ASYMPTOTIC APPROXIMATION FOR THE PARAMETER SPACE STABILITY BOUNDARY FOR TWO-PHASE FLOWS IN PIPELINE-RISER SYSTEMS

Karl Peter Burr, karl.burr@ufabc.edu.br
Centro de engenharia, modelagem e ciencias sociais aplicadas (CECS) da Universidade federal do ABC. Rua Santa Adélia, 166. Bairro Bangu. Santo André - SP - Brasil . CEP 09.210-170 .

Jorge Luis Baliño
Depto. de Engenharia Mecânica da Escola Politêcnica da Universidade de São Paulo. Av. Prof. Mello Moraes, 2231. Cidade Universitária. CEP 05508-970 - São Paulo - SP

Abstract.*Intermitent flow regimes may appear in offshore production facilities for low rate between gas and liquid mass flow rates when the pipeline preceeding the riser has a downward inclination angle. Stability analysis for an appropriate model for the multifase flow in pipeline-riser system reveals the regions in the parameter space which have a steady state, or a stable stationary point in the dynamic system language. We consider the no pressure wave model for the two-fase flow in a pipeline-riser system with vertical riser. We perform the linear stability analysis of this flow model. We use an analytic asymptotic approximation for the stationary state. We derive the stationary state perturbation governing equations and the spectrum of this linear boundary value problem is given by an eigenvalue equation in terms of the system parameters. By setting the real part of the eigenvalue equal to zero, the eigevalue equation gives the boundary between regions where the stationary state is stable or unstable in the system parameter space. We illustrate this boundary in the system parameter space for pipeline-riser system used in experiments reported in the literature.*

Keywords:*hydrodynamic instability, linear stability analysis, two-phase flows, pipeline-riser system, stability boundary, asymptotic approximation*

1. INTRODUCTION

Offshore oil production systems may not have a steady state operation regime for parameters configurations which may appear during their lifespan. For example, along the lifespan of an offshore production facility, the stock of gas in the reseirvour reduces with time. This implies in low ratio between gas and liquid flow rates. Under such conditions, for pipeline-riser system where the pipeline presents downward inclination just before the riser, the operation flow regime is intermitent and has a cyclic nature, like the severe slugging phenomenon. These cyclic flow regimes may have a tremendous impact in oil production. They may cause reservoir flow oscillations, high average back pressure at the well head, high instantaneous flow rates, which are difficult to control and eventually may cause the offshore oil production facility shutdown. Therefore, it would be usefull to know the regions in the pipeline-riser system parameter space for which the two-phase flow has a steady state operation regime.

The linear stability analysis of an appropriate model of the two-phase flow in a pipeline-riser system should provide tools to identify the regions in the pipeline-riser system parameter space where the two-phase flow has a steady state. The linear stability analysis of a dynamic system can be described as follows. First, we need to obtain the stationary states. Second, we need to study the stability of the stationary states under small perturbations. To study the stability of a stationary state, we write the dependent variables as their stationary state value plus a perturbation, and then substitute them into the system governing equations. We linearize the resulting equations to obtain the governing equations for the stationary state perturbations. If the solution of this set of equations grows with time, the stationary state is unstable, but if the solution decays with time, the stationary state is stable and it represents a steady state of the dynamic system. Once we are able to decide if a stationary state is stable or not for a configuration of the pipeline-riser system parameters, we can search in the system parameter space for the stability boundary, which is the boundary between the regions where the considered stationary state is stable and the regions where it is unstable.

The objective of this work is to obtain an approximation for the boundary in the pipeline-riser system parameter space between the regions where the two-phase flow has a steady state and the region where a steady state is not possible. We obtained an approximation for the stability boundary given as the graph of an implicit equation in the pipeline-riser system parameters space.

In the next section, we discuss the model for two-phase flow in pipeline-riser systems adopted in this work. This model is basically the model presented in Baliño et al. (2007) and is reproduced here to make the paper self contained. The third section contains the linear stability analysis of the adopted model for two-phase flow in pipeline-riser systems. We present the equations for the stationary state and their asymptotic solution with respect to a small system parameter. We use the zero-th order approximation for the stationary state as the base flow for the linear stability analysis. We obtain

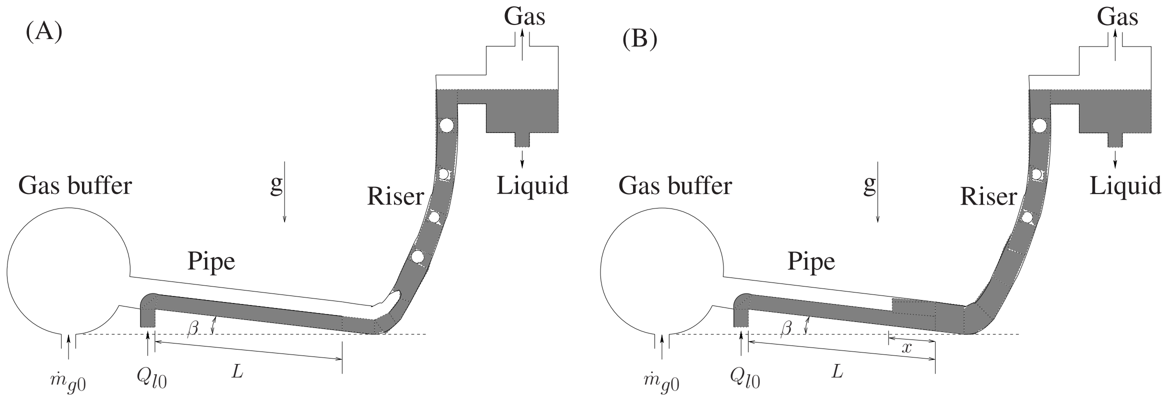

Figure 1. Part (A) - First configuration: $x = 0$. Part (B) - Second configuration: $x > 0$.

the governing equations for the perturbation of the adopted base flow (approximation for the stationary state), we solve them and we impose that their solution has zero growth rate. This condition furnishes an implicit equation for the stability boundary in the system parameter space. In the fourth section we illustrate the stability boundary obtained in the previous section for pipeline-riser systems used experiments reported in the literature. The fifth section has some discussion and conclusions.

2. TWO-PHASE FLOW MODEL.

The pipeline-riser system is composed basically of two parts. The pipeline plus a gas buffer and the riser (see Fig. 1). The pipeline and riser are connected at the bottom of the riser. The pressure at the top of the riser is assumed given and we have liquid and gas mass flowing into the pipeline.

The gas-liquid flow in the pipeline is assumed as always stratified. This flow behavior extends either to the whole pipeline (see part (A) of Fig. 1) or it extends until the liquid penetration position in the pipeline (see part (B) of Fig. 1). The configuration illustrated in part (A) of Fig. 1 corresponds to continuous gas flow from the pipeline into the riser and the configuration illustrated in part (B) of Fig. 1 corresponds to no gas flow from the pipeline into the riser and partial liquid flooding of the pipeline. Variables Q_{l0}, $\dot{m}_{g0}$, β, L, g and x are illustrated in Fig. 1 and represent, respectively, the volumetric flow rate of liquid into the pipeline, the gas mass flow rate into the pipeline, the pipeline inclination angle, the distance of the liquid inlet from the bottom of the riser, the gravity acceleration constant and the pipeline liquid flooding distance from the bottom of the riser (parts (B) of Fig. 1). We consider an isothermal drift-flux model assuming quasi-equilibrium momentum balance for the two-phase flow in the riser.

In summary, we consider a set of two different configurations. The first one is illustrated in part (A) of Fig. 1. In this configuration we have stratified flow in the pipeline and continuous gas penetration from the pipeline into the riser.

The second configuration is illustrated in part (B) of Fig. 1, where we have stratified flow in part of the pipeline with liquid flooding until a distance x from the bottom of the riser.

The set of governing equations is not the same for the two different configurations represented in Fig. 1. Below we give governing equations for the different configurations illustrated in Fig. 1.

2.1 Governing Equations for the Two-Phase Flow.

We give the governing equations for the two-phase flow in pipeline-riser system in non-dimensional form. We define the following non-dimensional variables according to the set of equations below.

$$x^* = \frac{x}{L_r}, \quad (1) \qquad P^* = \frac{P}{\rho_l R_g T_g}, \quad (3) \qquad t^* = t\frac{Q_{l0}}{AL_r}, \quad (5)$$

$$s^* = \frac{s}{L_r}, \quad (2) \qquad j^* = j\frac{A}{Q_{l0}}, \quad (4) \qquad \dot{m}^* = \frac{\dot{m}}{\rho_l Q_{l0}}, \quad (6)$$

where L_r is the riser length, A is the cross-sectional area of the pipeline and riser, s is the space parameterization along the riser length, T_g is the absolute temperature of the gas, ρ_l is the liquid phase density, R_g is the gas constant, j stands for superficial velocity, $\dot{m}$ stands for mass flow rate, P stands for pressure and t stands for time. The variables with * as a superscript are non-dimensional variables.

2.1.1 Pipeline Governing Equations.

We first give the non-dimensional governing equation for the pipeline. We consider the gas in the pipeline behaving as a pressure cavity at non-dimensional pressure P_g^*, constant in position and evolving isothermically as a perfect gas. We consider a fixed control volume with the pipeline and gas buffer contours as the control volume surface. For this control volume, we obtain the mass conservation equation for each of the two phases. We have to consider two different situations at the pipeline. We have either continuous gas penetration from the pipeline into the riser ($x^* = 0$, see part (A) of Fig. 1) or partial liquid flooding of the pipeline ($x^* > 0$, see part (B) of Fig. 1).

Below follows the governing equations for the pipeline for the conditions $x^* > 0$ and $x^* = 0$. We start with the equations for the case where $x^* > 0$. The mass conservation equation for the liquid phase is

$$-(\delta - x^*)\frac{d\alpha_p}{dt^*} + \alpha_p \frac{dx^*}{dt^*} + j_{lb}^* - 1 = 0, \quad (7)$$

where $\delta = L/L_r$. α_p is the pipeline void fraction and j_{lb}^* is the non-dimensional liquid phase superficial velocity at the bottom of the riser. The mass conservation equation for the gas phase is

$$[(\delta - x^*)\alpha_p + \delta_b]\frac{dP_g^*}{dt^*} + P_g^*(\delta - x^*)\frac{d\alpha_p}{dt^*} - \alpha_p P_g^* \frac{dx^*}{dt^*} - \dot{m}_{g0}^* = 0, \quad (8)$$

where we used the perfect gas relation $P_g = \rho_g R_g T_g$. $\delta_b = V_b/(AL_r)$ is the non-dimensional length equivalent to the gas buffer volume V_b divided by the product of the pipeline cross sectional area A by the riser length. We consider variations of pressure in the pipeline only due to hydrostatic effects. Then, the momentum equation is

$$P_g^* = P_b^* + \Pi_L x^* \sin(\beta), \quad (9)$$

where P_b^* is the non-dimensional pressure at the bottom of the riser and the non-dimensional number Π_L is given by the equation

$$\Pi_L = \frac{gL_r}{R_g T_g}. \quad (10)$$

This non-dimensional number is the ratio between the hydrostatic pressure at the bottom of the riser when it is filled completely with liquid and the gas pressure times the ratio between the gas and liquid densities.

We can eliminate the gas non-dimensional pressure P_g^* in favor of the riser bottom non-dimensional pressure P_b^*, by using the equation (9). Then the liquid phase mass conservation equation is not affected, but the gas phase mass conservation equation assumes the form

$$[(\delta - x^*)\alpha_p + \delta_b]\left(\frac{dP_b^*}{dt^*} + \Pi_L \sin(\beta)\frac{dx^*}{dt^*}\right) + (P_b^* + \Pi_L x^* \sin(\beta))\left[(\delta - x^*)\frac{d\alpha_p}{dt^*} - \alpha_p \frac{dx^*}{dt^*}\right] - \dot{m}_{g0}^* = 0. \quad (11)$$

Next, we present the equations for the case $x^* = 0$. The liquid phase mass conservation equation is

$$-\delta\frac{d\alpha_p}{dt^*} + j_{lb}^* - 1 = 0. \quad (12)$$

Notice that in this case, the gas non-dimensional pressure P_g is equal to the non-dimensional pressure at the bottom of the riser. Then, we use the riser bottom non-dimensional pressure P_b^* instead of the gas non-dimensional pressure P_g^* in the gas phase mass conservation equation, which is

$$(\delta\alpha_p + \delta_b)\frac{dP_b^*}{dt^*} + \delta P_b^* \frac{d\alpha_p}{dt^*} + P_b^* j_{gb}^* - \dot{m}_{g0}^* = 0, \quad (13)$$

where j_{gb}^* is the gas non-dimensional superficial velocity at the bottom of the riser.

To close the model for the pipeline, we use an implicit algebraic relation for the pipeline void fraction α_p which relates it with the non-dimensional gas superficial velocity at the bottom of the riser j_{gb}^*, with the non-dimensional liquid superficial velocity at the bottom of the riser j_{lb}^* and with the non-dimensional gas pressure P_g^*, and is derived from local momentum equilibrium for each phase of a stratified flow in a pipeline (Yemada and Dukler 1976, Kokal and Stanislav 1989 and others). For the case $x^* = 0$ we write

$$A_p(\alpha_p, j^*_{lb}, j^*_{gb}, P^*_b) = 0, \tag{14}$$

since in this case $P^*_b = P^*_g$. For the condition $x^* > 0$ we write the algebraic relation for α_p as

$$A_p(\alpha_p, j^*_{lb}, x^*, P^*_b, \frac{dx^*}{dt^*}) = 0. \tag{15}$$

To derive these algebraic relations we assume stratified flow in the pipeline. We consider local momentum equilibrium for each phase and assume that the pressure gradient is the same for both phases. Then we eliminate the pressure gradient and end up with an algebraic relation for the quantities mentioned in the above paragraph. This procedure leads to an algebraic relation similar to Eq. (3) of Yemada and Dukler (1976).

2.1.2 Equations for the Riser.

For the riser, non-dimensional equations are derived from an isothermal drift-flux model assuming quasi-equilibrium momentum balance for the two-phase flow in the riser. The mass conservation equation for the liquid phase is

$$-\frac{\partial \alpha_r}{\partial t^*} + \frac{\partial j^*_l}{\partial s^*} = 0, \tag{16}$$

where j^*_l is the non-dimensional liquid superficial velocity along the riser and α_r is the void fraction along the riser. The mass conservation equation for the gas phase is

$$\frac{\partial}{\partial t^*}(P^*\alpha_r) + \frac{\partial}{\partial s^*}(P^* j^*_g) = 0, \tag{17}$$

where P^* and j^*_g are, respectively, the non-dimensional pressure and the non-dimensional gas superficial velocity along the riser.

We assume the inertia forces small and neglect them. We consider pressure variation due to the hydrostatic force and friction. The shear stress at the riser wall was modeled using a homogeneous two-phase flow model (Kokal and Stanislav 1989) for the fluid and a Fanning friction coefficient f_m. Then, the linear momentum equation is

$$\frac{\partial P^*}{\partial s^*} = -\Pi_L[1 - \alpha_r + P^*\alpha_r]\left(\sin(\theta(s^*)) + \frac{4}{\Pi_D} f_m j^* |j^*|\right), \tag{18}$$

where $\theta(s)^*$ is the local riser inclination angle at position along the riser arc length, j^* is the sum of the liquid and gas superficial velocities and $f_m = f_m(R_{e,m}, \epsilon_r/D)$. The quantity ε_r represent the riser wall roughness, D represents the riser diameter and $R_{e,m}$ is the liquid-gas mixture Reynolds number given by

$$R_{e,m} = \frac{Q_{l0} D}{A \nu_l} \frac{(1 - \alpha_r + P\alpha_r)|j^*|}{1 - \alpha_r + \delta_\mu \alpha_r}, \tag{19}$$

where δ_μ is the ratio between the gas and liquid dynamic viscosities. The non-dimensional number Π_L is already defined by Eq. (10) and the non-dimensional number Π_D is defined as

$$\Pi_D = \frac{2gDA^2}{Q^2_{l0}}. \tag{20}$$

We consider the constitutive law corresponding to the drift flux model (Zuber and Findlay 1965) to relate the void fraction along the riser with the local values of the gas and liquid non-dimensional superficial velocities. Along the riser we have the relation

$$j^*_g = \alpha_r[C_d(j^*_l + j^*_g) + U^*_d]. \tag{21}$$

For the drift flux coefficients C_d and U^*_d we use the following correlation based on experimental data (Bendiksen 1984)

$$C_d = \begin{cases} 1,05 + 0,15\sin(\theta(s^*)) & \text{for } |j^*| < 3,5\frac{\sqrt{gD}A}{Q_{l0}} \\ 1,2 & \text{for } |j^*| \geq 3,5\frac{\sqrt{gD}A}{Q_{l0}} \end{cases} \tag{22}$$

$$U_d^* = \begin{cases} \frac{\sqrt{gD}A}{Q_{l0}}(0,35\sin(\theta(s^*)) + 0,54\cos(\theta(s^*))) \\ \qquad \text{for } |j^*| < 3,5\frac{\sqrt{gD}A}{Q_{l0}} \\ 0,35\frac{\sqrt{gD}A}{Q_{l0}}\sin(\theta(s^*)) \text{ for } |j^*| \geq 3,5\frac{\sqrt{gD}A}{Q_{l0}} \end{cases} \tag{23}$$

Not all equations above are valid for the two configurations defined previously and illustrated in Fig. 1. For the first configuration, the governing equations are Eqs. (12)-(14), (16)-(23), and the dependent variables are α_p, j_{lb}^*,j_{gb}^*, P_b^*, α_r, j_l^*, j_g^* and P_b^*. For the second configuration, the governing equations are Eqs. (7), (11), (15), (16)-(23), and the dependent variables are $\alpha_p, x^*, j_{lb}^*, P_b^*, \alpha_r, j_l^*, j_g^*$ and P^*. For this configuration $j_{gb}^* = 0$. Next, we have to describe when we switch from one configuration to another, or from one set of equations to another. The first (second) configuration is characterized by $x^* = 0, j_{gb}^* \neq 0 (x^* > 0, j_{gb}^* = 0)$, and we switch from the first (second) to the second (first) configuration when $j_{gb}^* \to 0, x^* > 0 (j_{gb}^* > 0, x^* \to 0)$.

The boundary conditions are the pressure P_t at the top of the riser which is given, the gas mass flow rate $\dot{m}_{g0}$ and the liquid volumetric flow rate Q_{l0} (see Fig. 1 for details). The boundary condition at the top of the riser in non-dimensional form is $P_t^* = P_t/(\rho_l R_g T_g)$.

Since we are working only with non-dimensional variables, and for simplicity, from now on we omit the superscript * from the equations.

3. LINEAR STABILITY ANALYSIS

Here we describe the linear stability analysis of the two-phase flow model presented in the previous section. The purpose of the linear stability analysis performed, besides being able to decide if the stationary sate is stable or not for a given configuration of the system parameters, is to obtain an approximation for the stationary state stability boundary in the system parameters space. An outline follows. First, we obtain the stationary states. We will see below that we have a single stationary state. We use the asymptotic theory presented in Burr and Baliño (2007) and in Burr and Baliño (2008) to obtain a closed form approximation for the stationary state. Second, we obtain the governing equations for the perturbations of the stationary states. We write the dependent variables as their stationary state value plus a perturbation and subsitute into the two-phase flow governing equations. The resulting equations are linearized with respect to the perturbation variables. As the stationary state (base flow) we use the zero-th order approximation for the stationary state given by Burr and Baliño (2007). Third, we obtain an implicit equation in terms of the system parameters for the stability boundary in the system parameter space. For the base flow used, the perturbation governing equations can be solved in closed form. The time growth rate for the perturbation of the stationary state is given as the solution of an implicit equation in terms of the pipeline-riser system parameters. If we impose zero growth rate, the graph of this implicit equation in the system parameter space is the stability boundary in the system parameter space. Since we use an approximation of the stationary state, the stability boundary obtained here is just an approximation for the actual stability boundary.

3.1 Stationary State

Only the first configuration ($x = 0$ for the pipeline) of the model for two-phase flows in pipeline-riser systems presented in the previous section has a stationary state. The equations for the stationary state are given by the Eqs. (12)-(14) and by the riser governing equations with the time partial derivatives set to zero ($\partial/\partial t = 0$). Liquid mass conservation equation for the pipeline ($x = 0$) reduces to

$$j_{lb} = 1. \tag{24}$$

Gas mass conservation equation for the pipeline ($x = 0$) reduces to

$$P_b j_g = \dot{m}_{g0}, \tag{25}$$

and the pipeline void fraction α_p is given by Eq. (14). Liquid mass conservation for the riser reduces to

$$\frac{\partial j_l}{\partial s} = 0 \to j_l = 1, \tag{26}$$

since $j_l(s = 0) = j_{lb} = 1$ (continuity condition between pipeline and riser bottom ($s = 0$) variables) according to Eq. (16). Gas mass conservation equation for the riser reduces to

$$\frac{\partial}{\partial s}(Pj_g) = 0 \rightarrow Pj_g = \dot{m}_{g0}, \tag{27}$$

since $P(s = 0)j_g(s = 0) = P_b j_{gb} = \dot{m}_{g0}$ (continuity between pipeline and riser bottom ($s = 0$) variables) according to Eq. (17). The linear momentum equation for the riser used to obtain the stationary state is Eq. (18). The constitutive law corresponding to the drift flux model used to determine the stationary state is given by Eq. (21).

The main difficulty to solve the governing equations for the stationary state results from Eq. (18) which is non-linear. For general riser geometries and no further simplifying assumptions, this set of equations has no closed form solution.

3.2 Asymptotic Approximation for The Stationary State

We use the asymptotic theory presented in Burr and Baliño (2007) and Burr and Baliño (2008) to construct an approximation for the stationary state. Their asymptotic theory is based on the hypothesis that the adimensional number $\Pi_L << 1$, which is true for all two-phase flows experiments in risers that we found in the literature. We also consider the additional hypothesis of a straight riser. The zero-th order approximation for a straight riser given by the asymptotic theory presented in Burr and Baliño (2007) does not depend on riser length parameterization variable s and is given by the set of Equations

$$P_0(s) = P_t(\text{constant}), \tag{28}$$

$$j_{g,0}(s) = \frac{\dot{m}_{g0}}{P_t}, \tag{29}$$

$$\alpha_{r,0}(s) = \frac{\frac{\dot{m}_{g0}}{P_t}}{C_d(s)\left[1 + \frac{\dot{m}_{g0}}{P_t}\right] + U_d(s)}. \tag{30}$$

We will use the zero-th order approximation given above as the approximation of the stationary state.

3.3 Perturbation Equations

Here we give the governing equations for perturnations of the stationary state. We write the dependent varibles which appear in Eqs. (12)-(14) and in Eqs. (16)-(21) as their stationary state values plus a perturbation and substitute them into the system governing equations. We linearize the resulting equations and obtain the perturbation governing equations. Next, we perform the first step. We write the dependent variables as

$$\alpha_r(s,t) = \bar{\alpha_r}(s) + \hat{\alpha_r}(s,t), \tag{31}$$

$$P(s,t) = \bar{P}(s) + \hat{P}(s,t), \tag{32}$$

$$j_l(s,t) = 1 + \hat{j_l}(s,t), \tag{33}$$

$$j_g(s,t) = \bar{j_g}(s) + \hat{j_g}(s,t), \tag{34}$$

$$P_b(t) = \bar{P_b} + \hat{P_b}(t), \tag{35}$$

$$j_{lb}(t) = \bar{j}_{lb} + \hat{j}_{lb}(t), \tag{36}$$

$$j_{gb}(t) = \bar{j}_{gb} + \hat{j}_{gb}(t). \tag{37}$$

We assume α_p constant and equal to its stationary state value. Next, we substitute Eqs. (35)-(37) into the governing equations for the pipeline for the case $x = 0$. The liquid and gas phase conservation equations assume, respectively, the form

$$\hat{j_{lb}} = 0 \tag{38}$$

and

$$\left(\frac{L}{L_r}\bar{\alpha_p} + \frac{L_b}{L_r}\right)\frac{d\hat{P}_g}{dt} + \hat{j_{gb}}\bar{P}_g + \bar{j_{gb}}\hat{P}_g = 0. \tag{39}$$

These equations are the boundary contions for the perturbation governing equation for the riser at $s = 0$. The boundary condition at the top of the riser is given by the equation

$$\hat{P}(s = 1, t) = 0. \tag{40}$$

Next, we derive the perturbations governing equation for the riser. We substitute Eqs. (31)-(34) into the riser governing Eqs. (16)-(21) and linearize the resulting equations. Regarding the linear momentum equation, we disregard the friction term under the assumption that the friction force compared to the hydrostatic force is small and keep only the hydrostatic contribution to the pressure gradient. We do so to simplify the linear momentum equation. We use the linearized drift relation for the perturbation variables to write the perturbation for the riser void fraction $\hat{\alpha}_r(s,t)$ in terms of the perturbations for the liquid and gas superficial velocities $\hat{j}_l(s,t)$ and $\hat{j}_g(s,t)$. This allow us to eliminate one of the governing equations for the perturbations of the stationary state and the perturbation variable $\hat{\alpha}_r(s,t)$. We end up with only to three partial differential equations, which are given in matrix form by

$$[M(s)]\left\{\begin{array}{c}\frac{\partial\hat{j}_l}{\partial t}\\ \frac{\partial\hat{j}_g}{\partial t}\\ \frac{\partial\hat{P}}{\partial t}\end{array}\right\}+[K(s)]\left\{\begin{array}{c}\frac{\partial\hat{j}_l}{\partial s}\\ \frac{\partial\hat{j}_g}{\partial s}\\ \frac{\partial\hat{P}}{\partial s}\end{array}\right\}+[Q(s)]\left\{\begin{array}{c}\hat{j}_l\\ \hat{j}_g\\ \hat{P}\end{array}\right\}=0, \tag{41}$$

where the matrices $[M(s)]$, $[K(s)]$ and $[Q(s)]$ are given below by the equations

$$[M(s)]=\left[\begin{array}{ccc}-\bar{\alpha_r}^2C_d & \bar{\alpha_r}(1-C_d\bar{\alpha_r}) & 0\\ -\bar{P}\bar{\alpha_r}^2C_d & \bar{P}\bar{\alpha_r}(1-C_d\bar{\alpha_r}) & \bar{j_g}\bar{\alpha_r}\\ 0 & 0 & 0\end{array}\right], \tag{42}$$

$$[K(s)]=\left[\begin{array}{ccc}\bar{j_g} & 0 & 0\\ 0 & \bar{j_g}\bar{P} & \bar{j_g}^2\\ 0 & 0 & \bar{j_g}\end{array}\right], \tag{43}$$

$$[Q(s)]=\left[\begin{array}{ccc}0 & 0 & 0\\ 0 & \bar{j_g}\frac{\partial\bar{P}}{\partial s} & \frac{\partial\bar{j_g}}{\partial s}\\ \pi_l(1-\bar{P})\bar{\alpha_r}^2C_d & -\pi_l(1-\bar{P})(1-\bar{\alpha_r}C_d)\bar{\alpha_r} & \pi_l\bar{j_g}\bar{\alpha_r}\end{array}\right]. \tag{44}$$

We assume the time dependence of the variables $\hat{j}_l(s,t), \hat{j}_g(s,t), \hat{P}(s,t), \hat{j}_{lb}(t), \hat{j}_{gb}(t)$ and $\hat{P}_b(t)$ to be of the form $\exp(\omega t)$, and as a result, the system of partial differential equations given by Eq. (41) becomes a system of differential equations with respect to the independent variable s. Since we approximate the stationary state by Eqs. (28)-(30), we substitute the variables $\bar{P}(s), \bar{j_g}(s)$ and $\bar{\alpha_r}(s)$, respectivelly, by $P_0(s), j_{g,0}(s)$ and $\alpha_{r,0}(s)$, which results in the following system of differential equations

$$\left\{\begin{array}{c}\frac{\partial\hat{j}_l}{\partial s}\\ \frac{\partial\hat{j}_g}{\partial s}\\ \frac{\partial\hat{P}}{\partial s}\end{array}\right\}=\left[\begin{array}{ccc}\omega\frac{P_t}{\dot{m}_{g0}}C_{0,0}^2C_d & \omega\frac{P_t}{\dot{m}_{g0}}C_{0,0}(C_dC_{0,0}-1) & 0\\ -\omega\frac{P_t}{\dot{m}_{g0}}C_{0,0}^2C_d & \omega\frac{C_{0,0}P_t}{\dot{m}_{g0}}(1-C_dC_{0,0})-\pi_l\frac{(1-P_t)}{P_t}C_dC_{0,0}^2 & \omega\frac{C_{0,0}}{P_t}-\pi_l\frac{\dot{m}_{g0}}{P_t}\\ \pi_l\frac{P_t}{\dot{m}_{g0}}(1-P_t)C_{0,0}^2C_d & \pi_l\frac{(1-P_t)}{\dot{m}_{g0}}(1-C_{0,0}C_d)C_{0,0} & \pi_lC_{0,0}\end{array}\right]\left\{\begin{array}{c}\hat{j}_l\\ \hat{j}_g\\ \hat{P}\end{array}\right\} \tag{45}$$

where $C_{0,0}=\bar{\alpha}_{r,0}(s)$. Most of the boundary conditions are unaffected by the assumed time dependence above, except the boundary condition give by Eq. (39), which now has the form

$$\omega\left(\frac{L}{L_r}\bar{\alpha_p}+\frac{L_b}{L_r}\right)\hat{P}_g+\hat{j_{gb}}\bar{P}_g+\bar{j_{gb}}\hat{P}_g=0. \tag{46}$$

The matrix of the system of differential Eqs. (45) has eigenvalues

$$\lambda_1=0, \tag{47}$$

$$\lambda_2=\omega\frac{P_t}{\dot{m}_{g0}}C_{0,0}, \tag{48}$$

$$\lambda_3=\frac{C_{0,0}}{P_t}\pi_l[C_{0,0}C_d(P_t-1)+1], \tag{49}$$

and the left eigenvectors form the rows of the matrix

$$[ML]=\left[\begin{array}{ccc}\frac{1}{P_t} & 1 & -\frac{P_t\omega-\pi_l\dot{m}_{g0}}{P_t^2\pi_l}\\ \frac{C_{0,0}C_d}{C_{0,0}C_d-1} & 1 & \frac{\dot{m}_{g0}}{P_t^2}\\ \frac{C_{0,0}C_d}{C_{0,0}C_d-1} & 1 & \frac{\omega P_t-\dot{m}_{g0}\pi_l}{P_t\pi_l[C_{0,0}C_d(P_t-1)+1-P_t]}\end{array}\right]. \tag{50}$$

With the inverse of the matrix $[ML]$ and the eigenvalues λ_j we can built the solution of the system of Eqs. (45). The general solution of the systems of equations is given by

$$\hat{j}_l(s) = A_1([ML]^{-1})_{1,1} + A_2([ML]^{-1})_{1,2}\exp(\lambda_2 s) + A_3([ML]^{-1})_{1,3}\exp(\lambda_3 s) \quad (51)$$

$$\hat{j}_g(s) = A_1([ML]^{-1})_{2,1} + A_2([ML]^{-1})_{2,2}\exp(\lambda_2 s) + A_3([ML]^{-1})_{2,3}\exp(\lambda_3 s) \quad (52)$$

$$\hat{P}(s) = A_2([ML]^{-1})_{3,2}\exp(\lambda_2 s) + A_3([ML]^{-1})_{3,3}\exp(\lambda_3 s) \quad (53)$$

The general solution are functions of the pipeline-riser system parameters and of the time growth rate ω, which is unknown so far. To obtain ω, we need to substitute the general solution into the boundary conditions, which are given by the Eqs (38), (46) and (40). Notice that at the base of the riser $\hat{j}_{lb} = \hat{j}_l(s=0)$, $\hat{j}_{gb} = \hat{j}_g(s=0)$ and $\hat{P}_b = \hat{P}(s=0)$. As a result, we obtain the system of algebraic equations

$$\underbrace{\begin{bmatrix} ([ML]^{-1})_{1,1} & ([ML]^{-1})_{1,2} & ([ML]^{-1})_{1,3} \\ P_t([ML]^{-1})_{2,1} & \left[\omega\left(\frac{L}{L_r}\bar{\alpha_p} + \frac{L}{L_b}\right) + \frac{\dot{m}_{g0}}{P_t}\right]([ML]^{-1})_{3,2} + P_t([ML]^{-1})_{2,2} & \left[\omega\left(\frac{L}{L_r}\bar{\alpha_p} + \frac{L}{L_b}\right) + \frac{\dot{m}_{g0}}{P_t}\right]([ML]^{-1})_{3,3} + P_t([ML]^{-1})_{2,3} \\ 0 & ([ML]^{-1})_{3,2}\exp(\lambda_2) & ([ML]^{-1})_{3,3}\exp(\lambda_3) \end{bmatrix}}_{[MB]} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = 0 \quad (54)$$

To avoid the trivial solution for the constants A_1, A_2 and A_3, it is necessary that the determinant of the matrix $[MB]$ to be zero. This implies in that ω is a solution of the implicit equation

$$-\exp(\lambda_3 - \lambda_2) = \frac{a + \omega b}{1 + \omega c}, \quad (55)$$

where the coeficients a, b and c are given in terms of the pipeline-riser system parameters according to the equations

$$a = -\frac{(1 - \dot{m}_{g0}^2/P_t^2)}{1 + \frac{\dot{m}_{g0}^2}{P_t(1-C_dC_{0,0})(1-P_t)}} \quad (56)$$

$$b = -\frac{1}{L_r}\left(\frac{\dot{m}_{g0}}{P_t}\right)\frac{(P\bar{\alpha}_p + L_b)}{1 + \frac{\dot{m}_{g0}^2}{P_t(1-C_dC_{0,0})(1-P_t)}} \quad (57)$$

$$c = -b - \left(\frac{\dot{m}_{g0}}{\Pi_L}\right)\frac{1}{(1 - C_dC_{0,0})(1 - P_t) + \dot{m}_{g0}^2/P_t} \quad (58)$$

Equation (55) for ω in terms of the system parameters cannot be solved in closed form. It need to be solved numerically. The time growth rate ω may assume complex values. We write $\omega = \omega_r + i\omega_i$ and substitute into the equation above and obtain a system of equations for ω_r and ω_i, given by

$$\frac{a + \omega_r b(a+1) + (\omega_r^2 + \omega_i^2)bc}{(1 + \omega_r c)^2 + \omega_i^2 c^2} = -\exp(\lambda_3 - \tilde{\lambda_2}\omega_r)\cos(\tilde{\lambda}_2\omega_i) \quad (59)$$

$$\frac{\omega_i(b - ac)}{(1 + \omega_r c)^2 + \omega_i^2 c^2} = -\exp(\lambda_3 - \tilde{\lambda_2}\omega_r)\sin(\tilde{\lambda}_2\omega_i), \quad (60)$$

where we define $\tilde{\lambda}_2 = \lambda_2/\omega$.

3.4 Stability Boundary Implicit Equation

At the stability boundary, the solutions of the system of Eqs. (59)-(60) should present one of the following possibilities:

1. All solutions ω should have negative real part ($\omega_r < 0$), except for one that should have $\omega_r = 0$;
2. All solutions ω should have negative real part ($\omega_r < 0$), except for a pair of complex conjugate solutions that should have $\omega_r = 0$.

Since the stability lost of the stationary state reported in the literature is followed by a cyclic phenomenon, which is tipical of a Hopf bifurcation in the language of dynamic systems, we expect the second possibility above to be the case at the stability boundary. Therefore, to obtain the stability boundary in the system parameter space, we impose the condition that $\omega_r = 0$ in the Eqs. (59) and (60). If we impose this condition to the Eqs. (59) and (60) and use the trigonometric relation $\cos^2(\tilde{\lambda}_2\omega_i) + \sin^2(\tilde{\lambda}_2\omega_i) = 1$, we obtain that ω_i is a solution of the polynomial equation

$$d_4\omega_i^4 + d_2\omega_i^2 + d_0 = 0 \tag{61}$$

where the coefficients d_4, d_2 and d_0 are given by the equations

$$d_4 =[(bc)^2 - \exp(2\lambda_3)c^4] \tag{62}$$
$$d_2 =b^2 + (ac)^2 - 2c^2\exp(2\lambda_3) \tag{63}$$
$$d_0 =a^2 - \exp(2\lambda_3) \tag{64}$$

If we divide Eq. (60) by Eq. (59) and impose that $\omega_r = 0$, we obtain another equation for ω_i to satisfy. This equation is given by

$$\tan(\tilde{\lambda}_2\omega_i) = \frac{\omega_i(ac - b)}{a + bc\omega_i^2}. \tag{65}$$

Once we eliminate ω_i from the system of Eqs. (61) and (65), we obtain an implicit equation for the stability boundary in terms of the system parameters. This ω_i elimination process could be carried out by solving the polynomial Eq. (61) for ω_i, which gives four different branches for ω_i in terms of the coefficients d_4, d_2 and d_0. We have to chose among branches, and we do not know before hand which one is appropriate. To make things worst, as the system parameters are varied, the right branch choice for ω_i may change, and we end up with a cumbersome way to obtain an implicit equation for the stability boundary in terms of the system parameters. Another way to proceed is to approximate the $\tan(\tilde{\lambda}_2\omega_i)$ in the Eq. (65) by a polynomial. This can be done by a Taylor series expansion of the left side of the identity in Eq. (65) with respect to $\omega_i = 0$ (for $ac - b > 0$) or to $\omega_i = \pi/\tilde{\lambda}_2$ (for $ac - b < 0$). This approach leads to a polynomial approximation of degree $2N + 1$ for Eq. (65), given by

$$\sum_{n=0}^{2N+1} e_n\omega_i^n = 0, \tag{66}$$

where the coefficients e_n for the expansion around $\omega_i = 0$ are $e_n = 0, n = 0, 2, 4, \ldots, 2N$, $e_1 = \tilde{\lambda}_2 p_1(0)a - (ac - b)$ and $e_n = ap_{n+1}(0)\tilde{\lambda}_2^n + bcp_n(0)\tilde{\lambda}_2^{n-2}$ for $n = 3, 5, \ldots, 2N + 1$, where

$$p_n(x) = \frac{1}{n!}\left(\frac{d^n\tan(z)}{dz^n}\right)_{z=x}, \tag{67}$$

and the expansion for the coefficients around $\omega_i = \pi/\tilde{\lambda}_2$ are

$$e_0 =- a\sum_{n=0}^{N} p_n(\pi)\pi^{2n+1} \tag{68}$$
$$e_1 =a\tilde{\lambda}_2\left\{\sum_{n=0}^{N} p_n(\pi)\begin{pmatrix} 2n+1 \\ 1 \end{pmatrix}(-\pi)^{2n}\right\} + (ac - b) \tag{69}$$
$$e_2 =- bc\left\{\sum_{n=0}^{N} p_n(\pi)\pi^{2n+1}\right\} + a\tilde{\lambda}_2\left\{\sum_{n=1}^{N} p_n(\pi)\begin{pmatrix} 2n+1 \\ 2 \end{pmatrix}(-\pi)^{2n-1}\right\} \tag{70}$$
$$e_n =- bc\tilde{\lambda}_2^{n-2}\left\{\sum_{l=[n/2]-1}^{N} p_n(\pi))\begin{pmatrix} 2l+1 \\ n-2 \end{pmatrix}\pi^{2l+1-n+2}\right\} + a\tilde{\lambda}_2^n\left\{\sum_{l=[n/2]}^{N} p_n(\pi)\begin{pmatrix} 2l+1 \\ n \end{pmatrix}(-\pi)^{2l+1-n}\right\}, \tag{71}$$

for n from 3 to $2N + 1$. $[n/2]$ stands for the larger integer less than $n/2$. To eliminate ω_i from the polynomial Eqs. (61) and (67), their resultant $R(d_j, e_j)$ (see Appendix 3 of Jones and Singerman (1987)) should be zero (both polynomial equations should have a common root). The resultant $R(d_j, e_j)$ is given by the determinant of the $2N+1+4 \times 2N+1+4$ matrix $[M(d_j, e_j)]$, given by

$$[M(d_j, e_j)] = \underbrace{\begin{bmatrix} d_4 & & & & e_{2N+1} & & & \\ d_3 & d_4 & & & e_{2N} & e_{2N+1} & & \\ d_2 & d_3 & \dots & & e_{2N-1} & e_{2N} & e_{2N+1} & \\ d_1 & d_2 & \dots & \dots & e_{2N-2} & e_{2N-1} & e_{2N} & e_{2N+1} \\ d_0 & d_2 & \dots & \dots & e_{2N-3} & e_{2N-2} & e_{2N-1} & e_{2N} \\ & d_0 & \dots & \dots & e_{2N-4} & e_{2N-3} & e_{2N-2} & e_{2N-1} \\ & & \dots & \dots & \dots & \dots & \dots & \dots \\ & & d_0 & d_1 & & & e_0 & e_1 \\ & & & d_0 & & & & e_0 \end{bmatrix}}_{2N+1\text{columns} \quad\quad 4\text{columns}} \tag{72}$$

Notice that the coefficients $d_3 = d_1 = 0$. All coefficients in the matrix above are functions only of the system parameters. Therefore $R(d_j, e_j) = 0$ gives an implicit equation for the stability boundary in the pipeline-riser system parameter space.

4. Result

We illustrate the implicit equation $R(d_j, e_j) = 0$ for the stability boundary in the plane $Q_{l0} \times \dot{m}_{g0}$. We vary this two boundary conditions and keep the other boundary condition and parameters constant. We use pipeline-riser system parameters used in the experiment described in Schmidt et al. (1980). The parameters and their values are

$$\begin{array}{lll} P_t = 10^5\ Pa, & D = 0.0508\ m, & T = 293.15\ K, \\ L = 30,48\ m, & \theta = 90^o, & R = 287.336\ m^2 sec^{-2} K^{-1}, \\ L_r = 15,24\ m, & \beta = 5^o, & \rho_l = 824.952\ kg.m^{-3}. \end{array}$$

An illustration of the stability boundary in the $Q_{l0} \times \dot{m}_{g0}$ plane for values $10^{-5} < Q_{l0} < 10^{-3}$ and $10^{-5} < \dot{m}_{g0} < 10^{-3}$ is given in the Fig. 2 below. We evaluated the surface generated by $R(d_j, e_j)$ over the $(Q_{l0}, \dot{m}_{g0})$ plane for the range specified above and ploted the contour level curve with zero value ($R(d_j, e_j) = 0$) in Fig. 2. We used $N = 7$ in the Eq. (66) to generate Fig. 2.

5. Discussion and Conclusions

The stability boundary represented by the implicit equation $R(c_j, e_j) = 0$ should give good results for short risers, where the pressure variation between the base and top of the riser is small (almost constant). In this case, the approximation of the stationary state given by the zero-th order solution of the asymptotic theory presented in Burr and Baliño (2007) should be a reasonable approximation. For longer risers, the approximation of the stationary state given by Eqs. (28)-(30) is no longer valid, and higher order terms of the asymptotic solution should be included to obtain a good approximation for the stationary state. This adds extra complexities, since the higher order approximations are polynomial functions of the riser parameterization variable s. If we approximate the stationary state in such a way, the perturbation approach used to derive the asymptotic approximations of the stationary state given by Burr and Baliño (2007) and Burr and Baliño (2008) should be extended to the solution of the perturbation equations derived in the linear stability analysis process to make further progress in deriving an implicit equation to approximate the stability boundary of the stationary state in the system parameter space. The results obtained here are valid only for straight vertical risers.

To evaluate the usefullness of the stability boundary given by $R(d_j, e_j) = 0$, we need to compare, for example, the contour curve displayed in Fig. 2 with stability boundary given by other stability criteria found in the literature and with experimental results. This point is the next step in the work reported here.

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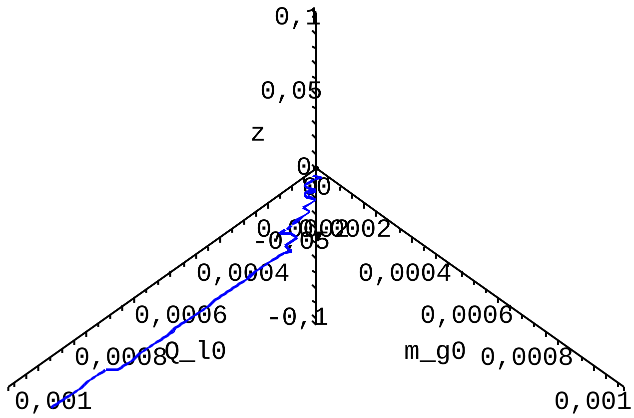

Figure 2. Resultant contour level curve of zero value in the plane $(Q_{l0}, \dot{m}_{g0})$. Left (right) horizontal axis stands for Q_{l0} $(\dot{m}_{g0})$.

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