubes, and to suggest an extension to bars or tubes made with hereditary materials. In the present first part, the focus will be in **high frequency wave propagation and low frequency vibrations in rods**.

The key concept for modelling is a phenomenon very important in many branches of engineering and physics: wave coupling (Pierce, 1954). If the dispersion curves of two uncoupled modes of propagation cross each other one or several times, when a coupling between them is introduced, the resulting dispersion curves for the coupled modes split from the crossover points. If the coupling is mild, the new dispersion curves nearly coincide, far from these intersections points, with part of the curves corresponding to the uncoupled modes. However, if the coupling is strong enough, a coupled mode may behave very differently from an uncoupled one, even far from the abovementioned cross section points. All these properties of wave coupling will be used to construct the mathematical models proposed in this paper. But let us make first a brief review of the previous work related with the construction of simple analytical models of longitudinal vibrations and waves in rods.

CRITICAL REVIEW OF SOME ASPECTS OF THE THEORY OF GIEBE AND BLECHSCHMIDT AND THE ALMOST FLUID WAVEGUIDE

Between 1876 and 1930 several efforts were done to identify longitudinal propagation modes in elastic bars of circular cross section, making analytical approximations in the original dispersion relation (obtained first by Pochhammer, and later by Chree), as was done by Raleigh in 1894, or even posing new and simplified field equations as was done by Love at the beginning of the XX century (Suárez-Ántola, 1998). These efforts were successful only for one particular mode and when $k.R$ is much smaller than one (long wavelengths relative to the bar radius R). Soon after that, Giebe and Blechschmidt (1933) developed a highly simplified theory (GB theory) to study mechanical resonance in bars of any cross section (Kolsky, 1963, pp. 64-65). The bar was treated as two mechanical systems coupled together: an elastic bar of infinitesimal cross-section in longitudinal vibration and an elastic plate, of infinitesimal thickness in radial vibration. The coupling produced two modes. In the ω versus k plane, each mode is given by a curve (a branch of the corresponding dispersion relation) (Fig. 1).

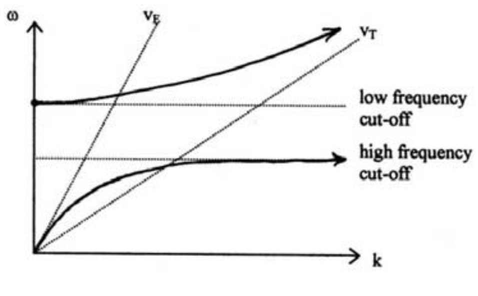

Figure1. Upper and lower modes in the Giebe and Blechschmidt theory

The slope (that is, the group velocity) of the lower branch when k is near zero (long wavelengths) is the extensional wave velocity $v_E = \sqrt{E_0/\rho}$ (E_0 is Young´s modulus and ρ is density). When k grows, ω in the lower branch tends to a horizontal asymptote (high frequency cut-off). The upper branch presents a low frequency cut-off typical of waveguide modes. When k grows ω approaches the asymptote $\omega = v_T k$ where $v_T = \sqrt{G_0/\rho}$ is the shear wave velocity in the medium (G_0 is the shear modulus). Between the upper and the lower cut-off frequencies, there is a forbidden interval of frequencies (death zone).

At small wave-numbers, the lower branch is a very good approximation to the lower mode predicted by Pochhammer - Chree´s dispersion equation. However, this last exact mode doesn't have a high-frequency cut-off as predicted by the GB theory. In fact when k grows the dispersion relation of this exact mode shows a tilted asymptote $\omega = v_R k$, being v_R the velocity of Rayleigh surface waves for the given material (Kolsky, 1963, pp. 60-62).

Neither the experimental results nor the detailed numerical analysis of Pochhammer-Chree´s exact dispersion equation, show evidence for a forbidden interval of frequencies at least in the case of cylindrical rods. Besides, the upper branch of GB theory does not describe properly the propagation of high frequency longitudinal pulses. Later numerical analysis of the exact dispersion equation (Mason, 1958, pp. 40-45) showed that, in order to describe the propagation of longitudinal pulses at high frequencies, it is necessary to take into account certain modes of fairly high order. For these modes, ω is asymptotic to $\omega = v_L . k$ when k tends to infinity. Here v_L is the velocity of bulk dilatational waves in the material under consideration. These modes could explain the known experimental results about the propagation of high frequency longitudinal pulses in bars, but they are outside the scope of G.B. theory. Because it failed at high frequencies, and because the death zone that it predicted was not found, the theory had to be dismissed.

However, when purely longitudinal vibrations of relatively low frequencies are excited in tubes and rods of any cross-section, in order to study mechanical resonance, the relation between ω and k follow the lower branch of the G.B. dispersion relation, for relatively small values of k . As a consequence, the G.B. theory can be applied, as a simplified mathematical model, to the design and interpretation of mechanical resonance experiments. Indeed, as it can be applied to tubes or bars of any cross section, at low frequencies the dispersion relation of the G.B. theory has a wider scope than the exact dispersion relation of Pochhammer and Chree, because this last one was obtained for and can be applied only to, cylindrical waveguides.

Modal analysis of wave propagation is amenable to an analytical approach, as the one intended here, if it can be done using only a few modes (two or three). **So, let us retain from the theory of Giebe and Blechschmidt the idea of describing longitudinal vibrations in a rod or tube using only a few modes (modes of propagation, in our case) that emerge from the coupling of others more basic modes of propagation**.

There is another idea that we will borrow in order to construct a simplified mathematical model for longitudinal wave propagation. It stems from some experimental results obtained in 1948 by Mc Skimin, working with trains of high frequency pulses in elastic rods (Mc Skimin, 1956). This author found that, when the radius to wavelength ratio is quite large, a short duration packet of dilatation waves propagates along the rod similar to a wave-packet in a pressure release fluid waveguide. Of course, mode conversion at the boundary of the elastic medium always produces an energy loss from the dilatation wave-packet. But even if we neglect mode conversion, reflection and interference of the dilatational waves at the boundary of the equivalent fluid waveguide produces a set of propagation modes each of which shows geometric dispersion. According to the results obtained by the computing group of Bell Telephone Laboratories, in connection with early delay line studies (Mc Skimin, 1956), the dispersion relation of the lowest mode of the equivalent fluid waveguide behaves very similar to the dispersion relation of the lowest mode corresponding to predominantly longitudinal waves, in the solid rod, as predicted by the equations of dynamics elasticity. This suggested the introduction of the concept of "**almost-fluid waveguide**" to describe with a maximum of simplicity the propagation of high frequency longitudinal waves in solid bars (Suárez-Antola, 1990 and 1998, Suárez-Ántola and Suárez-Bagnasco, 1999). Then, in order to comply with the almost-fluid waveguide concept, we will require that the dispersion relation for the highest mode of propagation of our intended model must be: **(a)** asymptotic to $\omega = v_L . k$ when k tends to infinity, and (b) near the dispersion relation of the first mode of the almost fluid waveguide for k big enough.

Besides the aforementioned **lower extensional-surface mode** and the **higher shear-dilatational modes**, the exact dispersion equation shows **a third kind of propagation mode**. The phase velocity is in this case asymptotic to v_T when k tends to infinity (asymptotic shear mode). But the group velocity of the main asymptotic shear mode, considered as a function of $k.R$, presents a significant maximum (less than v_E) followed by a less significant minimum (higher than the minimum of the group velocity of the extensional surface mode) before approaching asymptotically to v_T (Kolsky, 1963, Fig.15). As a consequence of all this, to study the propagation of short longitudinal pulses (like very short ultrasonic pulses with a wide spectrum of frequencies), we will require three branches to the dispersion equation obtained from our model: a lower extensional-surface branch, an asymptotically shear branch in the middle, and an upper asymptotically dilatational branch.

A SIMPLE MATHEMATICAL MODEL WITH THREE COUPLED MODES OF PROPAGATION IN THE ELASTIC CASE

Let us begin by considering a pure extensional mode with dispersion equation $\omega = v_E . k$, and two shear modes with dispersion equations: $\omega^2 = \omega_{c1}{}^2 + v_T{}^2 . k^2$ $\omega^2 = \omega_{c2}{}^2 + v_T{}^2 . k^2$ For a rod the cut-off frequencies are given by the equations: $\omega_{c1} = \frac{\alpha_{T1} \cdot v_T}{R}$ $\omega_{c2} = \frac{\alpha_{T2} \cdot v_T}{R}$ α_{T1} and α_{T2} are pure numbers, with α_{T1} less than α_{T2}. For the two lower and usually dominant modes we have $\alpha_{T1} = 3.83$ and $\alpha_{T2} = 7.02$. These dispersion equations for the shear modes are obtained after solving the equations of elasto-dynamics for an infinite waveguide of circular cross-section, but using mixed boundary conditions instead of the stress-free boundary conditions imposed by Pochhammer and Chree (Auld, 1973, p.113). Mixed boundary conditions means here that the radial component of displacement s_r and the axial component of stress $\sigma_{r,z}$ are both zero at the boundary of the waveguide, that is, when r = R. The extensional mode also verifies mixed boundary conditions, because in this case only s_z and $\sigma_{z,z}$ may be different from zero. However, this mode is not an exact solution of the full equations of dynamics elasticity, since it neglects the lateral contraction or dilatation due to Poisson´s effect.

As v_E is greater than v_T , we see that the curves that represent the dispersion equations of the shear modes intersect the straight line that represents the dispersion equation of the extensional mode in the ω-k plane.

Now, the **strong coupling** of intersecting modes obtained using mixed boundary conditions produce new emerging modes that can be used to describe several features of the case with stress-free boundary conditions (Achenbach, 1973; Auld, 1973). However, each shear mode intersects the extensional mode (the extensional mode couples with both shear modes), but the shear modes don't intersect (each shear mode couples only with the extensional mode). So, let us construct a model for the propagation of longitudinal pulses in linear elastic rods of uniform transverse sections, coupling three one dimensional wave equations that correspond to intersecting modes of propagation, as follows:

$$\frac{\partial^2 A}{\partial z^2}-\frac{1}{v_E^2}\frac{\partial^2 A}{\partial t^2}=K_1\frac{\partial^2 B_1}{\partial z^2}+K_2\frac{\partial^2 B_2}{\partial z^2} \qquad (1a) \qquad \frac{\partial^2 B_1}{\partial z^2}-\frac{1}{v_T^2}\frac{\partial^2 B_1}{\partial t^2}-\frac{\omega_{c1}^2}{v_T^2}B_1=K_1\frac{\partial^2 A}{\partial z^2} \qquad (1b)$$

$$\frac{\partial^2 B_2}{\partial z^2}-\frac{1}{v_T^2}\frac{\partial^2 B_1}{\partial z^2}-\frac{\omega_{c2}^2}{v_T^2}B_2=K_2\frac{\partial^2 A}{\partial z^2} \qquad (1c)$$

Here K_1 , K_2 are phenomenological coupling constants. They will be suitably restricted later. If $K_1 = K_2 = 0$, the coupling disappears and $A(t,z)$ becomes the amplitude of the aforementioned pure extensional mode, while $B_1(t,z)$ and $B_2(t,z)$ becomes the amplitudes of certain pure shear modes. Let us search new emerging modes of propagation substituting the following **ansatz** in Eqs.(1):

$$A(t,z)=C(k)\cdot e^{j(\omega t-kz)} \quad (2a) \qquad B_1(t,z)=D_1(k)\cdot e^{j(\omega t-kz)} \quad (2b) \qquad B_2(t,z)=D_2(k)\cdot e^{j(\omega t-kz)} \quad (2c)$$

This gives a system of linear homogeneous equations for the emergent mode amplitudes: $C(k)\ D_1(k)\ D_2(k)$

These homogeneous equations have non- zero solutions if and only if the following **dispersion equation** is verified:

$$\left(\frac{\omega^2}{v_E^2}-k^2\right)\cdot\Delta_1(k)\cdot\Delta_2(k)=k^4\left(K_1^2\cdot\Delta_2(k)+K_2^2\cdot\Delta_1(k)\right) \qquad (3)$$

By definition: $$\Delta_1(k)=\frac{\left(\omega^2-\omega_{C1}^2\right)}{v_T^2}-k^2 \quad (4a) \qquad \Delta_2(k)=\frac{\left(\omega^2-\omega_{C2}^2\right)}{v_T^2}-k^2 \quad (4b)$$

When both coupling constants are different from zero, this dispersion equation always has three real, positive and non-intersecting solutions $\omega=\omega_\mu(k)$ defined for every real k with μ = 1 (lower), m (middle), u (upper). If k approaches zero, the coupling of the modes disappear. The lower branch $\omega=\omega_l(k)$ approaches the dispersion equation of the uncoupled extensional mode $\omega=v_E.k$. The middle one $\omega=\omega_m(k)$ approaches the dispersion equation of the uncoupled first shear mode and the upper branch $\omega=\omega_u(k)$ approaches the dispersion equation of the second shear mode.

Mode amplitudes, phase and group velocities in the linear elastic case for two coupled modes

Nevertheless, to study longitudinal vibrations, either forced or free, and low frequency wave propagation, or at the other extreme, very high frequency wave propagation, a lower and an upper branch could be enough. So, let us begin by coupling only two modes of propagation: the extensional mode with dispersion equation $\omega=v_E.k$, and one of the shear modes with dispersion equation: $\omega^2=\omega_c{}^2+v_T{}^2.k^2$ So, we put: $K_2=0$, $K_1=K$, $B_1(t,z)=B(t,z)$, $D_1(k)=D(k)$

Then the dispersion equation reduces to (now $\Delta(k)=\frac{\left(\omega^2-\omega_C^2\right)}{v_T^2}-k^2$) : $$\left(\frac{\omega^2}{v_E^2}-k^2\right)\cdot\Delta(k)=k^4K^2 \qquad (5)$$

The system of linear equations for the mode amplitudes $C(k)\ D(k)$ reduces to:

$$\left(\frac{\omega^2}{v_E^2}-k^2\right)C(k)=-k^2\left(K\cdot D(k)\right) \quad (6a) \qquad \Delta(k)D(k)=-k^2KC(k) \quad (6b)$$

When the coupling constant is different from zero, this dispersion equation always has two real, positive and non-intersecting solutions $\omega = \omega_\mu(k)$ defined for every real k, with $\mu = l$ (lower), u (upper). If k **approaches zero**, the lower branch $\omega = \omega_l(k)$ approaches the dispersion equation of the uncoupled extensional mode $\omega = v_E.k$. The upper branch $\omega = \omega_u(k)$ approaches the dispersion equation of the shear mode, also when k tends to zero. But when k **tends to infinity**, the lower branch is asymptotic to $\omega = v_S.k$, and if we select K properly, the upper branch will be asymptotic to $\omega = v_L.k$. The asymptotic phase velocities v_S , v_L can be found from equation (6). If v_a =lim ω/k for $k \to \infty$, taking the limit in the dispersion equation we obtain : $\left(\frac{v_a^2}{v_T^2} - 1\right)\left(\frac{v_a^2}{v_E^2} - 1\right) = K^2$ (7)

If we establish that $v_a = v_L$ must be a positive root of Eq(7), then we must choose

$$K^2 = \left(\frac{v_L^2}{v_T^2} - 1\right)\left(\frac{v_L^2}{v_E^2} - 1\right) \quad (8)$$

In that case the second positive root of Eq(5) is $v_a = v_S$ and it verifies $v_S^2 = v_E^2 + v_T^2 - v_L^2$ (9)

This is the abovementioned asymptotic phase velocity of the lower branch. As will be pointed in the discussion, it should be equal to the velocity of Rayleigh waves in the material.

If $\omega = \omega_\mu(k)$ (μ = l, u) represents the two branches of the dispersion Eq.(5), and if we introduce the phase velocity: $v_{f,\mu} = \omega_\mu(k)/k$ then for each mode of propagation the following relation is obtained between $C(k)$ and $D(k)$

$$\frac{D_\mu(k)}{C_\mu(k)} = -\frac{1}{K}\left[\left(\frac{v_{f,\mu}}{v_E}\right)^2 - 1\right] \quad (10)$$

Note that $C_\mu(k)$ and $D_\mu(k)$ always remain bounded. In order to take into due account Poisson´s effect (axial elongation (compression) appears together with lateral contraction (dilatation)), K must be considered to be positive. Thus if $v_{f,\mu} > v_E$, and if $C_\mu(k)$ and $D_\mu(k)$ are taken as real functions, then they have opposite signs. If they are complex functions of k , their arguments are equal or differ in π , according to the sign of the right hand member of Eq.(10). For $\mu = l$, the lower extensional-surface mode, when k approaches to zero (long wavelengths) the phase velocity $v_{f,l}$ approaches to v_E , so that $D_l(k)$ approaches to zero. For $\mu = u$, the upper shear-dilatational mode $D_u(k)/C_u(k)$ approaches to $-\sqrt{\left(\frac{v_L^2}{v_E^2}\right) - 1 \Big/ \left(\frac{v_L^2}{v_T^2}\right) - 1}$ when k tends to infinity (short wavelengths). To obtain this last result we must substitute K in Eq. (10) by its expression as a function of v_L , v_T and v_E given by Eq.(8).

If we define non-dimensional variables $x = R\omega/v_L$ and $y = k.R$, the dispersion Eq.(5) can be rewritten as follows, with $\omega_C = \alpha_T v_T / R$:

$$\left(\frac{v_L^2}{v_E^2} x^2 - y^2\right) \cdot \left(\frac{v_L^2}{v_T^2} x^2 - \alpha_T^2\right) = K^2 y^4 \quad (11)$$

From Eq.(11) we can obtain x as a function of y (that is, ω as a function of k) or y as a function of x (that is, k as a function of ω). We are going to solve ω as a function of k, for purpose of comparison with the other approaches to the dispersion relation. Thus, we obtain two branches, $f_l(y)$ and $f_u(y)$ being

$$f_{\substack{l \\ (u)}}(y) = \sqrt{\frac{1}{2}\left[\alpha y^2 + \beta \underset{(+)}{-} \sqrt{(\beta + \gamma y^2)^2 + \delta y^4}\right]} \quad (12)$$

Here $\alpha = \left(\frac{v_E}{v_L}\right)^2 + \left(\frac{v_T}{v_L}\right)^2$ $\beta = \alpha_T^2 \left(\frac{v_T}{v_L}\right)^2$ $\gamma = \left(\frac{v_T}{v_L}\right)^2 - \left(\frac{v_E}{v_L}\right)^2$ $\delta = 4K^2 \left(\frac{v_E}{v_L}\right)^2 \left(\frac{v_T}{v_L}\right)^2$

Figure 2 shows the two branches of the simplified dispersion equation, calculated from equation (12), with α_T = 3.83 and for the case of a steel bar: v_T = 3230 m/s, v_E = 5192 m/s and v_L = 5900 m/s.

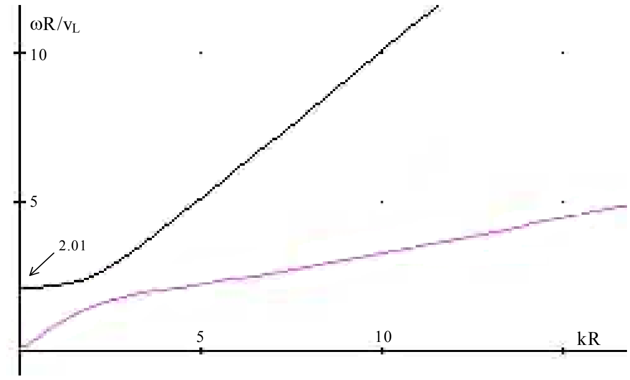

Figure2. The dispersion relations: upper and lower branch.

Figure 3 shows the dimensionless group velocity $v_g/v_L = \left(d/dy\right) f(y)$ is a function of y for each branch of the dispersion equation, where $f(y)$ is $f_l(y)$ or $f_u(y)$ As $y = kR$ we can obtain from these curves not only how the group velocity depends of the wave-number k, but also how it depends of the bar radius R.

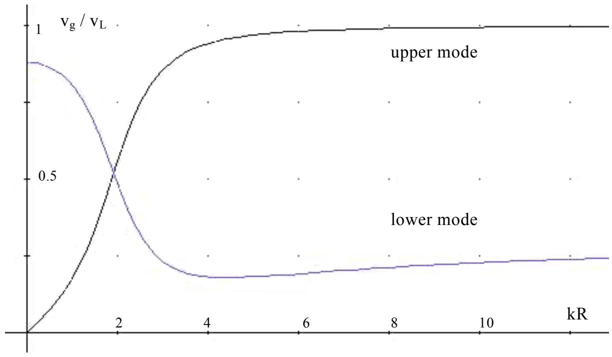

Figure3. Upper and lower mode: group velocities as functions of the product of the radius and wave-number.

The curve that gives the group velocity as a function of $k.R$ for the **extensional-surface mode** present a minimum exactly as the corresponding curve found by a detailed numerical analysis of the solutions of the Pochhammer-Chree´s exact dispersion equation. The other curve, that gives the group velocity as a function of $k.R$ for the **shear-dilatational mode**, begins from zero and grows towards a horizontal asymptote that corresponds to $v_g = v_L$.

Vibration and waves in the elastic case with two coupled modes of propagation

In principle we could consider three idealized situations: (a) First we have a wave guide that may be considered infinite in both directions. For t = 0 we assume that we know the spatial pattern of mode amplitudes of the **extensional-surface** mode $A(0,z)$, and the **shear-dilatational** mode $B(t,z)$ and after that we want to follow the propagation of the elastic disturbance. The localized initial disturbance (a wave-packet) can be expressed as a Fourier integral using the wave-number k. Then, to follow the propagation of the pulse we need ω as a function of k. (b)Second we have a half infinite waveguide, with an emitting transducer located at z = 0. Now we know the fields as a function of t, and we can express them as Fourier integrals using the frequency ω. To follow the propagation of the disturbance thus generated, we need k as a function of ω. (c)Third situation, we have a finite rod vibrating with given boundary conditions at both ends, and we obtain a numerable infinity of possible k values. In our case, the second and third situations are the most interesting.

Almost all the numerical calculations done for plates, rods and other types of solid waveguides give ω as a function of k, and then give both phase and group velocities as functions of k. This allows us to calculate the frequencies of rod vibration for each possible wave-number. For **free vibrations**, the fields $A(t,z)$ and $B(t,z)$ are represented by a linear combination (series) of standing waves for the allowed wave-numbers k_n, with the frequencies of the lower and upper modes given by Eq.(12) as well defined functions of k_n. The corresponding mode amplitudes are determined from the initial conditions and the restriction imposed by Eq.(10). **Forced vibrations** will be considered later.

On the other side, for each mode of propagation the relation between ω and k is given by a strictly monotonic function (fig.2). So, instead of solving Eq.(5) to find ω as a function of k, it is possible to solve it to find k as a function of ω:

$$2.\left(1-K^2\right)k_\mu^{\,2}(\omega)=k_E^{\,2}(\omega)+k_T^{\,2}(\omega)+\delta_\mu.\sqrt{\left(k_E^{\,2}(\omega)+k^2{}_T(\omega)\right)^2-4.\left(1-K^2\right)k^2{}_E(\omega).k_T^{\,2}(\omega)} \qquad (13)$$

($\delta_\mu=+1$ for the lower mode $\mu=l$, and $\delta_\mu=-1$ for the upper mode $\mu=u$). $k_E^{\,2}(\omega)=\dfrac{\omega^2}{v_E^{\,2}}$, $k_E^{\,2}(\omega)=\dfrac{\omega^2-\omega_C^{\,2}}{v_T^{\,2}}$

We can calculate $C_\mu(\omega)$ and $D_\mu(\omega)$ the same as before, now putting k as a function of ω. Then we can apply these last relations to the half infinite waveguide mentioned before, adding the contributions of the lower and upper modes to construct a wave-packet.

In terms of the frequencies, the wave fields of the coupled modes are given by the following equations:

$$A(t,z)=\int_{-\infty}^{+\infty}C_l(\omega).e^{i\phi_{l,C}(\omega)}.e^{i(\omega.t-z.k_l(\omega))}d\omega+\int_{-\infty}^{+\infty}C_u(\omega).e^{i.\phi_{u,C}(\omega)}.e^{i.[\omega.t-z.k_u(\omega)]}.d\omega \qquad (14a)$$

$$B(t,z)=\int_{-\infty}^{+\infty}D_l(\omega).e^{i\phi_{l,D}(\omega)}.e^{i(\omega.t-z.k_l(\omega))}d\omega+\int_{-\infty}^{+\infty}D_u(\omega).e^{i.\phi_{u,D}(\omega)}.e^{i.[\omega.t-z.k_u(\omega)]}.d\omega \qquad (14b)$$

In this case both $C_\mu(\omega)$ and $D_\mu(\omega)$ for $\mu=l,u$ are positive. The arguments $\phi_{\mu,C}$ and $\phi_{\mu,D}$ are either equal or differ in π by the same reasons explained before, in relation with Eq.(10).

When t is big enough, the method of stationary phase (Segel, 1987, pp.412-417; Pilant, 1979, pp.174-180) can be applied to the evaluation of the integrals (13). For given values of (t , z) the main contribution to the integral comes from the frequencies $\omega=\omega_e(t,z)$ that verify: $\dfrac{\partial k_\mu(\omega_e)}{\partial\omega}=\dfrac{t}{z} \quad \mu=l,u$ (15)

If (14) does not have real roots, neither for $\mu=l$ nor for $\mu=u$, both mode amplitudes A and B are negligible.

Applying the method of the stationary phase, the scalar field (14a) can be approximated by :

$$A(t,z)\approx\frac{2.C_l(\omega_{e,l})}{\left[\frac{1}{2\pi}.\left|\frac{\partial^2k_l(\omega_{e,l})}{\partial\omega^2}\right|.z\right]^{\frac{1}{2}}}.\cos\left(\omega_e.t-k_l(\omega_{e,l})z+\phi_l(\omega_{e,l})+s_l.\frac{\pi}{4}\right)+\frac{2.C_u(\omega_{e,u})}{\left[\frac{1}{2\pi}.\left|\frac{\partial^2k_u(\omega_{e,u})}{\partial\omega^2}\right|.z\right]^{\frac{1}{2}}}.\cos\left(\omega_{e,u}.t-k_u(\omega_{e,u})z+\phi_u(\omega_{e,u})+s_u.\frac{\pi}{4}\right) \qquad (16)$$

Here s_μ is the sign of $\dfrac{\partial^2k_\mu(\omega_e)}{\partial\omega^2}\neq0$. For the other field, (14b), we have the same expression but with D_μ instead of C_μ. Formula (16) is a local approximation by a harmonic wave of frequency ω_e and wave number $k(\omega_e)$.

But when z/t varies, ω_e changes as well (according to formula (15)), so that we obtain a wave modulated in amplitude, frequency and phase. When $\dfrac{\partial^2k_\mu(\omega_e)}{\partial\omega^2}\approx0$, the asymptotic method of the stationary phase fails, and another approach must be applied, like the Airy phase approximation method (Pilant, 1979, pp 178-180, Suárez-Ántola, 1998).

A MODEL WITH THREE COUPLED MODES OF PROPAGATION IN THE HEREDITARY ELASTIC CASE

Equations (1) are already in the form often used in the study of mode coupling in engineering. However, if the relations $v_E=\sqrt{E_0/\rho}$ $v_T=\sqrt{G_0/\rho}$ are taken into account, it is possible to generalize and reformulate Eqs. (1), as follows:

$$\rho\frac{\partial^2A}{\partial t^2}=\frac{\partial\sigma}{\partial z}+\rho.f \quad (17a) \qquad \rho\frac{\partial^2B_1}{\partial t^2}+\rho\omega_{c,1}^{\,2}B_1=\frac{\partial\tau_1}{\partial z}+\rho.g_1 \quad (17b) \qquad \rho\frac{\partial^2B_2}{\partial t^2}+\rho\omega_{c,2}^{\,2}B_2=\frac{\partial\tau_2}{\partial z}+\rho.g_2 \quad (17c)$$

By definition, the **equivalent** normal stress σ and the **equivalent** shear stresses τ are given by the relations:

$$\sigma=E.\left(\frac{\partial A}{\partial z}-K_1.\frac{\partial B_1}{\partial z}-K_2\frac{\partial B_2}{\partial z}\right) \quad (18a) \qquad \tau_1=G.\left(\frac{\partial B_1}{\partial z}-K_1.\frac{\partial A}{\partial z}\right) \quad (18b) \qquad \tau_2=G.\left(\frac{\partial B_2}{\partial z}-K_2.\frac{\partial A}{\partial z}\right) \quad (18c)$$

The additional terms $\rho.f(t,z)$ $\rho.g_1(t,z)$ $\rho.g_2(t,z)$ represent distributed forces that may excite the corresponding propagation mode. For example: magnetostrictive, electrostrictive or thermo-elastic forces, amongst others. The effect of forces due to transducers or other excitation devices located at the ends of the rods or tubes appear in the boundary conditions, as usual. The terms $\rho.\omega_c{}^2.B$ that already appeared in Eqs. (1), now look like a kind of elastic restoring forces associated with the displacements B, and they are added to the effects of the "equivalent stresses" σ and τ and the volume forces $\rho.f$ and $\rho.g$ in the above coupled set of non-homogeneous partial differential equations.

Equations (17) can be applied to linear elastic rods or tubes if the moduli E and G are substituted by the scalars E_0, G_0 But they can be applied in the **linear hereditary** case substituting E and G by the corresponding **linear operators** $\hat{E} = E_0\left(\hat{I} - \hat{L}\right)$ and $\hat{G} = G_0\left(\hat{I} - \hat{M}\right)$ The integral operators $\hat{L}$ and $\hat{M}$ act in the time domain according to Boltzmann's principle of superposition (Rabotnov, 1980, chapters 1 and 2). K_1, K_2 are phenomenological **coupling constants** that, as in the elastic case, have to be suitably restricted. If the coupling disappears, $A(t,z)$ becomes the amplitude of the aforementioned pure extensional mode, while $B(t,z)$ becomes certain amplitudes representative of the lower pure shear modes. If $L(t)$ and $M(t)$ are the kernels of the operators $\hat{L}$ and $\hat{M}$ respectively, if $h(t,z)$ is a field, then by definition:

$$\hat{L}[h] = \int_{-\infty}^{t} L(t-u).h(u,z).du) \qquad (19a) \qquad\qquad \hat{M}[h] = \int_{-\infty}^{t} M(t-u).h(u,z).du) \qquad (19b)$$

In general, when both $K_\alpha \neq 0$ we obtain three coupled partial integral-differential equations in the unknown coupled mode amplitudes $A(t,z)$, $B_\alpha(t,z)$ $(\alpha = 1,2)$ The linear term in $B_\alpha(t,z)$, that appears with the cut-off frequency, summarizes the effects, on the longitudinal propagation, of the shear strains in the cross sections of the rod or tube.

Let us consider forced longitudinal vibrations of a rod or tube in a description with only two modes: $A(t,z)$ $B(t,z)$ In this case it is convenient to apply the method of Finite Fourier Transforms (Zauderer, 1989, pp.207-218). All the fields are represented as linear combinations of a complete set of known orthonormal eigenfunctions weighted by time unknown dependent amplitudes. In our case the eigenfunctions $\varphi_n(z)$ verify the equation $-\dfrac{d^2\varphi_n}{dz^2} = k_n{}^2.\varphi_n$ with suitably chosen homogeneous boundary conditions at the beginning $z = 0$ and the end $z = b$ of the rod, so that the given stress or displacement at the boundary of the rod or tube can be taken into account in the dynamic equations for the amplitudes. If $h(t,z)$ represents any field (displacements, strains, stresses, or volume force densities) we define its projections onto the eigenfunctions: $h_n(t) = \int_0^b h(t,z).\varphi_n(z).dz$ Then, the field can be thus decomposed: $h(t,z) = \sum_n h_n(t).\varphi_n(z)$. Also note: (a) for any field the following equality is verified:

$$\int_0^b \left(\frac{\partial^2 h(t,z)}{\partial z^2}\right).\varphi_n(z).dz = \frac{\partial h(t,b)}{\partial z}.\varphi_n(b) - \frac{\partial h(t,0)}{\partial z}.\varphi_n(0) + h(t,0).\frac{d\varphi_n(0)}{dz} - h(t,b).\frac{d\varphi_n(b)}{dz} - k_n{}^2.h_n(t)$$

And: (b) the time dependent operators $\hat{E}$ and $\hat{G}$ **commute** with the partial derivatives taken relative to z . After these observations, we integrate Eqs. (17), (written for two coupled modes only), multiplied by $\varphi_n(z)$, over the interval $[0,b]$, and taking into account Eqs. (18), we obtain the following set of integral-ordinary differential equations:

$$\frac{d^2A_n}{dt^2} + \frac{k_n{}^2}{\rho}.(\hat{E}[A_n] - K\hat{E}[B_n]) = F_n \qquad (20a) \qquad\qquad \frac{d^2B_n}{dt^2} + \omega_C{}^2.B_n + \frac{k_n{}^2}{\rho}.(\hat{G}[B_n] - K\hat{G}[A_n]) = G_n \qquad (20b)$$

The functions $F_n(t)$ and $G_n(t)$ comprehend the sum of the projections of the volume forces onto the corresponding eigenfunction, with terms that stem from the boundary conditions at $z = 0$ and $z = b$. These terms are obtained after integrating twice by parts the second partial derivatives of the displacement fields and taking into account the equivalent stresses (Eqs. (18)). If the equivalent stresses are given at the ends of the rod or tube, then eigenfunctions are constructed to verify $\varphi_n(0) = \varphi_n(b) = 0$. If the displacements are given there, we impose the condition $\dfrac{d}{dz}\varphi_n(0) = \dfrac{d}{dz}\varphi_n(b) = 0$. To study forced vibrations in a stable regime in this linear mechanical system, we begin with: $F_n(t) = F_{o,n}.e^{j\omega.t}$ $G_n(t) = G_{0,n}.e^{j\omega.t}$ In a steady oscillating regime: $A_n(t) = A_{0,n}.e^{j\omega.t}$ $B_n(t) = B_{0,n}.e^{j\omega.t}$

This ansatz is substituted in Eqs. (20), and taking into account Eqs. (19), an algebraic system of two linear equations is derived for the two unknowns $A_{0,n}$ $B_{0,n}$. After solving it we obtain, after having re-introduced v_E, v_T

$$A_{0,n} = \frac{\left(\omega_c{}^2 + v_T{}^2\left(1 - \widetilde{M}(\omega)\right).k_n{}^2 - \omega^2\right).F_{o,n} + K.v_E{}^2\left(1 - \widetilde{L}(\omega)\right).k_n{}^2.G_{0,n}}{(k_n{}^2.v_E{}^2\left(1 - \widetilde{L}(\omega)\right) - \omega^2).\left(\omega_c{}^2 + v_T{}^2\left(1 - \widetilde{M}(\omega)\right).k_n{}^2 - \omega^2\right) - K^2.v_E{}^2.v_T{}^2.\left(1 - \widetilde{L}(\omega)\right)\left(1 - \widetilde{M}(\omega)\right).k_n{}^4} \qquad (21a)$$

$$B_{0,n} = \frac{\left(v_T^{\,2}\left(1-\tilde{M}(\omega)\right)k_n^{\,2}\right)K.F_{o,n} + (v_E^{\,2}\left(1-\tilde{L}(\omega)\right)k_n^{\,2} - \omega^2).G_{0,n}}{(k_n^{\,2}.v_E^{\,2}\left(1-\tilde{L}(\omega)\right)-\omega^2).\left(\omega_c^{\,2} + v_T^{\,2}\left(1-\tilde{M}(\omega)\right)k_n^{\,2} - \omega^2\right) - K^2.v_E^{\,2}.v_T^{\,2}.\left(1-\tilde{L}(\omega)\right)\left(1-\tilde{M}(\omega)\right)k_n^{\,4}} \qquad (21b)$$

$\tilde{L}(\omega)$, $\tilde{M}(\omega)$ are the Fourier Transforms of the causal kernels $L(t)$ and $M(t)$ respectively. If the solid is elastic, the kernels are zero, and in this case the denominator of the right hand members of Eqs. (21), reduces to the left hand member of the dispersion equation (Eq. (5)). If the amplitudes $F_{0,n}$ $G_{0,n}$ are both zero, there may be non-zero vibration amplitudes at the wave-number k if the frequency ω satisfies the equation:

$$(k^2.v_E^{\,2}\left(1-\tilde{L}(\omega)\right)-\omega^2).\left(\omega_c^{\,2} + v_T^{\,2}\left(1-\tilde{M}(\omega)\right)k^2 - \omega^2\right) - K^2.v_E^{\,2}.v_T^{\,2}.\left(1-\tilde{L}(\omega)\right)\left(1-\tilde{M}(\omega)\right)k^4 = 0 \qquad (22)$$

As $k = k_n$ **is real**, and $\tilde{L}(\omega)$ $\tilde{M}(\omega)$ are complex, ω for free vibrations must be a **complex frequency** $\tilde{\omega}_n$. We have now two complex attenuation-dispersion branches, $\tilde{\omega} = \tilde{\omega}_\mu(k)$ ($\mu = \tilde{l}, \tilde{u}$). Its imaginary part gives the temporal attenuation of the vibration mode, and its real part gives the vibration frequency. If the rod or tube is unbounded and ω **is real**, then $k = \tilde{k}$, which verifies Eq. (22), **is complex**. Its imaginary part gives the spatial attenuation, while its real part gives the wave-number. We have two branches $\tilde{k} = \tilde{k}_\mu(\omega)$ that correspond to free propagation modes with attenuation.

DISCUSSION AND CONCLUSIONS

(a)The coupling constant K is a function of Poisson's modulus only. Taking into account the well known relations between young's and Poisson's moduli, from one side, and v_L, v_T and v_E from the other (Segel, 1987, pp259-260), it follows from Eq. (8) that: $K = \frac{\nu}{1-2\nu}.\sqrt{\frac{2}{1+\nu}}$. Most solids have values of ν between 0.25 and 0.30. The corresponding values of K increase when ν increases and verify: $0.632 < K < 0.930$

(b) Let us consider again the dispersion equation (Eq.(3)) for three coupled modes in the elastic case.

The asymptotic phase velocities v_a can be found from Eq. (3) taking the limit of ω/k for $k \to \infty$. We obtain:

$$\left(\frac{v_a^2}{v_T^2}-1\right)^2\left(\frac{v_a^2}{v_E^2}-1\right)^2 = \left(K_1^2 + K_2^2\right)\left(\frac{v_a^2}{v_T^2}-1\right)$$

One positive root of this equation is $v_a = v_T$. But there are another two positive roots. If $v_a \neq v_T$, we obtain $\left(\frac{v_a^2}{v_T^2}-1\right)\left(\frac{v_a^2}{v_E^2}-1\right) = \left(K_1^2 + K_2^2\right)$ Now, let us choose $K_1^2 + K_2^2$ equal to $\left(\frac{v_L^2}{v_T^2}-1\right)\left(\frac{v_L^2}{v_E^2}-1\right)$, so that $v_a = v_L$ is a second positive root. Then the third positive root $v_a = v_S$ verifies $v_S^{\,2} = v_E^2 + v_T^2 - v_L^2$. According to the numerical studies of the Pochhammer and Chree's dispersion equation, it should be equal to the velocity of Rayleigh waves in the material. But as the lower mode is not excited at high frequencies, (with the exception of very special cases), the error in this case is not important. In the second part of this work we derive the dispersion relations, the group velocities, and the mode amplitudes for the three coupled modes in due detail.

(c)The upper branch for high frequencies ($k.R$ much bigger than 1) may be approximated by: $\omega^2 = \omega_{MS}^{\,2} + v_L^{\,2}.k^2$
Here $\omega_{MS} = \frac{\gamma.v_L}{R}$, ($\gamma$ is a dimensionless parameter) is the Mc- Skimin frequency (Suárez-Ántola, 1998). For this asymptotic dispersion relation the local frequency of the stationary phase approximation (Eq.(15)) of a propagating wave packet is: $\omega_e(t,z) = \frac{\omega_{MS}}{\sqrt{1-\left(\frac{z}{v_L.t}\right)^2}}$ (We have a single real root if z is less than $v_L.t$, and no root and no wave at all if it is greater). Let $\omega_{e,u}$ be the biggest frequency whose amplitude is over the detection threshold of the measurement system. Then from the equation for $\omega_e(t,z)$ it follows that for a certain position z of the receiving transducer, the first signal detected would be registered in an instant t such that: $\frac{(z/t)}{v_L} = \sqrt{1-\frac{\omega_{MS}^2}{\omega_{e,u}^2}}$ This would give us

the **apparent dilatational pulse velocity**, measured with a propagating wave-packet. Usually $\omega_{e,u}$ is greater than the carrier frequency ω_0 of the pulse.

(d) Finally, we can establish some conclusions:

- The analytical model that we began to construct in the present work may be considered as an improvement and a significant generalization of the old theory of Giebe and Blechschmidt. Our model retains certain advantages of the G.B. theory (description of the longitudinal oscillations using a few modes, very good behaviour at relatively low frequencies, and applicability to bars and tubes of any cross section (after a suitable selection of the cut-off frequencies of the basic shear modes)) and avoids its main pitfalls (death interval of frequency for rods with circular cross-section, horizontal asymptote in the ω-k plane for the lower mode, and shear wave velocity as asymptotic phase velocity for the upper mode). Besides, it allows us to take into account friction processes in the materials and could be used as an intermediate step towards a generalization of the concept of region of influence of defects (Suárez-Ántola, 2005).

- The model has to be completed in order to obtain a simple specification of the distribution of elastic energy between the different modes of propagation in the high frequency case. For low frequency vibrations of rods and tubes this can be considered as already done through the volume forces and the boundary conditions that appear together with Eqs. (17).With this addition and with the use of asymptotic methods, it should be possible to predict the propagation of longitudinal pulses in different solid waveguides and for different inputs.

- The full calculation for the amplitudes, phase, and group velocities with three modes of propagation remains to be done. In the case of the model with three modes, we need another relation between the phenomenological constants K_1 and K_2. As we shall see in the second part of the present work, this relation can be obtained imposing a condition over the maximum of the group velocity corresponding to the shear-shear mode.

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