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NONLINEAR DYNAMICS AND CHAOS IN SYSTEMS WITH DISCONTINUOUS SUPPORT USING A SWITCH MODEL

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Abstract: *Non-smooth systems are usually related to physical systems with dry friction, impact and backlash. These systems operate in different modes, and the transition from one mode to another can often be idealized as an instantaneous or discrete transition. Since the time scale of the transition from one mode to another is much smaller than the scale of the individual mode dynamics, the mathematical modeling of these systems can be lead as discontinuous. This contribution uses a switch model in order to analyze the dynamical behavior of a single-degree of freedom oscillator with a discontinuous support. The procedure seems to be effective in order to deal with non-smooth system. Numerical simulations show a rich response, presenting dynamical jumps and chaos.*
Keywords:*Nonlinear dynamics, discontinuous support, switch model, non-smooth systems, chaos, basin of attraction.*

1. Introduction

Physical systems with dry friction, impact and backlash operate in different modes, and the transition from one mode to another can often be idealized as an instantaneous or discrete transition. Since the time scale of the transition from one mode to another is much smaller than the scale of the individual modes dynamics, the mathematical modeling of this kind of system can be done by discontinuous systems.

An alternative to deal with the mathematical modeling of discontinuous systems is the definition of a smoothened vector field [Hinrichs et al., 1998; Van de Vrande et al., 1999]. The smooth approximation has many advantages and normally yields good results. The main disadvantage of the smoothing method is the fact that it yields stiff differential equations which implies difficulties for numerical simulations.

The objective of this research effort is the analysis of non-smooth systems employing a ***switch model***. Specifically speaking, a single-degree of freedom system with a discontinuous support is concerned. The procedure here presented is originally proposed by Leine and Van Campen [Leine and Van Campen, 1998] on the study of stick-slip vibrations. Basically, the switch model treats non-smooth systems defining different sets of ordinary differential equations. Each set is related to a subspace of the physical system. The innovative idea is the definition of transition regions, used in order to govern the dynamical response during the transition to one set of equation to another. Therefore, each subspace has its own *smooth* ordinary differential equation.

The use of this approach allows the use of classical numerical procedures for each subspace. Numerical investigations of the single-degree of freedom system with a discontinuous support shows efficiency and allows one to analyze many aspects related to the system dynamics. Results show rich response, presenting dynamical jumps and chaos.

2. Discontinuous Support Model

This contribution analyzes the dynamical response of a single-degree of freedom system with a discontinuous support, shown in Figure 1a. The support is massless, having a linear spring stiffness k_s and linear damping coefficient c_s. The displacement of the mass is denoted by x, relative to the equilibrium position, while the displacement of the support is denoted by y. The system has two possible modes, represented by a situation where the mass presents contact with the support and other situation when there is no contact. Calling f_s as the contact force between mass and support, there is no contact between the mass and the support when:

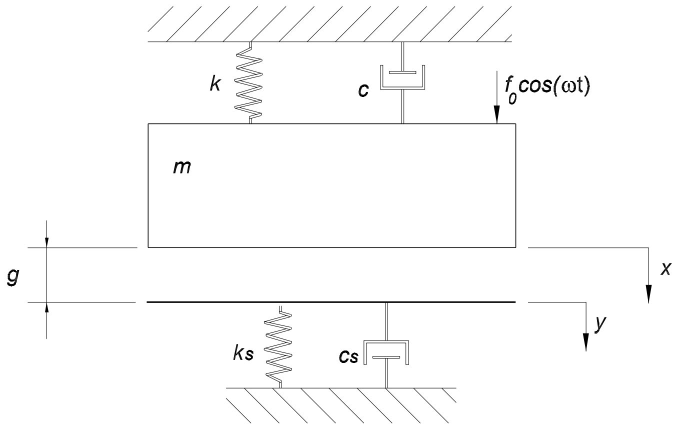


Figure 1: Mass-spring-damper with a discontinuous support.

$$x < y \text{ and } f_s = 0. \tag{1}$$

On the other hand, there is contact when:

$$x = y \text{ and } f_s = k_s\, y + c_s\, \dot{y} \geq 0. \tag{2}$$

The support relaxes to the equilibrium state, $y = g$, when there is no contact between the mass and the support. Assuming that the relaxation time of the support is much smaller than the time between two contact events, the dynamics of the support can be neglected. This assumption reduces the system dynamics to the dynamics of a mass-spring-dashpot, associated with a second-order differential equation. Therefore, the governing equations of the system may be defined by two different equation. One, for situations without contact:

$$m\,\ddot{x} + k\,x + c\,\dot{x} = f_0\, \cos(\omega\, t). \tag{3}$$

And the other when there is contact:

$$m\,\ddot{x} + k\,x + k_s\,(x - g) + (c + c_s)\,\dot{x} = f_0\, \cos(\omega\, t). \tag{4}$$

According to the Filippov theory, the state space of a system $\dot{u} = f(u), u \in \mathbf{R}^n$ may be split into two subspaces Γ_- and Γ_+ separated by a hyper-surface Σ. The hyper-surface Σ is defined by a scalar indicator function $h(u)$. Therefore, the state u is in Σ when:

$$h(u) = 0. \tag{5}$$

Hence, it is possible to define the subspaces Γ_- and Γ_+, and also the hyper-surface Σ, using the following sets:

$$\begin{aligned} &\Gamma_- = \{u \in \mathbf{R}^n \mid h(u) < 0\} \\ &\Sigma = \{u \in \mathbf{R}^n \mid h(u) = 0\} \\ &\Gamma+ = \{u \in \mathbf{R}^n \mid h(u) > 0\} \end{aligned} \tag{6}$$

Some physical systems need different interfaces in order to perform a correct description of the transitions. The single-degree of freedom system of Figure 1 is an example. In this system, one denotes $u = (x, \dot{x})$ and the contact between the mass and the support occurs if the displacement becomes equal to contact distances g.On the other hand, the mass loses contact with the support when the contact force vanishes, *i.e.* if $f_s = k_s\,(x - g) + c_s\,\dot{x} = 0$. Therefore, it is necessary two indicator functions in order to define the system subspaces:

$$\begin{aligned} &h_\alpha(x, \dot{x}) = x - g; \\ &h_\beta(x, \dot{x}) = k_s\,(x - g) + c_s\,\dot{x}; \end{aligned} \tag{7}$$

The mass is not in contact with the support if the state-vector $u = (x, \dot{x}) \in \Gamma_-$, i.e.

$$\Gamma_- = \{u \in \mathbf{R}^2 \mid h_\alpha(x, \dot{x}) < 0 \text{ or } h_\beta(x, \dot{x}) < 0\}. \tag{8}$$

On the other hand, there is contact between the mass and the support if $u = (x, \dot{x}) \in \Gamma_+$:

$$\Gamma_+ = \{u \in \mathbf{R}^2 \mid h_\alpha(x, \dot{x}) > 0 \text{ and } h_\beta(x, \dot{x}) > 0\}. \tag{9}$$

The hyper-surface Σ consists of the conjunction of two surfaces Σ_α and Σ_β. The hyper-surface Σ_α defines the transition from Γ_- to Γ_+ because the contact is made when u becomes large than g,

$$\Sigma_\alpha = \{u \in \mathbf{R}^2 \mid h_\alpha(x, \dot{x}) = 0,\ h_\beta(x, \dot{x}) \geq 0\}. \tag{10}$$

The hyper-surface Σ_β defines the transition from Γ_+ to Γ_- as the contact is lost when the force of the support vanishes.

$$\Sigma_\beta = \{u \in \mathbf{R}^2 \mid h_\alpha(x, \dot{x}) \geq 0,\ h_\beta(x, \dot{x}) = 0\}. \tag{11}$$

All subspaces related to the cited system, which involves the subspace Γ_- and Γ_+, and also the hyper-surfaces Σ_α and Σ_β, are shown in Figure 2.

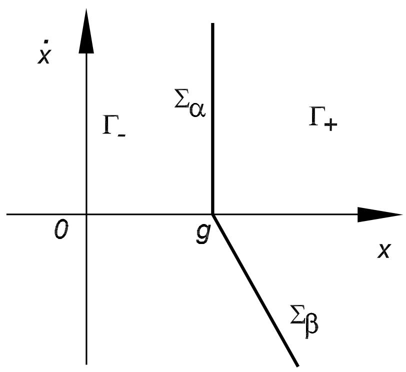


Figure 2: Phase space of the discontinuous support system.

The state equation of this discontinuous system can be written as follows

$$\dot{u} = f(u,t) = \begin{cases} f_-(u,t) \ u \in \Gamma_- \\ \overline{co}\{f_-(u,t), f_+(u,t)\},\ u \in \Sigma \\ f_+(u,t) \ u \in \Gamma_+ \end{cases} \tag{12}$$

with,

$$f_-(u,t) = \begin{bmatrix} \dot{x} \\ -\frac{k}{m}x - \frac{c}{m}\dot{x} + \frac{f_0}{m}\cos(\omega t) \end{bmatrix}; \tag{13}$$

$$f_+(u,t) = \begin{bmatrix} \dot{x} \\ -\frac{k}{m}x - \frac{k_s}{m}(x-g) - \frac{c+c_s}{m}\dot{x} + \frac{f_0}{m}\cos(\omega t) \end{bmatrix}; \tag{14}$$

and the closed convex set,

$$\overline{co}\{f_-(u,t), f_+(u,t)\} = \begin{bmatrix} \dot{x} \\ -\frac{k}{m}x - \frac{c+c_s}{m}\dot{x} + \frac{f_0}{m}\cos(\omega t) \end{bmatrix} \text{in the hyper-surface } \Sigma_\alpha; \tag{15}$$

or

$$\overline{co}\{f_-(u,t), f_+(u,t)\} = \begin{bmatrix} \dot{x} \\ -\dfrac{k}{m}x - \dfrac{c}{m}\dot{x} + \dfrac{f_0}{m}\cos(\omega t) \end{bmatrix} \text{in the hyper-surface } \Sigma_\beta. \tag{16}$$

This approach allows one to deal with non-smooth systems employing a smoothened system. Leine [Leine , 2000] also defines a finite region in order to consider transition subspaces. Therefore, a region of small relative displacement is define as $|x - g| < \eta$, where $\eta \ll 1$. The finite region is useful for numerical simulations since an exact value of zeros will rarely be computed. The thickness parameter of the narrow band η needs to be chosen according to the physical problem.

3. Numerical Simulation

Numerical procedure developed to deal with nonlinearities of the formulation analyzes, at each step of time, the subspace related to the system state. It allows one to define the system mode and, as a consequence, the governing equation. The change from the contact mode to the mode without contact operates as switches between different systems. The thickness of the narrow band η and the Runge-Kutta Fehlberg tolerance TOL need to be appropriated chosen [Leine, 2000].

The switch model is introduced as an efficient numerical method for interacting a system with discontinuous support. The concept of the switch model with a finite narrow band, during the transition, allows the decrease of the time steps being essential for the efficiency of this method. This article considers numerical simulations related to the parameters presented in Table 1.

Table 1: Parameter Values

$m = 1$ kg	$k = 1$ N/m	$c = 0.05$ N/(ms)	$f_0 = 0.5$ m
$c_s = 0.3$ N/(ms)	$k_s = 10$ N/m	$g = 4.0$	

At first, the analysis of numerical parameters is performed. Figure 3 presents an example for $\eta = 10^{-4}$ and $TOL = 10^{-8}$. Under this condition, the system presents an evolution where part of the movement occurs without contact and the other part occurs with contact (Figure 3a). Therefore, the system passes for all subspaces previously defined. Figure 3b shows the time evolution of the system response showing its position in the subspaces. It is worthwhile to observe that the system passes in all regions, *i.e.*, the subspaces Γ_- and Γ_+, and the hyper-surfaces Σ_α and Σ_β.

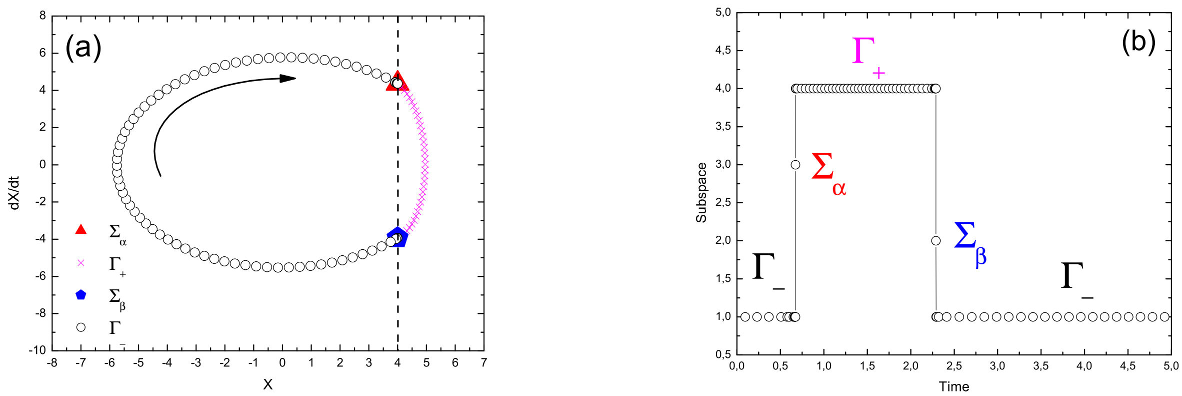


Figure 3: a)Phase space, $\omega = 1.0$ rad/s; b) Subspaces and hyper-surfaces *versus* time.

Now, different stiffness support values, k_s, are considered. As expected, the decrease on the stiffness of the springs causes more penetration of the mass through the support. It should be pointed out that the proposed procedure presents good results for different values of spring stiffness, allowing the analysis of impact systems, which can be done introducing high values of stiffness k_s. Figure 4 presents some phase spaces related to different values of k_s, confirming this conclusion.

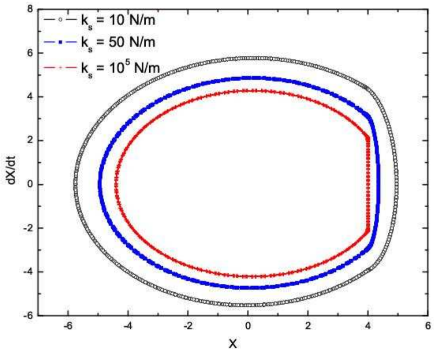


Figure 4: Phase spaces related to different values of k_s: $k_s = 10$; 50 and 10^5 N/m, respectively.

Frequency domain analysis is now in focus considering the resonance curve of the system with discontinuous support (Figure 5). These results is obtained considering two different values of the support stiffness: $k_s = 10$ N/m (Figure 5a and $k_s = 50$ N/m (Figure 5b. The resonance curve has a typical characteristic of nonlinear systems, presenting dynamical jumps. Besides, it presents a discontinuity related to the contact with the support. This is caused by the hardening related to the contact with the support. Notice that the hardening is high when k_s is increased, Figure 5b. Moreover, the amplitude becomes equal to g when $\omega = \omega_a$ and $\omega = \omega_b$, on both sides of the peak.

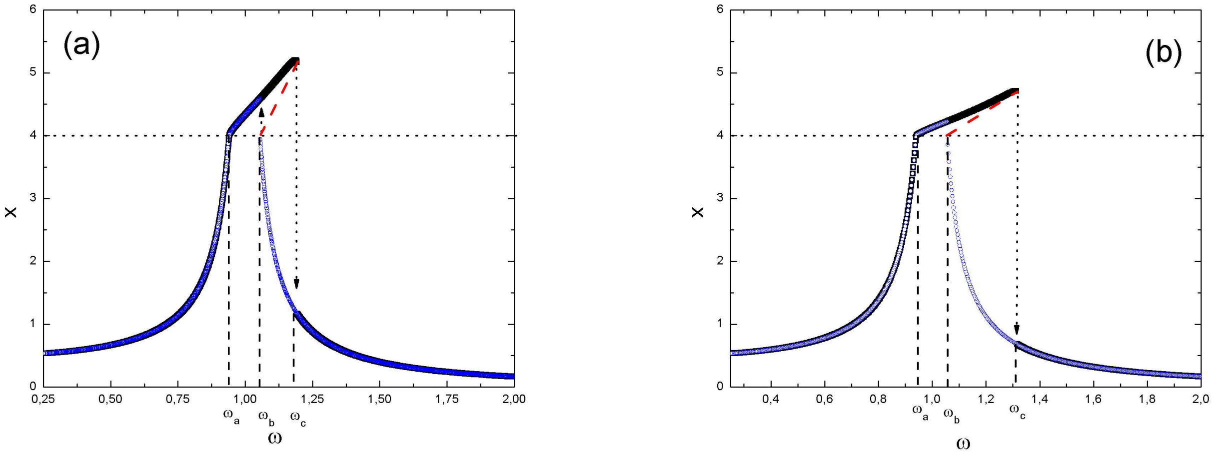


Figure 5: Frequency response of the discontinuous support system: a) $k_s = 10$ N/m; b) $k_s = 50$ N/m.

In order to analyze dynamical jumps, different situations are now considered. When $\omega_b < \omega < \omega_c$, a bifurcation is observed and three periodic solution or attractors may co-exist: two stable and one unstable (dashed line). Leine et al. shows that this bifurcation is caused by the jump of the Floquet multiplier through the unit circle, being denoted as *discontinuous fold bifurcation* [Leine and Van Campen, 2002].

The analysis of basins of attraction can be useful to observe the co-existence of attractors, for $\omega_b < \omega < \omega_c$. Figure 6 shows the evolution of the basin of attraction under increasing frequency ω. In each case, it is observed two stable co-existing periodic attractors, represented by two different colors (dark and light).

Each one of these behaviors can be observed in the phase space presented in Figure 7. In Figure 6a, the excitation frequency is $\omega = 1.1$ rad/s, while in Figure 6b $\omega = 1.18$ rad/s. The light color basin of attraction represents a behavior without contact, while the dark basin of attraction is associated with a behavior with contact. Initially, when $\omega = \omega_b$ all initial conditions lead to dark attractor. After that, it is interesting to notice that the light attractor increases, when excitation frequency is increased. The "dynamical jump", when $\omega = \omega_c$, is represented when all initial conditions lead to light basin of attraction.

4. Grazing Boundaries and Bifurcations

The dynamical response of the system for small amplitudes has no contact, and therefore a linear response is expected, represented by an ellipse in the phase space. Increasing the displacement amplitude, for example

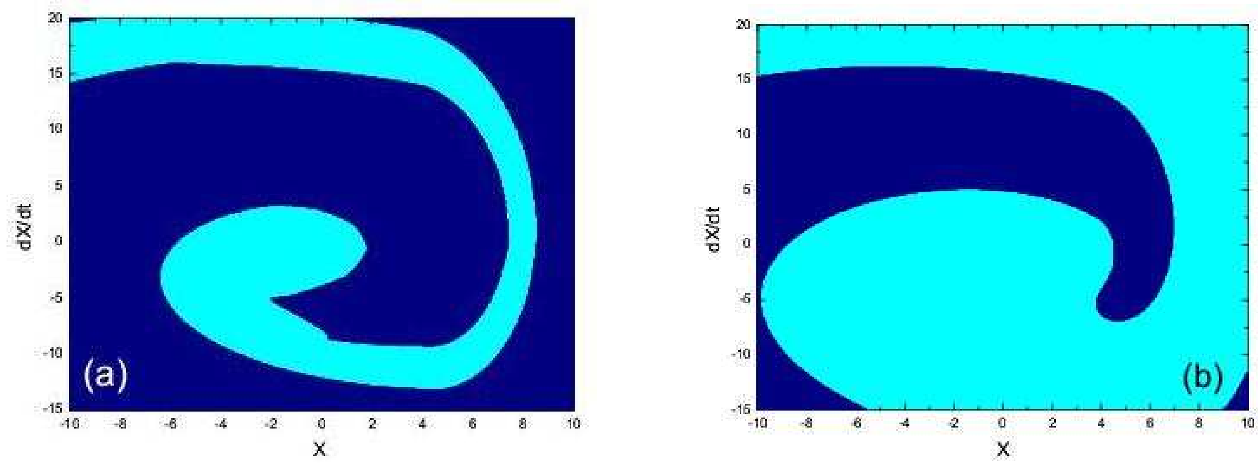


Figure 6: Basin of attraction: a)$\omega = 1.1$ rad/s; b) $\omega = 1.18$ rad/s.

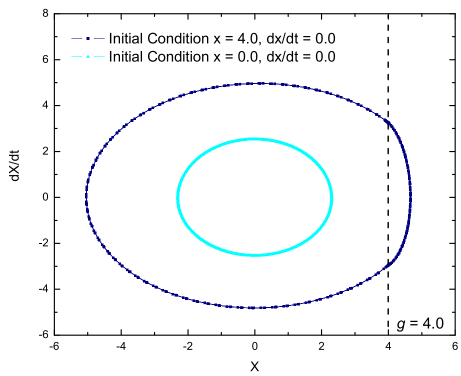


Figure 7: Phase space of each periodic attractor.

due to changes in the forcing characteristics, the elliptical trajectory tends to grow and, eventually, reaches the support. The situation where the phase space ellipse exactly fits within the contact or impact boundaries, is called *grazing orbit*. Therefore, the mass graze the support with zero velocity. Afterwards, a further incremental change to the excitation characteristics (typically amplitude or frequency) produces contact behavior, that is not easily predicted. For some combination of system parameters, the response may proceed very abruptly into chaos. Higher-order periodic response may interrupt the chaotic regime as the forcing characteristic continue to change. On the other hand, the grazing orbit may be replaced directly by a stable periodic response. This unpredictability in connection with the grazing phenomenon has sparked interest in defining and exploring the so-called *grazing bifurcation* [Nordmark, 1991, Casas et al., 1996, Nusse et al., 1994, Virgin and Begley, 1999].

The single-degree of freedom reported in this article allows one to present a demonstration of grazing phenomenon and also different kinds of motion related to this. On this basis, it is necessary to increase k_s and also change the gap, g. Figure 8 shows bifurcation diagram where the control parameter is the excitation frequency, ω. Notice that this diagram gives an indication that the motion quickly moves into a chaotic region after an graze contact. These diagrams are obtained considering the parameters presented in Table 2.

Table 2: Parameter Values

$m = 1$ kg	$k = 1$ N/m	$c = 0.05$ N/(ms)	$f_0 = 0.5$ m
$c_s = 0.6$ N/(ms)	$k_s = 300$ N/m	$g = 1.6$	$0.8 < \omega < 1.0$ rad/s

By considering a frequency lower than $\omega = 0.83$, the system presents a linear response, non-impacting. By increasing this frequency value, the system reaches the support, presenting contact, and the response becomes

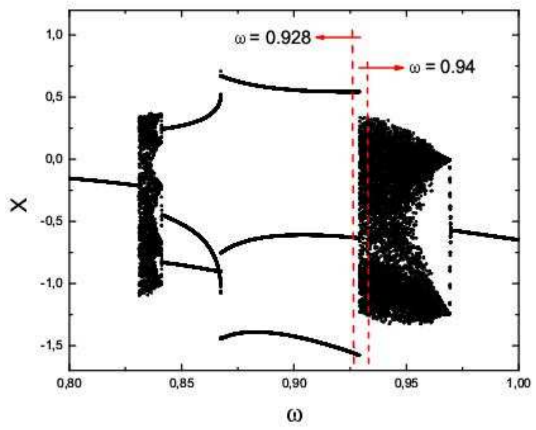
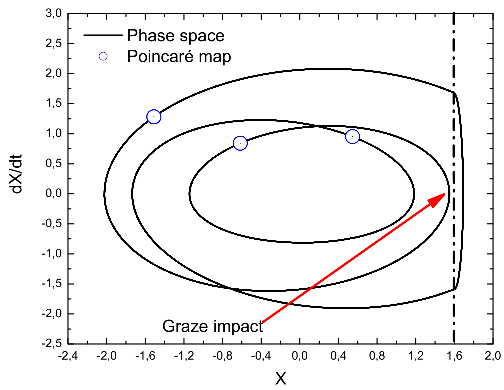


Figure 8: Bifurcation diagram with ω as control parameter, displacement $\times$ ω.

irregular over a range of excitation frequency. The phase space related to $\omega = 0.93$, is shown in Figure 9a, where it can be observed a period-3 orbit with a grazing contact. By promoting a small increase in the forcing frequency, $\omega = 0.94$, the response becomes chaotic, as shown in the *Poincaré* map of the Figure 9b.



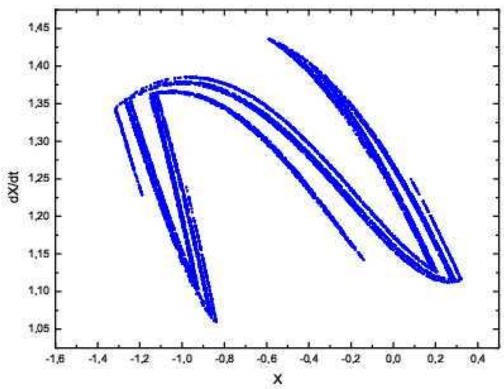


Figure 9: a) Phase space with graze contact; b) *Poincaré* map.

Now, a different bifurcation diagram is focused on, considering the forcing amplitude f_0 as a control parameter. The other parameters used are:$m = 1$ kg; $k = 1$ N/m; $c = 0.05$ N/(ms); $c_s = 0.6$ N/(ms); $\omega = 0.9$ rad/s; $f_0 = 0.5$ m; $k_s = 1000$ N/m and $g = 2.0$. This bifurcation diagram are shown in Figure 10.

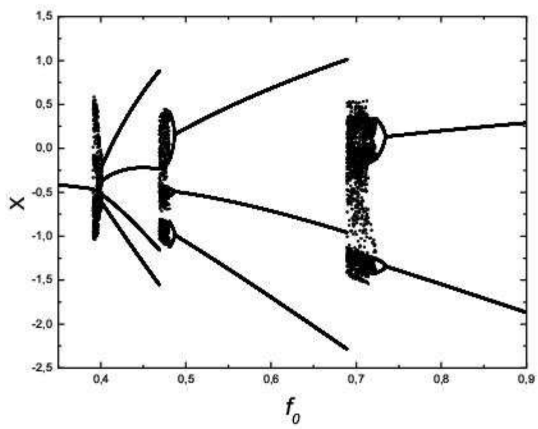


Figure 10: Bifurcation diagram, displacement $\times$ f_0.

Notice that, after a region related to periodic behaviors, chaotic regions appears. Once again, a graze orbit occurs just before the birth of the chaotic attractor. Phase spaces and *Poincaré* map of period-4, chaotic and period-2 is presented in Figure 11a, b, c, respectively.

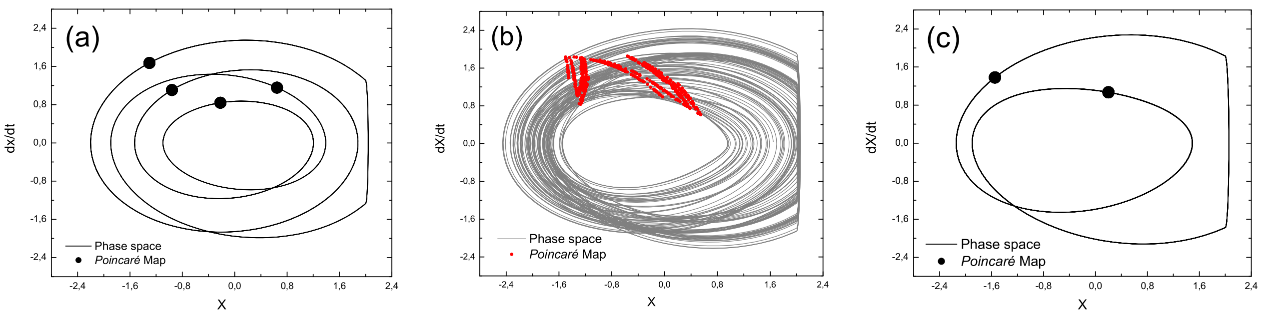


Figure 11: Phase space and *Poincaré* map, c) $f_0 = 0.45$; d) $f_0 = 0.7$ and e) $f_0 = 0.8$ m.

5. Conclusions

The objective of this research effort is the analysis of non-smooth systems employing a ***switch model***. As an application of the general formulation, a single-degree of freedom system with a discontinuous support is considered. The switch model treats the system as four different sets of ordinary differential equations and, therefore, the state space is split into subspaces. Moreover, a finite region of transition is defined being useful for numerical simulations. In this case, the thickness parameter of the narrow band needs to be chosen according to the physical problem. Numerical investigations of the single-degree of freedom system shows efficiency and allows one to analyze many aspects related to the dynamics of this system. The dynamical response is very rich, presenting dynamical jumps and chaos. It is shown the importance of the analysis of Graze contact related to this kind of system. Finally, the authors agree that the proposed procedure may be useful for the analysis of other non-smooth systems.

6. Acknowledgements

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