AN ERROR ESTIMATOR AND A MESH REFINEMENT STRATEGY FOR THE MODIFIED ELEMENT-FREE GALERKIN METHOD

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Abstract. *In this work an h-adaptive modified element-free Galerkin (MEFG) method is investigated. The proposed error estimator is based on a recovery by equilibrium of nodal patches where a recovered stress field is obtained by a moving least square approximation. The procedure generates a smooth recovered stress field that is not only more accurate then the approximate solution but also free of spurious oscillations, normally seen in EFG methods at regions with high gradient stresses or discontinuities.*

The MEFG method combines conventional EFG with extended partition of unity finite element (EPUFE) methods in order to create global shape functions that allow a direct imposition of the essential boundary conditions.

The re-meshing of the integration mesh is based on the homogeneous error distribution criterion and upon a given prescribed admissible error. Some examples are presented, considering a plane stress assumption, which shows the performance of the proposed methodology.

Keywords: *Mesh-free methods, EFG, Error Estimators, REP, h-adaptive.*

1. Introduction

Important contributions on error estimation for mesh-free/meshless methods have already been presented in the last years. In Duarte and Oden (1996) a residual-based error estimator is presented and applied in the context of the h-p clouds method. Other different contributions were proposed by Chung and Belytschko (1998), Gavete *et al*. (2001), Gavete *et al*. (2002), Gavete *et al*. (2003) and Rossi and Alves (2004), which considered a recovery based error estimator applied to the EFG method. Another contribution, presented by Liu and Tu (2002), considers a refinement of the background integration mesh based in the error that takes place when different integration strategies are performed. In Rossi and Alves (2004) a recovery based error estimator is proposed together with an h-adaptive strategy, where the recovered stress field is determined by considering the minimization of a potential that takes into account a least square error minimization of the stress difference, a distributed residual error and a prescribed traction residual error. The proposed error estimation was applied within the frame work of the Modified Element-Free Galerkin (MEFG) method, see Alves and Rossi (2003), that allows a direct imposition of the essential boundary conditions, as done in FEM. The MEFG method may be seen as a conventional element-free Galerkin (EFG) method, see Belytschko *et al*. (1994), that considers two different weight functions, which automatically selects at each particle the proper type of weight function and determines the adequate size of its support. As a result, the global shape functions, defining the approximation space, are derived for a given intrinsic base by the use of the Moving Least Square Approximation (MLSA), see Lancaster and Salkauskas (1981).

In this work, an h-adaptive modified element-free Galerkin (MEFG) method is investigated. The proposed error estimator is based on a recovery by equilibrium of nodal patches where a recovered stress field is obtained by a moving least square approximation. The procedure generates a smooth recovered stress field that is not only more accurate then the approximate solution but also free of spurious oscillations, normally seen in EFG methods at regions with high gradient stresses or discontinuities.

Results are presented, considering a plane stress assumption in the small stress-strain elasticity context, in order to investigate the efficiency of the proposed error estimator and of the h-adaptive procedure.

2. Modified element-free Galerkin method

• **Moving least square approximation:** By the use of a MLSA it is possible to construct an approximation function $u^h(\boldsymbol{X})$ that fits a discrete set of data $\{u_I, I=1\ldots n\}$ such that:

$$u^h(\boldsymbol{X}) = \sum_{I=1}^{n} \Phi_I(\boldsymbol{X}) u_I \ , \tag{1}$$

$$\Phi_I(\boldsymbol{X}) = \boldsymbol{p}(\boldsymbol{X}) \cdot \mathbf{A}(\boldsymbol{X})^{-1} \boldsymbol{b}_I(\boldsymbol{X}) \tag{2}$$

$$\mathbf{A}(\boldsymbol{X})=\sum_{I=1}^{n} w(\boldsymbol{X}-\boldsymbol{X}_I)\left[\boldsymbol{p}(\boldsymbol{X}_I)\otimes \boldsymbol{p}(\boldsymbol{X}_I)\right] \text{ and } \mathbf{b}_I(\boldsymbol{X})=w(\boldsymbol{X}-\boldsymbol{X}_I)\mathbf{p}(\boldsymbol{X}_I), \tag{3}$$

where $\{p_j(\boldsymbol{X}), j=1\ldots m\}$ represents the set of intrinsic base functions and $w(\boldsymbol{X}-\boldsymbol{X}_I)$ is a weight function centered at $\boldsymbol{X}_I$. Here, $\Phi_I(\boldsymbol{X})$ is the global shape function, defined at particle $\boldsymbol{X}_I$, and $\mathbf{A}(\boldsymbol{X})$ is the moment matrix.

• **Element-free Galerkin:** The conventional EFG method is characterized by the construction of a set of global shape functions, $\Phi_I(\boldsymbol{X})$ defined at particle $\boldsymbol{X}_I$, which defines the approximation space, used by the Galerkin method to solve a boundary value problem. The particle distribution that defines how the covering of the domain is performed, by the global shape functions $\Phi_I(\boldsymbol{X})$, is not arbitrary since it must satisfy the stability condition

$$card\left\{\boldsymbol{X}_J \left|\Phi_J(\boldsymbol{X})\neq 0\right.\right\}\geq \dim\left[\mathbf{A}(\boldsymbol{X})\right] \tag{4}$$

i.e., the number of particles $\boldsymbol{X}_J$ whose associated shape function $\Phi_J(\boldsymbol{X})$ have a nonzero value at $\boldsymbol{X}$, must be larger that the size of $\mathbf{A}(\boldsymbol{X})$, which is given by the number of intrinsic base functions in $\boldsymbol{p}(\boldsymbol{X})$. Moreover, for $\boldsymbol{X}\in R^n$, there must be $n+1$ particles, whose position vectors form a nonzero *n-th* rank simplex element. In order to obtain a particle distribution that comply with Eq.(4), we perform a partition of the domain, Ω, into a triangular integration mesh, where we consider each triangular partition/element to be an integration cell and each vertex node to be the position of a particle. One of the most common weight function is the quartic-spline function, w^{EFG}, given as:

$$w^{EFG}(r)=\begin{cases}1-6r^2+8r^3-3r^4, & \text{for } r\leq 1.0\\ 0, & \text{for } r>1.0\end{cases} \tag{5}$$

where $r=r_I/\overline{r}_I$ with $r_I=\left\|\boldsymbol{X}-\boldsymbol{X}_I\right\|$. The radius $\overline{r}_I$, defining the support of $w^{EFG}(\boldsymbol{X}-\boldsymbol{X}_I)$, is determined by

$$\overline{r}_I=\beta\cdot r_{I\max}, \qquad \beta>1,\ \beta\in R \text{ with } r_{I\max}=\max_i\left\|\boldsymbol{X}_i-\boldsymbol{X}_I\right\|, \quad i\in J_I, \tag{6}$$

where J_I represents the set of adjacent nodes associated with $\boldsymbol{x}_I$.

Now, in the conventional EFG method, the global shape functions $\{\Phi_I(\boldsymbol{X}), I=1\ldots n\}$, defining the approximation space, do not satisfy, in general, the kronecker delta property, i.e., $\Phi_I(\boldsymbol{X}_J)\neq\delta_{IJ}$. As a consequence, it is not possible to enforce the essential boundary conditions, by directly prescribing nodal values, as done in the FEM. However, special weight functions may be constructed in order to satisfy the kronecker delta property. Among the possible weight functions is the extended partition of unity finite element (EPUFE) weight function.

• **Extended Partition of unity finite element weight functions:** The global shape functions $\{\Phi_I(\boldsymbol{X}), I=1\ldots n\}$, employed in EPUFE, are obtained by a MLSA. In the case where a linear triangular finite element base function is used as a weight function for the MLSA, one derives:

$$w^{EPF}\left(\boldsymbol{X}^*-\boldsymbol{X}_I\right)=\begin{cases}\frac{1}{2A}\left[\left(x_i^* y_{i+1}^*-x_{i+1}^* y_i^*\right)+\left(y_i^*-y_{i+1}^*\right)x+\left(x_{i+1}^*-x_i^*\right)y\right], & \boldsymbol{X}\in\operatorname{supp}\left[\Phi_I(\boldsymbol{X})\right]\\ 0, & \text{otherwise}\end{cases} \tag{7}$$

Here, $\boldsymbol{X}_i^*$ and $\boldsymbol{X}_{i+1}^*$ are the elements of the adjacent extended node list set of $\boldsymbol{X}_I$, obtained in a counter clockwise sense of the triangular integration cell whose area is A. The usage of an intrinsic base $\boldsymbol{p}^T(\boldsymbol{X})=[1,x,y]$ together with a EPUFE weight function satisfy the requirement in Eq.(4), therefore, this extension ensures the regularity of $\mathbf{A}(\boldsymbol{X})$.

The extended points are determined as: $\boldsymbol{X}_i^*=\boldsymbol{X}_i+\varepsilon\left(\boldsymbol{X}_i-\boldsymbol{X}_I\right)$. Notice that, letting $\varepsilon\to 0$, we derive a global shape function that satisfy, in a limiting sense, at a given particle $\boldsymbol{X}_J$, the kronecker delta property, i.e., $\lim_{\varepsilon\to 0}\Phi_I\left(\boldsymbol{X}_J\right)=\delta_{IJ}$.

• **Modified element-free Galerkin method:** The objective of the MEFG method is to combine, in a suitable way, both weight functions, in order to explore the smoothness of w^{EFG} and the kronecker delta property of w^{EPF}. The strategy can be shown by considering a body with domain Ω and boundary ∂Ω, where $\partial\Omega=\Gamma_u\cup\Gamma_t$ and $\Gamma_u\cap\Gamma_t=\varnothing$. Here, Γ_u and Γ_t are respectively the part of ∂Ω with prescribed essential and natural boundary conditions. Notice that the EPUFE weight functions are specified at particles that belong to a neighborhood of Γ_u and the EFG weight functions are specified at the remaining particles of the mesh. This procedure enables the determination of an approximate solution that satisfies accurately the essential boundary condition and is smooth in the entire domain, except for a neighborhood of Γ_u. The MEFG method can be seen as a conventional EFG method, having a set of different weight functions, which is able to automatically select, at each particle, the proper type of weight function and to compute the adequate size of its support, as shown in Figure 1.

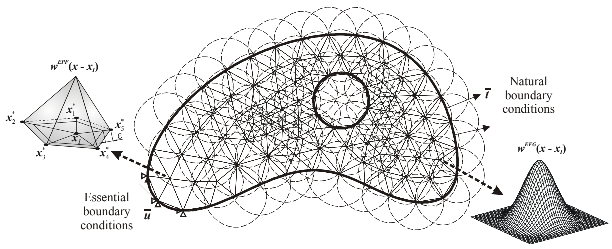

Fig. 1 An example of body coverage by the MEFG

3. Formulation of the problem and error analysis

• **Classical elastostatics:** Let $\Omega \subset R^2$ be a bounded domain with a Lipschitz boundary $\partial\Omega$, subjected to: a prescribed body force $\overline{\boldsymbol{b}}$ defined on Ω, a prescribed surface traction $\overline{\boldsymbol{t}}$ defined on Γ_t and a prescribed displacement $\boldsymbol{u} = \overline{\boldsymbol{u}}$ defined on Γ_u. The classical boundary value problem associated with elastostatics is stated as: Find $\boldsymbol{u}$ so that

$$div\,\boldsymbol{\sigma} + \overline{\boldsymbol{b}} = \mathbf{0}, \quad \forall \boldsymbol{x} \in \Omega, \quad \boldsymbol{\sigma n} = \overline{\boldsymbol{t}}, \quad \forall \boldsymbol{x} \in \Gamma_t \quad and \quad \boldsymbol{u} = \overline{\boldsymbol{u}}, \ \forall \boldsymbol{x} \in \Gamma_u \tag{8}$$

Here, $\boldsymbol{n}$ is the outer normal to the surface at Γ_t and $\boldsymbol{\sigma}$ is the Cauchy stress tensor, with $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$.

Now, let $H=\{\boldsymbol{u} \mid u_i \in H^1(\Omega), \boldsymbol{u}=\overline{\boldsymbol{u}}$ at $\Gamma_u\}$ denote the set of admissible displacements and $H_0=\{\boldsymbol{u} \mid u_i \in H^1(\Omega), \boldsymbol{u}=\mathbf{0}$ at $\Gamma_u\}$ the set of admissible variations. The weak formulation of Eq.(8) may be stated as: Find $\boldsymbol{u} \in H$ so that

$$a(\boldsymbol{u},\boldsymbol{v}) = L(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in H_0 \ \text{ where } \ a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, d\Omega \ \text{ and } \ L(\boldsymbol{v}) = \int_\Omega \overline{\boldsymbol{b}} \cdot \boldsymbol{v}\, d\Omega + \int_{\Gamma_N} \overline{\boldsymbol{t}} \cdot \boldsymbol{v}\, d\Gamma\,. \tag{9}$$

• **Error estimator:** Different error estimators have been proposed in the literature where most of the work has been applied in the framework of the FE methods. A relevant review on error estimators applied to FE methods is presented in Ainsworth and Oden (1997) and Zienkiewicz *et al.* (1999), where we see that most of the error estimators belong to one of the following two categories: The residual based error estimators and The recovery based error estimators.

However, much of the theory on error estimators employed in FE methods may be extended to all Galerkin methods. As a result, one may adapt these proposed methodologies to the particular case of EFG methods.

Recently, some important contributions on error estimation for meshless and/or EFG methods have been proposed. Chung and Belytschko (1998) presented a recovery based error estimator that uses the difference between the values of the projected/recovered stress $\boldsymbol{\sigma}^*$ and the approximate MEFG solution $\boldsymbol{\sigma}^h$. The recovered stress was obtained by considering a MLSA of the nodal stress values $\{\boldsymbol{\sigma}^h(\boldsymbol{x}_I), I=1,\ldots,n\}$, but employing a different domain of influence. In their work, the effectivity index was optimized by varying the domain of influence in the MLSA of the recovered stress. Belytschko *et al.* (1998) presented a residual-based error estimator that considered a multi-resolution analysis. Another important contribution, presented in Duarte and Oden (1996), is a residue based error estimator that was applied in their proposed *h-p clouds* method. In their work, due to the high regularity of the global shape functions, no jumps in the approximate stress field were present, what simplified considerably the implementation of the method. Gavete *et al.* (2002) presented a recovery based error estimator where the recovered solution was obtained by a MLSA using a Taylor series expansion around each particle, together with the four quadrant criteria to choose the neighborhood points. Gavete *et al.* (2003) presents an error approximation in EFG method using different moving least squares approximations.

The objective of this work is to propose a recovery-based error estimation that is able to: produce a recovered stress $\boldsymbol{\sigma}^*$, which is a more refined estimate of the exact stress solution $\boldsymbol{\sigma}$ than $\boldsymbol{\sigma}^h$; and be easily extended to arbitrary nonlinear inelastic problems. Notice that, unlike FE methods, there is no assurance of the existence of super convergent points, where the approximate stresses $\boldsymbol{\sigma}^h$ have a higher order of accuracy. Therefore, in order to assure that the recovered stress $\boldsymbol{\sigma}^*$ is more accurate than $\boldsymbol{\sigma}^h$, some additional criterion must be enforced.

The strategy of the proposed method is to combine the equilibrium by nodal patch criterion, proposed by Zienkiewicz *et al.* (1999), responsible for determining the patch recovered nodal stress values $\boldsymbol{\sigma}^{REP}(\boldsymbol{x}_I)$, for each nodal patch $\boldsymbol{x}_I$, with the optimal domain of influence criterion, proposed by Chung and Belytschko (1998), employed in the determination of the recovered stress, obtained by a MLSA of the nodal stress values $\{\boldsymbol{\sigma}^{REP}(\boldsymbol{x}_I),\ I=1,\ldots,n\}$, where the optimal domain of influence is obtained with the optimum value of an effectivity index. In addition to the above criteria, in order to filter out possible spurious oscillations, that may appear in EFG methods, an adequate low order polynomial interpolation procedure was also implemented in the equilibrium by nodal patch procedure.

In this work, the approximation of the error is computed in terms of a convenient energy norm, which depends on the difference between the recovered solution $\boldsymbol{\sigma}^*$ and the MEFG solution $\boldsymbol{\sigma}^h$. Moreover, the selection of a recovery-based method was based on the fact that these methods have shown to be extremely accurate, robust and that in most cases they appear to give a superior accuracy of estimation than that obtained by the residual based methods see Zienkiewicz *et al.* (1999). Here, we consider the stress error measure, in the energy norm, to be given by

$$\|\mathbf{e}_\sigma\|_E = \left[\int_\Omega \left(\boldsymbol{\sigma}-\boldsymbol{\sigma}^h\right)\cdot \mathbf{D}^{-1}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}^h\right)d\Omega\right]^{\frac{1}{2}} \tag{10}$$

where $\boldsymbol{\sigma}$ is the exact stress vector, $\boldsymbol{\sigma}^h$ the approximate solution vector and $\mathbf{D}$ the second order tensor of the elastic constitutive equation. Moreover, Once $\boldsymbol{\sigma}^*$ is determined, from a suitable post-processing of the MEFG stress field $\boldsymbol{\sigma}^h$, the approximate error estimator is determined as:

$$\|\mathbf{e}_\sigma\|_E \simeq \|\mathbf{e}^*_\sigma\|_E = \left[\int_\Omega \left(\boldsymbol{\sigma}^*-\boldsymbol{\sigma}^h\right)\cdot \mathbf{D}^{-1}\left(\boldsymbol{\sigma}^*-\boldsymbol{\sigma}^h\right)d\Omega\right]^{\frac{1}{2}}. \tag{11}$$

In order to evaluate the quality of the approximate error estimator and to compute the optimal domain of influence, employed in the determination of $\boldsymbol{\sigma}^*$, we introduce the effectivity index, which is defined as

$$\theta = \frac{\|\mathbf{e}^*_\sigma\|_E}{\|\mathbf{e}_\sigma\|_E}, \text{ and the relative error, given as } \eta = 100\frac{\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^h\|_E}{\|\boldsymbol{\sigma}\|_E}. \tag{12}$$

Moreover, replacing $\boldsymbol{\sigma}$ by $\boldsymbol{\sigma}^*$ in Eq.(12), one may define $\boldsymbol{\eta}^*$.

• **Recovery procedures:** In the finite element method the most effective, among the recovery methods, are: the super convergent patch recovery (SPR) and the recovery by equilibrium of patches (REP), see Zienkiewicz *et al.* (1999).

The SPR procedure is based on the assumption of the existence of points in the domain for which super-convergence occurs. In the FEM, the existence of such points can be shown in most of the cases. However, in mesh-free methods there is no assurance of the existence of such points. One way to circumvent the absence of super-convergent points in mesh-free methods is to apply the REP procedure. In the REP approach we derive, in each patch, a smooth patch recovered stress field $\boldsymbol{\sigma}^{REP}$ that satisfies in the least square sense the same patch equilibrium condition as the numerical MEFG solution, i.e., that

$$\int_{\Omega_p} \boldsymbol{\sigma}^h \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, d\Omega \simeq \int_{\Omega_p} \boldsymbol{\sigma}^{REP} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) d\Omega,\ \forall \boldsymbol{v} \in H_0(\Omega). \tag{13}$$

Now, as proposed by Zienkiewicz *et al.* (1999), in order to avoid singularity problems in the determination of the patch recovered solution $\boldsymbol{\sigma}^{REP}$ in Ω_p, the patch equilibrium condition is modified, by enforcing the equilibrium condition to each component of $\boldsymbol{\sigma}^{REP}$, which in the case of a plane stress condition is given by $\boldsymbol{\sigma}^{REP}=[\,\sigma^{REP}_{xx}, \sigma^{REP}_{yy}, \sigma^{REP}_{xy}\,]^T$. In this case the modified patch equilibrium condition is:

$$\int_{\Omega_p} \boldsymbol{\sigma}^h_i \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) d\Omega \simeq \int_{\Omega_p} \boldsymbol{\sigma}^{REP}_i \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) d\Omega,\ \forall \boldsymbol{v} \in H_0(\Omega) \tag{14}$$

where $\boldsymbol{\varepsilon}$ is the vector containing the components of the infinitesimal strain tensor, $\boldsymbol{\sigma}^h_i = \left(\boldsymbol{\sigma}^h \cdot \boldsymbol{e}_i\right)\boldsymbol{e}_i$, $\boldsymbol{\sigma}^{REP}_i = \left(\boldsymbol{\sigma}^{REP} \cdot \boldsymbol{e}_i\right)\boldsymbol{e}_i$, and Ω_p is the patch domain.

• **Definition of patches:** In general the patch can be designed in two different ways:

1. Patch of elements surrounding a vertex node, i.e., a node patch;

2. Patch defined as a union of elements surrounding an element, i.e., an element patch.

Here, it is considered only node patches. Moreover, since the partition of the domain is done with the use of triangular integration cells, a typical node/particle patch is of the type illustrated in Figure 2.

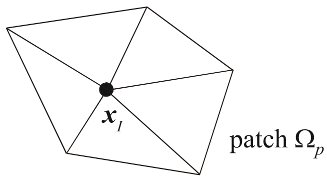

Figure 2: Node/particle patch defined at $\boldsymbol{x}_I$.

• **Node patch recovered stress approximation:** Before describing the procedure employed for the determination of the recovered stress field $\boldsymbol{\sigma}^*$, it is important to notice that, in regions covered by EPUFE global shape functions, the resulting MEFG stress solution $\boldsymbol{\sigma}^h$ have some edge discontinuities. Now, in regions covered only by the EFG global shape functions, the resulting stress $\boldsymbol{\sigma}^h$ is smooth. However, this smooth stress field may present some spurious oscillations at regions with high gradient stresses or discontinuities, Chung and Belytschko (1998).

In order to circumvent the discontinuities or spurious oscillations in the node patches, $\boldsymbol{\sigma}^h$ is approximate in each patch by a polynomial function of a suitable order. Thus, the resulting patch recovered stress $\boldsymbol{\sigma}^{REP}$ will not only be smooth but also free of spurious oscillations in each node patch.

The proposed procedure for the determination of the improved stress field $\boldsymbol{\sigma}^*$ can be described as follows:

(i) Approximate the patch recovered stress $\boldsymbol{\sigma}_i^{REP}$ in each nodal patch by a polynomial of a suitable order, i.e., $\boldsymbol{\sigma}_i^{REP} = \left(\boldsymbol{\sigma}^{REP}\cdot\boldsymbol{e}_i\right)\boldsymbol{e}_i = \sigma_i^{REP}\boldsymbol{e}_i$, where $\sigma_i^{REP} = \boldsymbol{q}\cdot\boldsymbol{a}_i$, $\boldsymbol{q}^T$=[1, x, y, x^2, xy, y^2,…] and $\boldsymbol{a}_i$ denotes the vector of unknown coefficients. In this case, the discrete form of the modified node patch equilibrium condition may be expressed as:

$$\int_{\Omega_p}\mathbf{B}^T\boldsymbol{\sigma}_i^h d\Omega \simeq \int_{\Omega_p}\mathbf{B}^T\boldsymbol{\sigma}_i^{REP} d\Omega \tag{15}$$

where $\mathbf{B}$ is the matrix relating the strain $\boldsymbol{\varepsilon}$ vector with the particle nodal values.

Now, in order to compute the coefficients $\boldsymbol{a}_i$ a criterion is enforced which consists in the least square minimization of:

$$\left\|\int_{\Omega_p}\mathbf{B}^T\boldsymbol{\sigma}_i^h d\Omega - \int_{\Omega_p}\mathbf{B}^T\boldsymbol{\sigma}_i^{REP} d\Omega\right\|^2. \tag{16}$$

As a result, by defining

$$\mathbf{H}_i = \int_{\Omega_p}\mathbf{B}^T\boldsymbol{e}_i\otimes\boldsymbol{q}\,d\Omega \text{ and } \boldsymbol{F}_i = \int_{\Omega_p}\left[\mathbf{B}^T\boldsymbol{e}_i\otimes\boldsymbol{e}_i\right]\boldsymbol{\sigma}^h d\Omega \text{ one derives: } \boldsymbol{a}_i = \left[\mathbf{H}_i^T\mathbf{H}_i\right]^{-1}\mathbf{H}_i\boldsymbol{F}_i\,. \tag{17}$$

The strategy employed for the determination of the patch recovered stress is to approximate $\boldsymbol{\sigma}^{REP}$ by a quadratic polynomial in all node patches associated with an EFG weight function. At the node patches associated with an EPUFE weight functions it is employed a trial linear polynomial. If the matrix resulting from the linear system, given in Eq.(17) , is singular, $\boldsymbol{\sigma}^{REP}$ is then approximated by a constant polynomial. This is due to the fact that in some particular cases a patch with a constant stress field $\boldsymbol{\sigma}^h$ may occur. Moreover, the quadratic approximation of $\boldsymbol{\sigma}^{REP}$ in the EFG node patches is possible since β>1, what may prevent singularity problems when solving for $\boldsymbol{a}_i$.

(ii) Once the patch recovered stress field is determined at, for example, the *k-th* node patch, the nodal stress value $\boldsymbol{\sigma}^{REP}(\boldsymbol{x}_k)$ is evaluated.

(iii) Finally, after the patch recovered nodal stresses $\boldsymbol{\sigma}^{REP}(\boldsymbol{x}_k)$ are obtained, for every particle $\boldsymbol{x}_k$, a recovered smooth stress field $\boldsymbol{\sigma}^*$ is computed by a MLSA using an appropriate domain of influence, defined by the selection of an optimal value for β in Eq.(7), denoted here by β_r, which is determined based on the effectivity index and on the relative error analysis. Notice that, in the construction of $\boldsymbol{\sigma}^*$, only conventional quartic-spline EFG weight functions are used. Thus, from Eq.(1), we have: $\boldsymbol{\sigma}^*(\boldsymbol{x}) = \sum_{k=1}^{n}\left\{\bar{\Phi}_k(\boldsymbol{x})\boldsymbol{\sigma}^{REP}(\boldsymbol{x}_k)\right\}$. Here $\bar{\Phi}_k$ are the shape functions constructed with a domain of influence β_r.

• **Adaptive refining procedure:** The h-adaptive procedure aims at achieving a prescribed a priory percentage error γ in terms of the global energy norm of the system, i.e., $\|\mathbf{e}_\sigma\|_E \leq \frac{\gamma}{100}\|\boldsymbol{\sigma}\|_E$ where $\|\boldsymbol{\sigma}\|_E$ is the global energy norm associated with the exact solution. The most commonly used optimality criteria, for the error distribution in the mesh, considers that a mesh is optimal, for a given prescribed error, if the error distribution is uniform throughout the entire domain, Bugeda (1991). Thus, according to this criterion, the required error for each element, i.e., integration cell, is considered to be $\|\mathbf{e}_\sigma^{req}\|_{E_e} = \frac{\gamma}{100\sqrt{n_e}}\|\boldsymbol{\sigma}\|_E$ where n_e is the number of elements in the integration mesh. Now, considering the approximation of $\boldsymbol{\sigma}$ by $\boldsymbol{\sigma}^*$ and that $\|\boldsymbol{\sigma}\|_E^2 = \|\boldsymbol{\sigma}^h\|_E^2 + \|\mathbf{e}_\sigma\|_E^2$, the required error becomes: $\|\mathbf{e}_\sigma^{req}\|_{E_e} = \frac{\gamma}{100}\sqrt{\frac{\|\boldsymbol{\sigma}^h\|_E^2 + \|\mathbf{e}_\sigma^*\|_E^2}{n_e}}$.

The h-adaptive procedure consists in the solution of a sequence of h-adaptive refinement steps that will converge once the approximate convergence criterion is achieved, which is given by $\|\mathbf{e}_\sigma^*\|_E \leq \frac{\gamma}{100}\|\boldsymbol{\sigma}^*\|_E$.

At each h-adaptive refinement step a new mesh is generated and the problem is solved. If the convergence criterion is not satisfied, all the elements that violate criterion are refined. Moreover, the necessary transition elements, required for the mesh compatibility, are introduced. This procedure is necessary for a proper construction of the EPUFE weight functions. With the objective of improving the mesh quality, after the refinement step, a Laplacian smoothing is performed.

3. Examples

In order to investigate the performance of the proposed h-adaptive method we solve a simple problem. The Young's Modulus E=210GPa and the Poisson's ratio ν=0.3. Also, it is considered, in all examples, a Gauss-Legendre rule with 7 integration points and ε=10^{-4}.

• **Plate with hole:** An infinite plate with a center hole, shown in Figure 3(a), is simulated, by imposing, over a finite plate, the proper symmetry conditions and the exact prescribed traction obtained from the analytical solution, Timoshenko and Goodier (1970).

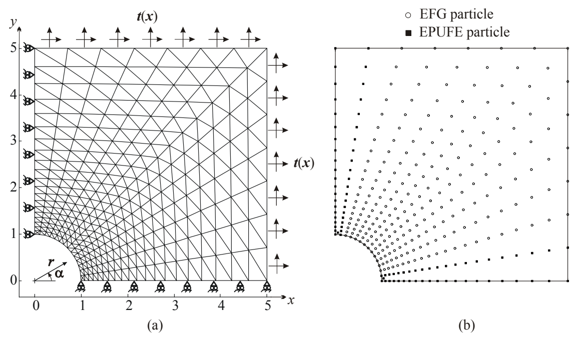

Figure 3: a) Infinite plate with hole model; b) Particle distribution - β_a=1.5.

In order to determine the proper values for β_a and β_r it is considered an initial mesh, illustrated in Figure 3(a). Table 1 shows the variation of the effectivity index. Table 2 shows the variation of the approximate relative error with respect to β_a and β_r. The last column in table 2 shows the variation of the relative error, with respect to the exact solution as a function of β_a.

β_a / β_r	1,01	1,05	1,10	1,25	1,50	2,0	2,50	3,0	3,50	4,0
1,25	1,7150	1,7137	1,7121	1,7119	1,7407	1,8411	1,9134	1,9954	2,1000	2,2452
1,5	1,0423	1,0416	1,0412	1,0425	1,0526	1,1108	1,2110	1,3552	1,5466	1,7858
1,75	1,0711	1,0704	1,0695	1,0673	1,0683	1,1003	1,1662	1,2707	1,4175	1,6082
2,0	1,0516	1,0510	1,0503	1,0475	1,0450	1,0598	1,1063	1,1927	1,3220	1,4942
2,25	1,0500	1,0497	1,0493	1,0479	1,0476	1,0655	1,1093	1,1863	1,3008	1,4543
2,5	1,0546	1,0543	1,0539	1,0525	1,0526	1,0695	1,1056	1,1672	1,2591	1,3843
2,75	1,1092	1,1084	1,1071	1,1040	1,1027	1,1181	1,1464	1,1924	1,2621	1,3595
3,0	1,1003	1,0992	1,0974	1,0920	1,0871	1,0971	1,1215	1,1611	1,2203	1,3026
3,25	0,9995	0,9987	0,9976	0,9952	0,9957	1,0124	1,0424	1,0856	1,1453	1,2252

3,5	0,9851	0,9841	0,9829	0,9810	0,9839	1,0043	1,0357	1,0786	1,1366	1,2131
4,0	1,0722	1,0708	1,0692	1,0669	1,0703	1,0845	1,1046	1,1336	1,1740	1,2282

Table 1: Variation of θ with respect to β_a and β_r.

β_a / β_r	1,01	1,05	1,10	1,25	1,50	2,0	2,50	3,0	3,50	4,0	η (%)
1,25	3,8064	3,8034	3,7999	3,7996	3,8651	4,0925	4,2592	4,4493	4,6926	5,0298	2,2207
1,5	2,2179	2,2165	2,2157	2,2186	2,2402	2,3650	2,5801	2,8906	3,3040	3,8229	2,1299
1,75	2,6390	2,6372	2,6351	2,6296	2,6322	2,7118	2,8760	3,1370	3,5047	3,9838	2,4672
2,0	2,8260	2,8246	2,8225	2,8153	2,8092	2,8506	2,9782	3,2149	3,5692	4,0425	2,6882
2,25	3,0747	3,0738	3,0726	3,0688	3,0688	3,1230	3,2539	3,4839	3,8261	4,2865	2,9251
2,5	3,4883	3,4875	3,4861	3,4819	3,4831	3,5414	3,6640	3,8726	4,1842	4,6098	3,3027
2,75	4,0925	4,0894	4,0850	4,0742	4,0708	4,1304	4,2388	4,4145	4,6804	5,0519	3,6818
3,0	4,5239	4,5191	4,5120	4,4905	4,4718	4,5163	4,6208	4,7897	5,0421	5,3927	4,0996
3,25	4,3667	4,3633	4,3588	4,3486	4,3521	4,4273	4,5626	4,7571	5,0263	5,3875	4,3575
3,5	4,4132	4,4090	4,4038	4,3958	4,4098	4,5037	4,6484	4,8467	5,1154	5,4710	4,4685
4,0	5,3797	5,3726	5,3647	5,3532	5,3702	5,4425	5,5455	5,6955	5,9055	6,1890	5,0159

Table 2: Variation of η^* (%) with respect to β_a and β_r.

To illustrate the h-adaptive procedure in this case we considered the initial mesh, illustrated in Figure 4(a), and used β_a = 1.5 and β_r = 1.1. The choice of these suitable/optimal values was based in the analysis of tables 1 and 2 in the same way as done for the cantilever bean problem, already discussed.

The final refined mesh, for $\gamma \leq 1.5\%$, is illustrated in Figure 4(c). Figures 4(b) and 4(d) show the particle distribution for the initial mesh and final mesh respectively.

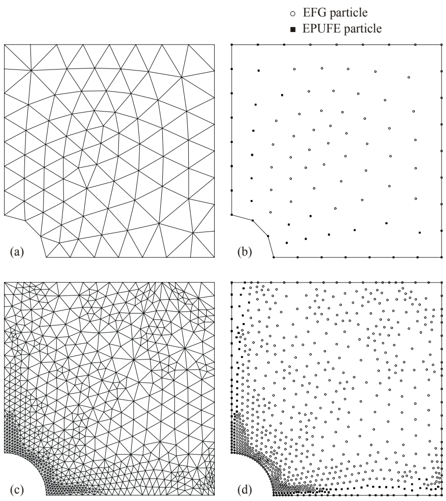

Figure 4: Initial and final integration meshes for $\gamma \leq$ 1,5% and theirs particle distributions.

A Convergence analysis in energy norm is presented in Figure 5. For comparison, it is also illustrated in Figure 5 the convergence achieved using a Tri3 finite element with the same integration meshes obtained with h-refinement procedure.

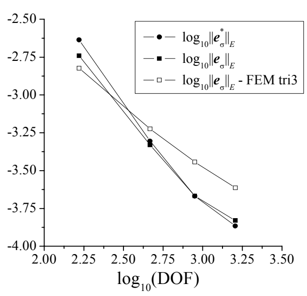

Figure 5: Error convergence analysis.

4. Conclusions

In this paper is proposed an error estimator and an h-adaptive strategy applied to a modified element-free Galerkin. The proposed procedure for the determination of the recovered stress field $\boldsymbol{\sigma}^*$ has shown to generate an accurate approximation of $\boldsymbol{\sigma}$, as illustrated in Figures 10(b), 12 and 14(b). The procedure is simple to implement and leads to a proper refinement of the integration mesh, as shown in the problem cases. Moreover, we can see that all the regions in the vicinity of Γ_u have been properly refined, reducing the stress error of the final approximate solution. Also, as seen in the derived results, the most effective domain of influence used in the MLSA of the recovered stress $\boldsymbol{\sigma}^*$ is the one where the weight function tends to have the smallest domain of influence, i.e., ${}_r$ should be slightly larger than 1.0, which leads to a fast computation of the shape functions in the post processing phase. Here, we have derived an approximate optimal value of β_r=1.1. In Figure 16 we can also see that the proposed method has shown to be more efficient, in terms of the required degrees of freedom to achieve the prescribed global error, than the one presented in Rossi and Alves (2004).

From the results obtained in this work, it is possible to conclude that the proposed h-adaptive MEFG method is a promising approach. More importantly, the proposed method may be extend, with some minor changes, to nonlinear inelastic problems, and should be investigated further.

5. Acknowledgements

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