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THE REGION OF INFLUENCE OF SIGNIFICANT DEFECTS AND THE MECHANICAL VIBRATIONS OF LINEAR ELASTIC SOLIDS

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Abstract. *The presence of cracks, voids or fields of pores, and their growth under applied forces or environmental actions, can produce a meaningful lowering in the proper frequencies of normal modes of mechanical vibration in machines and structures. A quite general expression for the square of mode´s proper frequency as a functional of displacement field, density field and elastic moduli fields is used as a starting point. The effect of defects on frequency are modeled as equivalent changes in density and elastic moduli fields, introducing the concept of* ***"region of influence"*** *of each defect. This region of influence is derived from the relation between the stress field of flawed components in machines or structures, and the elastic energy released from a suitable reference state, due to the presence of significant defects in the abovementioned mechanical components. An approximate analytical expression is obtained, which relates the relative variation in the square of mode´s proper frequency with position, size, shape and orientation of defects in mode displacement field. Some simple mathematical models of machine and structural elements with cracks or fields of pores are considered as examples. The connections between the relative lowering in the square of mode´s proper frequency and the stress intensity factor of a defect are discussed : the concept of region of influence of a defect is used as a bridge between (low frequency and low amplitude) vibration dynamics and linear elastic fracture mechanics. Some limitations of the present approach are discussed as well as the possibility of applying the region of influence of defects to the damping of normal modes of vibration.*

Keywords *: mechanical vibrations, modal analysis, linear elastic fracture mechanics, machine elements, structures.*

1. Introduction

The presence of cracks, voids or fields of pores, and their growth under applied forces or environmental actions, can produce a meaningfull lowering in proper frequencies of normal modes of mechanical vibrations in machines and structures. This lowering can be related with position, size, shape and orientation of defects in the corresponding normal mode's displacement field, introducing the concept of region of influence of each defect.

The main purpose of this paper is to discuss the abovementioned concept in the framework of mechanical vibrations of linear elastic solids. As will be shown below, the idea of a region of influence of a defect in a nominal stress field is already implicit in both, linear elastic fracture mechanics (LEFM) and in the static theories of the elastic moduli of heterogeneous media. But, it seems that the introduction of a region of influence for defects in a dynamic framework wasn't done until 15 years ago, during a proposal of a method for non-destructive testing of structures (Suárez-Antola, 1990; Suárez-Antola, 2003).

The origin of the concept of region of influence can be traced back to the concept of volume of influence of a defect in Griffith's theory of brittle fracture (Hertzberg, 1998).

According to Griffith, a certain crack will become unstable and lead to fracture, when the decrease in elastic energy of the stress field of the crack (the so called strain energy release rate) equals or exceds the increase of surface energy (or the work of plastic deformation at the crack's tip) due to the creation of new crack surfaces.

The presence of the crack reduces the stresses in some of the neighbouring material. This reduction of stresses produces a reduction in the elastic energy due to the nominal stress field.

Let u be the density of elastic strain energy at a certain location of a loaded solid body, in the abscence of a crack. Then, the reduction in elastic energy due to the presence of the defect can always be written as $u \cdot V_d$

Here V_d is the volume of a certain region determined by the location, shape, size and orientation of the defect in the nominal stress field. This region could be called the region of influence of the defect in a static stress field.

In the well known case of a plain crack of length $2 \cdot a$ in a thin plate of thickness h ($h << 2 \cdot a$), uniformly loaded perpendicular to crack's plane, we have $u = \sigma^2 / (2E)$ (plane unidirectional stress) and $V_d \cong \pi \cdot a^2 \cdot h$. Here σ is the nominal stress and E is Young's modulus. The region of influence is a cylinder of radius a and height h, that can be regarded as stress free.

If the crack's plane forms an angle β with the direction of stress, $V_d \cong \pi \cdot c^2 \cdot h$ with $c = a \cdot sin\beta$. The region of influence is also a cylinder, but now it's radius varies as $sin\beta$.

If an elastic cracked body has volume V_0 , if the defects are tubular holes of the same elliptical cross-sections, and if the length of defects per unit volume is L, then the static modulus is given by (Jaeger, 1969)

$$\frac{1}{E_c} = \frac{1}{E} + \frac{(\kappa + 1) \cdot \pi \cdot c^2 \cdot L}{12 \cdot \tilde{G}}$$

E_c is the effective Young's modulus of the cracked body, E is Young's modulus and $\tilde{G}$ is shear modulus of the uncracked material, and κ is a dimensionless number (a function of Poisson's modulus).

Here c is a characteristic linear dimension of the elliptical corss-section (Jaeger, 1969). So, if l is the length of a tubular defect, $\pi \cdot c^2 \cdot l$ is a corresponding volume of influence of the defect, and $\pi \cdot c^2 \cdot (L \cdot V_0)$ is the sum of the volumes of influence of all the tubular defects in the body.

2. The volume of influence and the region of influence in the static case

In the general case, the volume V_d of the region of influence can be defined as the quotient between the reduction ΔU of strain elastic energy U due to the defect, and the density u of strain elastic energy that would exist in the location of the defect but withouth the presence of the defect (ΔU is taken positive).

Then we define

$$V_d = \frac{\Delta U}{u} \tag{1}$$

From an analytical point of view, the volume V_d of a crack in a given arbitrary nominal stress field can be derived as follows. We begin with the strain energy release rate of LEFM, $G = -dU/dA$, where U is the elastic energy of the solid body and A is an area representative of the defect (the foundations of LEFM needed as a background for the present paper can be found, amongst other sources, in Hertzberg, 1998). If we integrate G in a constant nominal stress field, with A as integration variable, the reduction in strain energy U due to the presence of a crack of area A_d is

$$\Delta U = \int_0^{A_d} G dA \tag{2}$$

G can be obtained from the stress intensity factors K_I , K_{II} and K_{III} , corresponding to the opening mode, the shearing mode and the tearing mode of crack propagation. The stress intensity factors are given by $K = Y \cdot \tilde{\sigma} \cdot \sqrt{\pi \cdot a}$ where Y is a geometric factor, $\tilde{\sigma}$ is the component of the nominal stress field of the corresponding mode of crack propagation, and a is a linear dimension characteristic of the crack. For plane stress

$$G = \frac{1}{E} \cdot \left(K_I^2 + K_{II}^2 + (1 + \nu) K_{III}^2 \right) \tag{3}$$

and for plane strain

$$G = \frac{(1 - \nu^2)}{E} \cdot \left(K_I^2 + K_{II}^2 + \frac{\nu}{(1 - \nu)} K_{III}^2 \right) \tag{4}$$

(ν is Poisson's modulus).

From Eq.(3) and Eq.(4) it follows that G depends linearly of the characteristic dimensions of the defect. It is possible to reduce the integration relative to A to an integration relative to a .

The sizes and shapes of significant defects, as well as their stress intensity factors in given nominal stress fields of practical interest, can be taken from LEFM. So, V_d can be calculated. Furthermore, a three dimensional region can be defined in relation with this volume, using the symmetries of the defect model : the region of influence of the defect.

3. The frequencies of vibration of linear elastic solids

We are going to consider now a structure or machine element statically loaded and with a combination of free-stress and rigid (zero velocity) boundary conditions. Under these conditions, a proper mode of linear vibration is excited. The resultant displacement field is the superposition of a static and a dynamic field. In this linear case, the dynamic field can be considered independently of the static one.

It is possible to derive an analytical formula for the square of the frequency of a proper mode of vibration under the given boundary conditions. The dynamic displacement field $\vec{s}(t,\vec{r})$ for a proper mode can be written

$$\vec{s}(t,\vec{r}) = f(t)\cdot\vec{s}_0(\vec{r}) \tag{5}$$

Then, the local strain tensor $\hat{\varepsilon}(t,\vec{r})$ is given by

$$\hat{\varepsilon}(t,\vec{r}) = f(t)\hat{\varepsilon}_0(\vec{r}) \tag{6}$$

If we neglect anelastic behaviour, the local stress tensor $\hat{\sigma}(t,\vec{r})$ can be written

$$\hat{\sigma}(t,\vec{r}) = f(t)\hat{\sigma}_0(\vec{r}) \tag{7}$$

The dynamic displacement field verifies

$$\rho\frac{\partial^2\vec{s}}{\partial t^2}(t,\vec{r}) = \nabla\bullet\hat{\sigma}(t,\vec{r}) \tag{8}$$

because the volume forces acts only over the static displacement field. Here $\nabla\bullet\hat{\sigma}(t,\vec{r})$ is the divergence of the tensor $\hat{\sigma}(t,\vec{r})$ and ρ is the local density of the solid body (the mathematical foundations of elasticity theory needed as a background for the present paper can be found, amongst other sources, in Nadeau, 1964).

For free vibrations, the boundary conditions exclude any exchange of mechanical energy between the body and its environment. Then from Eq.(5), Eq.(7) and Eq.(8) it follows that

$$\frac{1}{f(t)}\frac{d^2}{dt^2}f(t) = -\omega^2 \tag{9}$$

Taking Eq.(9) into account, Eq.(8) reduces to

$$-\omega^2\rho(\vec{r})s_0(\vec{r}) = \nabla\bullet\hat{\sigma}_0(\vec{r}) \tag{10}$$

Now, $\hat{\varepsilon}_0(\vec{r})$ is given by

$$\hat{\varepsilon}_0(\vec{r}) = \frac{1}{2}\left[\nabla\vec{s}_0(\vec{r}) + (\nabla\vec{s}_0(\vec{r}))^T\right] \tag{11}$$

where $(\nabla\vec{s}_0(\vec{r}))^T$ is the trasposed tensor. Taking Eq.(11) into account, multiplying scalarly both members of Eq.(10) by $\vec{s}_0(\vec{r})$ after some operations we arrive to

$$-\omega^2\rho(\vec{r})\vec{\varepsilon}_0(\vec{r})\bullet\vec{s}_0(\vec{r}) = \nabla\bullet(\vec{s}_0(\vec{r})\bullet\hat{\sigma}_0(\vec{r})) - \hat{\sigma}_0(\vec{r}):\hat{\varepsilon}_0(\vec{r}) \tag{12}$$

where the symbol **:** represents tensor double scalar product. Integrating both members of this last equation in the region *V* occupied by the body, using the divergence theorem for tensors, and taking into accout the combination of stress free and rigid boundary conditions, we obtain

$$\omega^2 = \frac{\int_B \hat{\sigma}_0(\vec{r}):\hat{\varepsilon}_0(\vec{r})\,dV}{\int_B \rho(\vec{r})\vec{s}_0(\vec{r})\bullet\vec{s}_0(\vec{r})\,dV} \tag{13}$$

Multiplying the numerator and the denominator of Eq.(13) by $f^2(t)$ it follows that

$$\omega^2 = \frac{\int_B \hat{\sigma}(t,\vec{r}) : \hat{\varepsilon}(t,\vec{r})\, dV}{\int_B \rho(\vec{r})\, \vec{s}(t,\vec{r}) \bullet \vec{s}(t,\vec{r})\, dV} \tag{14}$$

The denominator is equal to twice the elastic strain energy of the solid body vibrating in the given normal mode.

Equations (13) and (14) can be used as a point of departure in a discussion about the relation between the frequency of a normal mode and the population of defects in body's material.

4. The region of influence in the dynamic case

A defect (crack cavity or even a solid inclusion) can be viewed as something that modifies the degree of connectivity of the body, changing its bounaries and adding new boundary conditions. As a consequence, the displacement field changes relative to the displacement field when the defect is absent. From this point of view, it is necessary to exclude from the region V the set of points corresponding to defects that are identified as such in the material.

But we can take an alternative viewpoint. We can substitute each defect by a local change in elastic moduli or density. These changes can be selected so that they have the same observable effects on the vibration frequencies that the faults in the material have. So, effective fields of elastic moduli and density can be introduced, differing from the true elastic moduli and density only in a certain dynamic region of influence of each defect. For low frequency and low amplitude vibrations, this dynamic region can be considered as equal to the static region of influence already introduced in Section 2 of this paper in connection with LEFM.

If the body is isotropic, from Eq.(14), introducing the incompresibility modulus $\tilde{K}$, the shear modulus $\tilde{G}$, the cubic dilatation $\Theta(t,\vec{r}) = \nabla \bullet \vec{s}(t,\vec{r})$ and the strain deviator $\hat{\varepsilon}_D(t,\vec{r})$, it follows that

$$\omega^2 = \frac{\int_B \tilde{K}\, \Theta^2\, dV + 2\int_B \tilde{G}\ \hat{\varepsilon}_D : \hat{\varepsilon}_D\, dV}{\int_B \rho\, \vec{s} \bullet \vec{s}\, dV} \tag{21}$$

For a body with a population of defects, the elastic moduli $\tilde{K}$ and $\tilde{G}$, as well as the density ρ, may be considered as effective fields of mechanical parameters. Introducing the dimensionless scalar fields $\varphi_K(\vec{r})$, $\varphi_G(\vec{r})$ and $\varphi_\rho(\vec{r})$, the influence of the defects can be described by the following equations:

$$\tilde{K}(\vec{r}) = \tilde{K}_0(\vec{r})\left(1 - \varphi_K(\vec{r})\right) \tag{22}$$

$$\tilde{G}(\vec{r}) = \tilde{G}_0(\vec{r})\left(1 - \varphi_G(\vec{r})\right) \tag{23}$$

$$\rho(\vec{r}) = \rho_0(\vec{r})\left(1 - \varphi_\rho(\vec{r})\right) \tag{24}$$

$\tilde{K}_0$ and $\tilde{G}_0$ are the moduli and ρ_0 is the density of the material withouth the population of defects.

From Eq.(21) to Eq.(24), after several re-arrangements we obtain :

$$\frac{\omega_0{}^2 - \omega^2}{\omega_0{}^2} = \frac{M - D}{1 - D} \tag{25}$$

This last equation gives the relative variation in the square of normal mode's frequency due to a population of defects in the vibrating body. ω_0 is the frequency of the considered normal mode in absence of defects, ω is the frequency of the same normal mode with the presence of the population of defects in the material, and M and D are given by

$$M = \frac{\int_B \tilde{K}_0(\vec{r})\, \varphi_K(\vec{r})\, \Theta_0{}^2(\vec{r})\, dV + 2\int_B \tilde{G}_0(\vec{r})\, \varphi_G(\vec{r})\, \hat{\varepsilon}_D : \hat{\varepsilon}_D dV}{\int_B \tilde{K}_0(\vec{r})\, \Theta_0{}^2(\vec{r})\, dV + 2\int_B \tilde{G}_0(\vec{r})\, \hat{\varepsilon}_D : \hat{\varepsilon}_D dV} \tag{26}$$

$$D=\left(\frac{\int_B \rho_0(\vec{r})\,\varphi_\rho(\vec{r})\vec{s}_0(\vec{r})\bullet\vec{s}_0(\vec{r})dV}{\int_B \rho_0(\vec{r})\vec{s}_0(\vec{r})\bullet\vec{s}_0(\vec{r})\,dV}\right) \tag{27}$$

Let us assume now that the spatial scale of the displacement field is at least an order of magnitude greater that the dimension of the defects. We also asume that the defects are far enough from each other so that their regions of influence can be considered disjoint (if this is not possible, it should be possible to define composite defects applying the guidelines of LEFM). With these asumptions, from Eq.(25) to Eq.(27) it follows that

$$\frac{\omega_0{}^2-\omega^2}{\omega_0{}^2}=\frac{\sum_{j=1}^{N}\left(M_K(\vec{r}_j)m_{Kj}+M_G(\vec{r}_j)m_{Gj}-\overline{D}(\vec{r}_j)\overline{d}_j\right)}{1-\sum_{j=1}^{N}\overline{D}(\vec{r}_j)\overline{d}_j} \tag{28}$$

In this equation *N* is the number of distinct defects in the material, $\vec{r}_j$ is the position vector of defect number j and

$$M_K(\vec{r}_j)=\left\{\frac{\tilde{K}_0(\vec{r}_j)\Theta_0{}^2(\vec{r}_j)}{\frac{1}{V(B)}\left(\int_B\tilde{K}_0(\vec{r})\Theta_0{}^2(\vec{r})+2\int_B\tilde{G}_0(\vec{r})\hat{\varepsilon}_{D0}(\vec{r}):\hat{\varepsilon}_{D0}(\vec{r})\,dV\right)}\right\} \tag{29}$$

$$M_G(\vec{r}_j)=\left\{\frac{\tilde{G}_0(\vec{r}_j)\hat{\varepsilon}_{D0}:\hat{\varepsilon}_{D0}(\vec{r}_j)}{\frac{1}{V(B)}\left(\int_B\tilde{K}_0(\vec{r})\Theta_0{}^2(\vec{r})+2\int_B\tilde{G}_0(\vec{r})\hat{\varepsilon}_{D0}(\vec{r}):\hat{\varepsilon}_{D0}(\vec{r})\,dV\right)}\right\} \tag{30}$$

$$m_{Kj}=\frac{1}{V(B)}\int_{BKj}\varphi_K(\vec{r})\,dV \tag{31}$$

$$m_{Gj}=\frac{1}{V(B)}\int_{BGj}\varphi_G(\vec{r})dV \tag{32}$$

$$\overline{D}(\vec{r}_j)=\frac{\rho_0(\vec{r}_j)\vec{s}_0(\vec{r}_j)\bullet\vec{s}_0(\vec{r}_j)}{\frac{1}{V(B)}\int_B\rho_0(\vec{r})\vec{s}_0(\vec{r})\bullet\vec{s}_0(\vec{r})} \tag{33}$$

$$\overline{d}_j=\frac{1}{V(B)}\int_{B\rho j}\varphi_\rho(\vec{r})\,dV \tag{34}$$

In these equations, $B_{K,j}$, $B_{G,j}$ and $B_{\rho,j}$ are the regions of influence of defect number *j* corresponding to the effective moduli $\tilde{K}$ and $\tilde{G}$, and the effective density ρ. $V(B)$ is the volume of the vibrating body. For cracks, cavities, porosities or inclusions less stiff and dense than the matrix that includes them, the dimensionless fields φ are non negative. As a consequence, in this case all the dimensionless functionals M, m, $\overline{D}$ and $\overline{d}$ are non negative. From Eq.(28) jointly with Eq.(29) to Eq.(34), it is in principle possible to calculate the relative lowering in the square of normal mode's frequency due to a population of defects.

If the local densities are not modified by the defects, Eq.(28) reduces to

$$\frac{\omega_0{}^2-\omega^2}{\omega_0{}^2}=\sum_{j=1}^{N}\left(M_K(\vec{r}_j)\,m_{Kj}+M_G(\vec{r}_j)\,m_{Gj}\right) \tag{35}$$

The terms $M_K\,m_{Kj}$ refer to dilatation field and the terms $M_G m_{Gj}$ refer to distortion field. Both dilatation field and distortion field correspond to the normal mode of vibration that has been excited. The contribution of each defect, for each type of field (dilatation or distortion) appears as a product of a factor *M* that depends of the normal mode only, and a factor *m* that depends mainly of the defect.

5. An alternative derivation of the equation for the relative lowering in the square of normal mode's frequencies and some applications

It is possible to derive an equation similar to Eq.(35) straigth forwardly from Eq.(14). The numerator verifies

$$\int_B \hat{\sigma}(t,\vec{r}) : \hat{\varepsilon}(t,\vec{r})\, dV = 2U \tag{36}$$

where U is the strain elastic energy of the body with defects. But, from LEFM, assuming that the static relation can be extended to the dynamic regime of interest :

$$U = U_0 - \sum_{j=1}^{N} u_j \cdot V_{d,j} \tag{37}$$

U_0 is the strain elastic energy of the body vibrating in a given normal mode, but without the population of defects, u_j is the local density of strain elastic energy due to the nominal stress field (that is without defects) of the normal mode that is being considered, and $V_{d,j}$ is the volume of the region of influence of defect number j. Then, if $\vec{s}_0(t,\vec{r})$ represents the displacement field of the body without defects, the frequency ω_0 of the normal mode of vibration is given by

$$\omega_0^{\;2} = \frac{2U_0}{\int_B \rho(\vec{r})\, \vec{s}_0(t,\vec{r}) \bullet \vec{s}_0(t,\vec{r})\, dV} \tag{38}$$

while the frequency ω of the same normal mode of vibration of the body with defects is given by

$$\omega^2 = \frac{2U}{\int_B \rho(\vec{r})\, \vec{s}(t,\vec{r}) \bullet \vec{s}(t,\vec{r})\, dV} \tag{39}$$

In a first approximation we can substitute $\vec{s}_0(t,\vec{r})$ in place of $\vec{s}(t,\vec{r})$ in Eq.(39). From the resulting equation and from Eq.(38) it follows that :

$$\frac{\omega_0^{\;2} - \omega^2}{\omega_0^{\;2}} = \sum_{j=1}^{N} \left[\frac{u_j}{\langle u_0 \rangle} \cdot \frac{V_{d,j}}{V_0} \right] \tag{40}$$

$V_0 = V(B)$ is the volume of the vibrating body, and $\langle u_0 \rangle = U_0 / V_0$ is the volume average of strain elastic energy in nominal stress field corresponding to the normal mode of vibration that is being considered.

Let us consider a bar of rectangular cross-section of area S and length l . The bar has a plane crack located in a point with coordinates $x = x_0$, $y = 0$, $z = 0$. The crack's plane forms an angle β with the axis of the bar. The crack can be represented by a rectangle having dimensions $2 \cdot a$ and h, as shown in Fig.1.

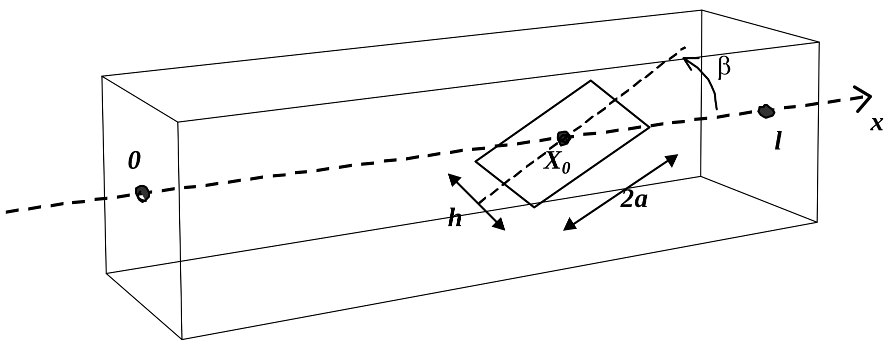

Figure 1. A cracked bar

We assume rigid boundary conditions at both ends. We excite longitudinal vibration and we neglect lateral contractions and dilatations. Then, the volume of influence of the defect is given by

$$V_d = \pi^2 \cdot a^2 \cdot h \cdot sin^2\beta \tag{41}$$

and the volume of the bar is $S \cdot l$.

The space dependent part of the displacement fields of the normal modes of longitudinal vibration, without defect, are, with $n = 1,2,3,...$

$$s_{0,n}(x) = A_n \cdot sin\left(\frac{n\pi x}{l}\right) \tag{42}$$

Then, the quotient between the density of strain elastic energy in the location of the defect and the mean strain elastic energy, both without defect, is given by :

$$\frac{u_j}{\langle u_0 \rangle} = \frac{\left(\frac{ds_{0,n}(x_0)}{dx}\right)^2}{\frac{1}{l} \cdot \int_0^l \left(\frac{ds_{0,n}(x)}{dx}\right)^2 dx} = 2 \cdot \cos^2\left(\frac{n \cdot \pi \cdot x_0}{l}\right) \tag{43}$$

Taking into accout these results, Eq.(40) is reduced to

$$\frac{\omega_{0,n}{}^2 - \omega_n{}^2}{\omega_{0,n}{}^2} = \frac{u}{\langle u_0 \rangle} \cdot \frac{V_d}{V_0} = \frac{2 \cdot \pi^2 \cdot a^2 \cdot h \cdot sin^2 \beta}{S \cdot l} \cdot \cos^2\left(\frac{n \cdot \pi \cdot x_0}{l}\right) \tag{44}$$

Eq.(44) gives a closed analytical formula that may be used to estimate the lowering in the square of the frequency of mode number *n* due to the presence of a defect in a given position along bar’s axis. If the defect is located in a node of the strain field, the cosine factor is zero and there is no lowering in frequency due to this defect. If the defect is in a maximum of the strain field, the lowering in the frequency is maximum. Measuring the lowering in the frequency of normal modes, relative to a reference state, the defects that are located near enough to strain maxima could be detected, if they are properly oriented relative to the nominal dynamic stress field. But the defects that are near enough to a node of the strain field are going to pass unoticed. If it is possible to excite several normal modes in the same structure or piece of machine, the defects that pass unoticed when one mode is excited can be detected when another mode is produced. From this observation the idea of a sequential method for non-destructive testing of solid bodies arose, combining modal analysis with local non-destructive testing such as ultrasonics (Suárez-Antola, 1990).

Furthermore, we can derive a formula for the stress intensity factor of the cracked bar under static tension. Let us consider a static tensile stress $\tilde{\sigma}$ superposed with the dynamic field of stresses. The corresponding stress intensity factor when the crack plane is perpendicular to the bar axis is $K_I = Y \cdot \tilde{\sigma} \cdot \sqrt{\pi \cdot a}$. Clearing away the dimension *a* of the defect between Eq.(44) and the formula for K_I we obtain the stress intensity factor as a function of the lowering in the square of normal mode’s frequency :

$$K_I{}^4 = \left(\frac{\pi^2 \cdot Y^4 \cdot S \cdot l \cdot \alpha^4 \cdot \tilde{\sigma}^4}{2 \cdot \pi \cdot h \cdot \cos^2\left(\frac{n \cdot \pi \cdot x_0}{l}\right)}\right) \cdot \left[\frac{\left(\omega_{0,n}{}^2 - \omega_n{}^2\right)}{\omega_{0,n}{}^2}\right] \tag{45}$$

Finally, as a simple application of Eq.(28), let us consider a solid body with a uniform porosity field. The pores are approximately spherical, with different radius, void or full of air. The wavelength of the corresponding normal mode of vibration is at least an order of magnitude greater than the least upper bound of pore’s radius. In that case, if ϕ represents a mean porosity, the effective moduli are given by (Kuster and Nafi, 1974) :

$$K = K_0 \frac{(1-\phi)}{(1+\beta_K \phi)} \tag{46}$$

$$G = G_0 \frac{(1-\phi)}{(1+\beta_G \phi)} \tag{47}$$

$$\beta_K = \frac{3}{4} \frac{K_0}{G_0} \tag{48}$$

$$\beta_G = \frac{6(K_0 + 2G_0)}{(9K_0 + 8G_0)} \tag{49}$$

The density is given by $\rho = \rho_0(1-\phi)$. Substituting all these relations in Eq.(28) we obtain

$$\frac{\omega_0{}^2 - \omega^2(\phi)}{\omega_0{}^2} = \frac{K_0 \int_B \Theta_0{}^2 dV}{2 \cdot U} \cdot \frac{(\beta_K \phi)}{(1+\beta_K \phi)} + \frac{2G_0 \int_B \hat{\varepsilon}_{D0} : \hat{\varepsilon}_{D0}\, dV}{2 \cdot U} \cdot \frac{(\beta_G \phi)}{(1+\beta_G \phi)} \tag{50}$$

where

$$2 \cdot U = K_0 \int_B \Theta_0{}^2 dV + 2G_0 \int_B \hat{\varepsilon}_{D0} : \hat{\varepsilon}_{D0}\, dV \tag{51}$$

The relative lowering in square of frequency is the sum of two hiperbolic functions of porosity. The weight factors of each hiperbolic function are the quotients between the dilatation energy and total energy in one case, and between distortion energy and total energy in the other case.

6. Discussion and conclusions

In the previous developments we neglected anelastic effects in the material. Anelastic effects in general, and internal friction in particular, can be taken into account introducing integral relationships between stress and strain. In order to simplify the discussion, we will work with a one dimensional model (Rabotnov, 1981) :

$$\rho \cdot \frac{\partial^2 s(t,x)}{\partial t^2} = \frac{\partial}{\partial x}\left[E(x)\left(\frac{\partial s(t,x)}{\partial x} - \int_{-\infty}^{t} L(t-t')\frac{\partial s(t',x)}{\partial x} dt' \right)\right] \tag{52}$$

L(t) is the stress relaxation kernel. A normal mode of vibration is again given by

$$s(t,x) = f(t) \cdot s_0(x) \tag{53}$$

but now $f(t)$ represents an exponentially damped vibration.

Superposed to a background of distributed internal friction that pervades the whole material in a more or less uniform way, we have the contribution of significant defects that increase the anelastic effects in its neighbourhood. This suggests the introduction of a region of influence for anelastic behaviour of defects. Then, the stress relaxation kernel could be written

$$L(t,x) = L_0(t) + \delta L(t,x) \tag{54}$$

$L_0(t)$ corresponds to the abovementioned uniform background of energy dissipation, and $\delta L(t,x)$ gives a small perturbation that is different from zero in a neighbourhood of a defect. The dissipation of mechanical energy in the normal modes of free vibration involves a very complex set of phenomena. Microscopic phenomena related with anelastic relaxation in polycrystalline solids (Nowick and Berry, 1972), and macroscopic mechanisms of energy dissipation in composite structures and machines (Lu and Yu, 2004).

Introducing regions of influence of significant defects, we obtained a closed analytical formula for the relative lowering of the square of o given normal mode's frequency. In a first approximation we may use the displacement field of the eigenmode calculated for a body without defects. This was done in the example of the loaded and cracked bar, with free longitudinal vibrations superposed to the static fields of displacement, strain and stress.

Initially we used the formulae of LEFM for strain energy release rate in order to calculate volumes of influence of defects in given stress fields, and to define the corresponding static regions of influence.

After that, we introduced the dynamic regions of influence in relation with the effective fields of elastic moduli in a vibrating body with a population of defects. For small amplitude and low frequency vibrations we assumed that the static regions can be a fairly good approximation to the dynamic ones. But this subject has to be studied in detail.

A relatively simplified analytical approach as intended in this paper, may be of interest for two reasons. First, as a guide to the design of ab-initio digital simulations of mechanical vibrations in defective structures and machine elements. Second, to ease the interpretation of certain experimental results obtained with non destructive testing methods.

7. References

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8. Responsibility notice

The author is the only responsible for the printed material included in this paper.