FREQUENCY RESPONSE FUNCTION OF STOCHASTIC STRUCTURES

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ABSTRACT. *In this paper we analyze the frequency response function of structures including random parameters. The Frequency Response Function (FRF) is an usual representation of the dynamic behaviour of systems, it is also a current form to identify them. Thus, in the study of stochastic structures it is interest to have the stochastic expression of the frequency response function. The mean value and the standard deviation of the FRF of the structure are obtained from the inputs expectations and cross-covariance. The stochastic finite element method base on perturbation technique, will be used to do so. The Stochastic Finite Element Method attempt to combine the finite element analysis and the stochastic analysis to study structures with random variation in geometry or material properties. The stochastic finite element method based on perturbation uses the development of Taylor about the mean values of random variable to represent the uncertainty of structural engineering problems. This technique will be used to compute the frequency response function of a simple stochastic structure. Limits of perturbation approach are studied in terms of sensitivity of the solution versus the random parameters values. The convergence of the method is analysed. A Monte Carlo Simulation is used to validate our results.*

Keywords: *stochastic analysis, finite element method, perturbation technique, frequency response function.*

1-Introduction

The extension of the finite element method to take in account the uncertainties in the geometry or material properties of a structure, as well as the applied loads, is spelled Stochastic Finite Element Method. This field has recently become an active area of research because of perception that in some structures the response is strongly sensitive to the small random variation in material properties or geometry of the structure. Eccentricity in cross-section, differences of mass density and/or Young's modulus are examples of randomness of structural engineering problems. Such uncertainties are usually spatially distributed over the region of the structure and should be modeled as random fields. The stochastic analysis refers to the explicit treatment of uncertainty in any quantity entering the corresponding deterministic analysis. The Stochastic Finite Element Method attempt to combine the finite element analysis and the stochastic analysis.

Several methodologies can be adopted to evaluate structural response uncertainties. Early applications used the Monte Carlo simulation (Astill et al., 1972), which computes the responses for a (large) set of random numbers representing the uncertainties. Such a method is time consuming and needs a lot of CPU. Later, the Taylor series expansions, sensitivity vectors methods and perturbation methods were used to compute the second-moment statistics of response quantities in structural applications. These methods are mathematically identical to the second order of perturbation method (Benaroya & Rehak, 1988). Spectral methods, based in polynomial expansion, coupled with Galerkin projection are used as well (Ghanem & Spanos, 1991).

The basic idea of the second-moment analysis of stochastic systems by perturbation method, is to expand, via Taylor series, all the stochastic field variables about the mean values of random variables, to retain only up to second-order terms (Kleiber & Hien, 1992). The output expectations and cross-covariances are obtained from the input expectations and cross-covariances. This method is much faster than the Monte Carlo one, but it increases the number of equations to be solved. However, the perturbation method has a simple formulation of the spectral methods (which require integral solutions), presenting the same result quality low values of dispersion of the random parameters (Diniz et all, 1999). The increase of the number of problem equations in the perturbation method can be minimized by employing a Component Mode Synthesis method (Diniz & Thouverez, 1999)

In this paper, the stochastic finite element method, based on perturbation technique, will be used to compute the frequency response function of stochastic structures. A general formulation is presented and a simple structural example is studied to illustrate the method. Limits of perturbation approach are studied in terms of sensitivity of the solution versus the random parameters values. A Monte Carlo simulation is used to validate our results.

Regarding that the achieved results present instability in the stochastic FRF behavior near resonance a convergence analysis of the Taylor series expansion using higher orders is performed.

2– Frequency response function expression for the stochastic case

The FRF is a well-adapted representation of the dynamic behavior of systems for experimental comparison, it is also a current form to identify them. In the study of stochastic systems, it is interesting to use the stochastic form of the Frequency Response Function. Using a perturbation approximation for the study of the stochastic FRF, the Taylor expansion o f the FRF will be performed as a function of the random variables.

A bar with only one random variable will be studied and, to simplify, with only one degree of freedom. The Frequency Response Function with respect to the mass and stiffness will be applied making the higher order Taylor development simpler. The Taylor development convergence of the FRF in this case will be studied.

The equilibrium equation of a "N" degree of freedom discrete system with hysteretic damping is:

$$((1+i\eta)[K]-\omega_i^2[M])u_i = f_i \qquad i = 1, 2,\ldots N \tag{1}$$

In the particular case of one degree of freedom, the Frequency Response Function "H" can be written as:

$$H = \frac{u}{f} = \frac{1}{(1+i\eta)K-\omega^2 M} \tag{2}$$

Assuming "*K*" is a direct function of the random parameters "*b*"(in this particular case: Young's modulus), the FRF will be an indirect function of this parameter only. It is desired, then, to attain the FRF mean and standard deviation values from the "*K*(*b*)" mean and standard deviation.

The Taylor development of stiffness "*K*" and of FRF "*H*" over the mean and standard deviation values of the random variable "*b*" are given by:

$$K(b) \cong K^0 + K^{(1)}\Delta b + \frac{1}{2}K^{(2)}(\Delta b)^2 + \ldots \frac{1}{n!}K^{(n)}(\Delta b)^n + \ldots \tag{3}$$

$$H(b) \cong H^0 + H^{(1)}\Delta b + \frac{1}{2}H^{(2)}(\Delta b)^2 + \ldots \frac{1}{n!}H^{(n)}(\Delta b)^n + \ldots \tag{4}$$

with:

$$K^{(n)} = \frac{\partial^n K\left(b^0\right)}{\partial b^n} \qquad H^{(n)} = \frac{\partial^n H\left(b^0\right)}{\partial b^n} \qquad \Delta b = b - b^0$$

where:

$$b^0 = \overline{b} \qquad K^0 = \overline{K} = K\left(\overline{b}\right) \qquad H^0 = \frac{1}{(1+i\eta)K^0 - \omega^2 M}$$

and, with only one random variable:

$$K^{(r)} = \frac{\partial^r K(b^0)}{\partial b^r} = 0 \qquad \forall r \geq 2 \tag{5}$$

Thus, the random system equation developed with the Taylor series is attained:

$$\left((1+i\eta)K^0 - \omega^2 M + (1+i\eta)K^{(1)}\Delta b\right)\left(H^0 + H^{(1)}\Delta b + \frac{1}{2}H^{(2)}(\Delta b)^2 + \cdots \frac{1}{\eta!}H^{(n)}(\Delta b)^n + \cdots\right) = 1 \tag{6}$$

Defining: $K^* = (1+i\eta)K^0 - \omega^2 M$ and $K^{**} = (1+i\eta)K^{(1)}$, if the same order terms are grouped, the following expressions are achieved:

$$K^* H^0 = 1 \tag{7}$$

$$K^{*}H^{(1)} + K^{**}H^{0} = 0 \tag{8}$$

$$\frac{1}{2}K^{*}H^{(2)} + K^{**}H^{(1)} = 0 \tag{9}$$

$$\frac{1}{n!}K^{*}H^{(n)} + \frac{1}{(n-1)!}K^{**}H^{(n-1)} = 0 \tag{10}$$

In general, the "n-th" derivative of "H(b)" will be given by:

$$H^{(n)} = -nH^{0}K^{**}H^{(n-1)} \tag{11}$$

Hence, the Taylor developed FRF can be written as:

$$H(b) \cong H^{0}\sum_{n=0}^{\infty}\underbrace{(-1)^{n}\left(H^{0}\right)^{n}\left(K^{**}\right)^{n}(\Delta b)^{n}}_{H_n} \tag{12}$$

3 – Taylor Development of the FRF's mean.

The mean value of the FRF is given by the expected value de "*H*(*b*)" in equation 12:

$$\overline{H}(b) = E[H(b)] \cong H^{0}\sum_{n=0}^{\infty}(-1)^{n}E[\Delta b]\left(H^{0}\right)^{n}\left(K^{**}\right)^{n} \tag{13}$$

The expected value $E[\Delta b] = E\left[\left(b - b^{0}\right)\right]$ depends on the kind of probability distribution adopted for random variable "b".

If a **gaussian probability distribution** is adopted for the random variable "*b*", the characteristic law, responsible for generating the statistic moments, will give by:

$$E\left[\left(b - b^{0}\right)^{2n+1}\right] = 0 \tag{14}$$

$$E\left[\left(b - b^{0}\right)^{2n}\right] = \prod_{l=1}^{n}(2l-1)\left(\sigma^{2}\right)^{n} \tag{15}$$

with: $\boldsymbol{\sigma}^{2} = E\left[\left(b - b^{0}\right)^{2}\right] =$ **Var**(b)

The odd expected value exponents are then zero and the even exponents are a function of the random variable variance.

The Taylor development of the mean value of the FRF for a gaussian distribution of the random variable is given by:

$$\overline{H}(b) \cong H^{0} + H^{0}\sum_{n=1}^{\infty}\underbrace{\left[\prod_{l=1}^{n}(2l-1)(\mathbf{Var}(b))^{n}\left(H^{0}\right)^{2n}\left(K^{**}\right)^{2n}\right]}_{\overline{H}_n} \tag{16}$$

or, in a recursive manner and considering only the even terms:

$$\overline{H}_{2n}(b) \cong \overline{H}_{2n-2}(b) + H^{0}\left[\prod_{l=1}^{n}(2l-1)(\mathbf{Var}(b))^{n}\left(H^{0}\right)^{2n}\left(K^{**}\right)^{2n}\right] \qquad \forall n = 1,2,3,\cdots \tag{17}$$

The order two development (n=1) provides:

$$\overline{H}_2(b) = H^0 + \left(H^0\right)^3 \left(K^{**}\right)^2 \mathbf{Var}(b) \tag{18}$$

For a **uniform probability distribution**, the characteristic law generating the statistic moments gives:

$$E\left[\left(b - b^0\right)^n\right] = \frac{\left(\sqrt{3}\boldsymbol{\sigma}(b)\right)^n}{n+1} \tag{19}$$

The Taylor development of the mean value of the FRF for the random variable uniform distribution is given by:

$$\overline{H}_u(b) \cong H^0 \left[\underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{\left(\sqrt{3}\boldsymbol{\sigma}(b)\right)^n}{n+1} \left(H^0\right)^n \left(K^{**}\right)^n}_{\overline{H}_{un}} \right] \tag{20}$$

where the index “u” is used to distinguish the mean value of the FRF in the uniform case from the mean value in the gaussian case.

The order two development of the FRF for the random variable uniform distribution gives then:

$$\overline{H}_{u_2}(b) = H^0 - \left(H^0\right)^2 K^{**}\boldsymbol{\sigma}(b) + \left(H^0\right)^3 \left(K^{**}\right)^2 \mathbf{Var}(b) \tag{21}$$

Applying the perturbation method to calculate the to a clamped-clamped bar of length 2 [m], diameter 0,02 [m] and density 7800 [kg/m^3] with random Young’s modulus with mean value E = 21*10^{10} [N/m^2] and standard deviation $\boldsymbol{\sigma}_{\%}$ = 5% we was obtained the results shows in figure 1. A 100 elements discretization is used. This figure presents the comparison between the FRF, calculated by using a Taylor expansion of order two, and the Monte Carlo simulation (30.000 iterations) to a gaussian probability distribution of the random Young modulus.

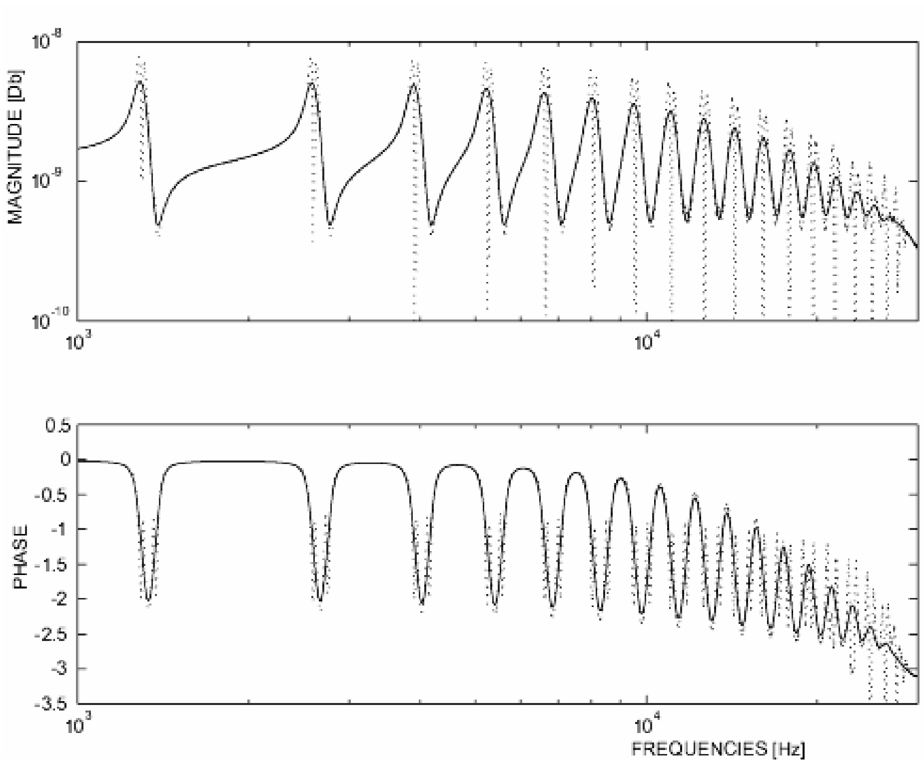

Figure 1. Mean FRF (magnitude and phase) - Monte Carlo simulation (———) and Perturbation method (......).

4. – Convergence Analysis of the Mean Value

The comparison between the FRF evaluated for both methods shows in figure 1, for frequencies close to the natural frequencies of the bar, a large difference between the results of Taylor expansion and Monte Carlo simulation.

In order to evaluate if the approximation of the stochastic FRF near resonance frequencies by higher order Taylor expansion series is effective, a convergence analysis of the Taylor series expansion was done. To determine the permissible values for the random variable variance, which would result in a converging Taylor development of the mean of the FRF magnitude, the following expressions are shown.

Gaussian Distribution

From equation (16), one can write:

$$\left|\frac{\overline{H}_{n+1}}{\overline{H}_{n}}\right| < 1 \Rightarrow \left|(2(n+1)-1)\cdot \mathbf{Var}(b)\left(H^{0}\right)^{2}\left(K^{**}\right)^{-2}\right| < 1 \tag{22}$$

Hence:

$$\mathbf{Var}(b) < \left|\frac{1}{2n+1}\left(H^{0}\right)^{2}\left(K^{**}\right)^{2}\right| \tag{23}$$

With the definitions of "K**" and "H^0" in resonance, it can be obtained for the bar with a random Young's modulus:

$$\mathbf{Var}(b) < \left|\frac{1}{2n+1}\left(\frac{\eta}{1+i\eta}\right)^{2} E^{2}\right| \tag{24}$$

Thus, to guarantee the convergence of the FRF's mean value Taylor development, the non-dimensional standard deviation "$\boldsymbol{\sigma}_{\%}(E)$"is constrained by:

$$\boldsymbol{\sigma}_{\%}(E) < \frac{1}{2n+1}\cdot\frac{\eta^{2}}{\left|(1+i\eta)^{2}\right|} \tag{25}$$

This expression show that the order increase in the Taylor development decreases the array of standard deviation values to which the FRF mean development converges.

The Young's modulus admissible standard deviation to different orders of Taylor expansion are show in table 1, for the gaussian distribution case.

Table 1 - Young's modulus admissible standard deviation to gaussian distribution.

Gaussian distribution					
Order of expansion	2	4	6	8	10
$\boldsymbol{\sigma}_{\%}(E)$ maximum	0.0080	0.0044	0.0031	0.0024	0.0019

Even though, if a comparison is made with the Monte Carlo's simulation, regarding an allowable error margin, it can be defined an array of standard deviation values where the Taylor series solution gives satisfactory results. That is shown in Figure 2, where the results attained with Monte Carlo's simulation with 30.000 iterations and with Taylor series development are compared. The curves were plotted for a 0.2% dumping for the different order terms in the Taylor development.

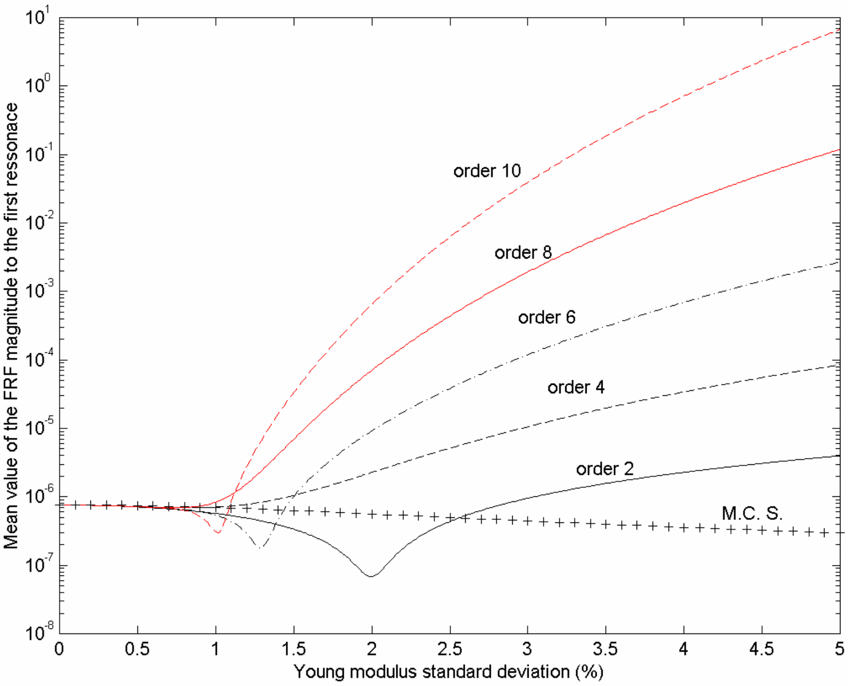

Figure 2: comparison of the mean value evolution in the case of a Young's modulus gaussian distribution.

Figure 2 shows that an order two Taylor development is already sufficient to represent the Frequency Response Function mean. The higher order developments get farther from the Monte Carlo's simulation for smaller Young's modulus standard deviation values. This trend is confirmed through figure 3.

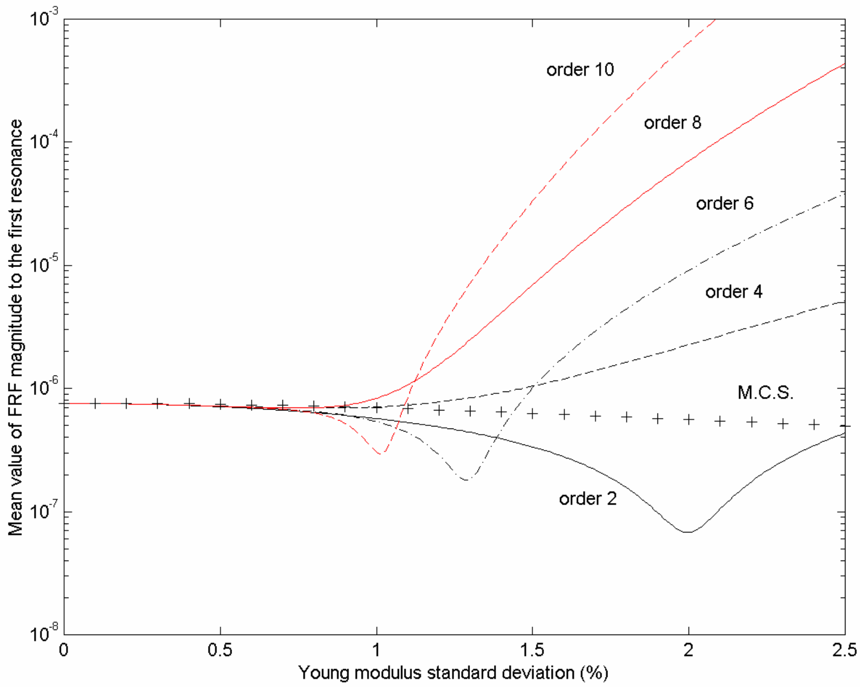

Figure 3: comparison of the mean value evolution in the interval [0; 2.5] – Young's modulus gaussian distribution case.

Using different values for the dumping coefficient, one can determine for each order in the Taylor development, the maximum standard deviation that ensures the series convergence. These values are shown in figure 4.

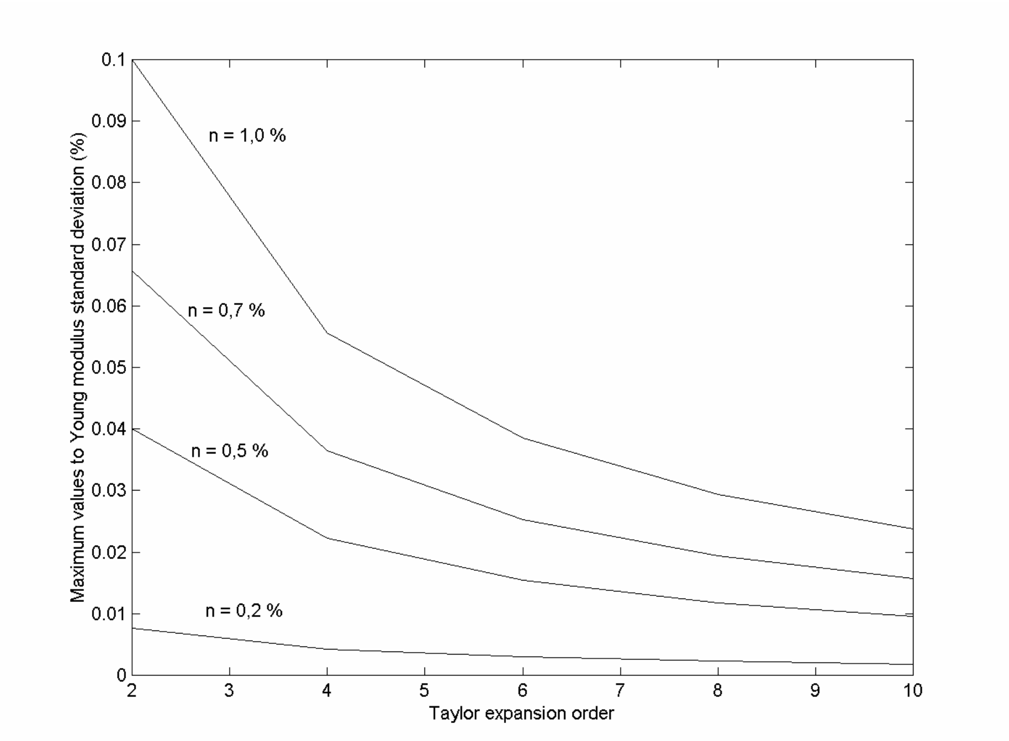

Figure 4: maximum standard deviation for a gaussian distribution

The admissible standart deviations to Young's modulus obtained in Tab. 1 and show in Fig. 4 are extremely small for pratical application of a bar with a gaussian distribution Young's modulus.

Uniform Distribution

From equation (21), it can be written:

$$\left|\frac{\overline{H}_{u_{n+1}}}{\overline{H}_{u_n}}\right| < 1 \Rightarrow \left|\frac{n+1}{n+2}\sqrt{3}\boldsymbol{\sigma}(b)H^0K^{**}\right| < 1 \tag{26}$$

With the definitions of "K**" and "H^0" in resonance, one can obtains for a random Young's modulus rod:

$$\left|\sqrt{3}\,\frac{n+1}{n+2}\cdot\frac{1+i\eta}{i\eta}\cdot\frac{\boldsymbol{\sigma}(b)}{E}\right| < 1 \tag{27}$$

Thus to ensure the FRF mean value Taylor development convergence, the standard deviation non-dimensional value "$\boldsymbol{\sigma}_{\%}(E)$" is constrained by:

$$\boldsymbol{\sigma}_{\%}(E) < \frac{n+2}{n+1}\cdot\frac{\eta}{\left|\sqrt{3}(1+i\eta)\right|} \tag{28}$$

The Young's modulus admissible standard deviation to different orders of Taylor expansion are show in table 2, for the uniform distribution case.

Table 2 - Young's modulus admissible standard deviation to uniform distribution

Uniform distribution					
Order of expansion	2	4	6	8	10
$\boldsymbol{\sigma}_{\%}$(E) maximum	0.1540	0.1386	0.1320	0.1260	0.1283

As for the gaussian distribution, the order increase of the Taylor development to the uniform distribution does not increase the series convergence. The order two development is then sufficient to represent the random FRF.

The Taylor series intervals of convergence are bigger in the Young's modulus uniform distribution than the gaussian distribution intervals. For example, in an order two development with uniform distribution the Taylor series is convergent until a 1.54% standard deviation value against a 0.01% in the gaussian case.

The FRF magnitude variation as function of the Young's modulus standard deviation in the uniform distribution case is shown in figure 5. The curves were plotted for a 0.2% dumping for different orders in the Taylor development. As in the gaussian case, comparison with Monte Carlo's simulation allows to increase the validation limits for the Taylor development (figure 5).

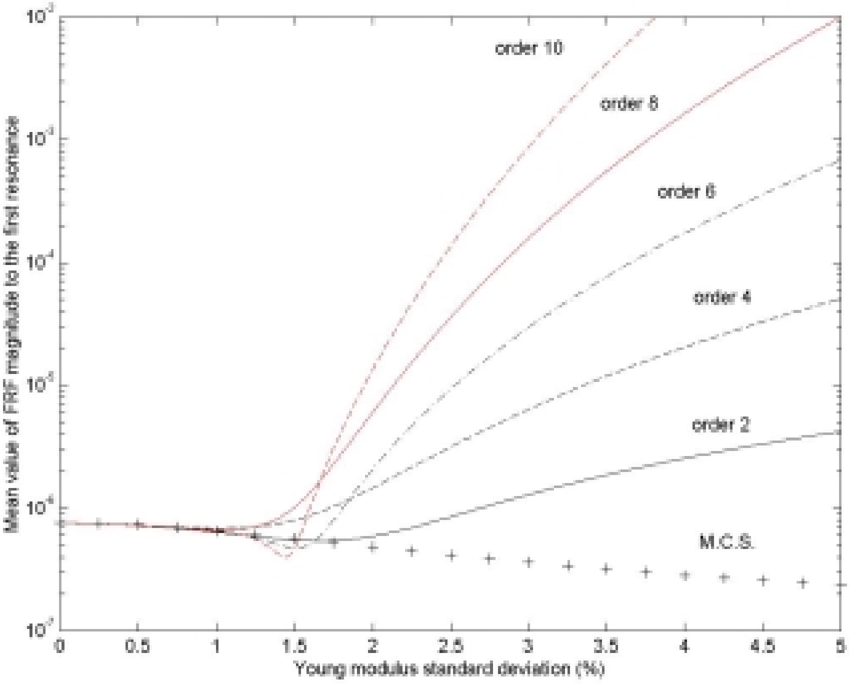

Figure 5: mean value evolution comparison in the case of a Young's modulus uniform distribution

Figure 5 also shows that the order two development of the Frequency Response Function's mean is closer to the Monte Carlo's simulation. Figure 6 confirms this observation for standard deviation lower than 2.5%.

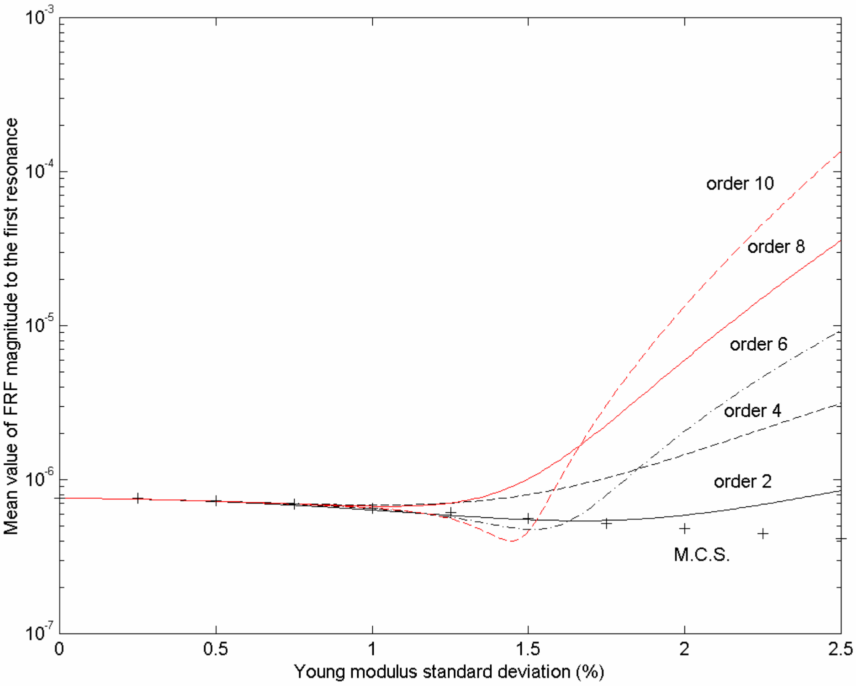

Figure 6: comparison of the mean value evolution in the interval [0; 2.5] – Young's modulus uniform distribution case.

To different values of the dumping coefficient figure 7 shows the maximum standard deviation for different orders of Taylor expansion in the uniform distribution case.

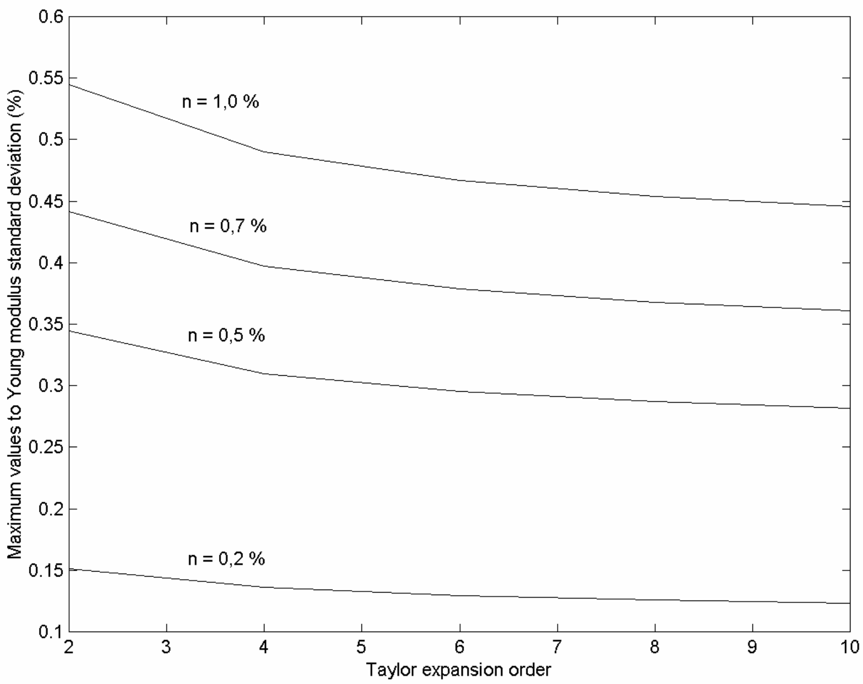

Figure 7: maximum standard deviation for the uniform distribution

5. Conclusion

The stochastic finite element method based on perturbation technique is used to compute the frequency response function of stochastic structures. A general formulation is presented and a simple structural example is studied to illustrate the method.

The ontained results shows that the use of a Taylor development in the study of the mean of the Frequency Response Function is constrained through the extremely poor convergence of the Taylor series near resonance. This convergence interval decreases when the development order is increased. It is also observed that for a development of the same order the intervals of convergence are larger for the uniform distribution.

It can also be observed that an order two development compared to the Monte Carlo gives a good representation of the FRF behavior even for standard deviations higher than the convergence limit if small error values are assumed. This behavior observed in both cases studied is more significant in the case of a uniform distribution.

One can conclude that the order two development is a better approximation for the FRF statistic behavior. Though, due to the limited Taylor series convergence to represent the FRF mean, this approximation is limited to the extremely small values of the Young's modulus standard deviation.

6. References

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