NUMERICAL SIMULATION OF INCOMPRESSIBLE MAGNETOHYDRODYNAMIC FLOWS

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Abstract. This paper presents numerical simulations of incompressible fluid flow problems in the presence of a magnetic field. The equations governing the flow consist of the Navier-Stokes equations of fluid motion coupled with Maxwell's equations of electromagnetics. The study of fluid flows under the influence of a magnetic field and with no free electric charges or electric fields is known as magnetohydrodynamics. The magnetohydrodynamics approximation is considered for the formulation of the non-dimensional problem and for the characterization of similarity parameters. A finite-difference technique is used to discretize the equations. The simulation of incompressible magnetohydrodynamic fluid flows is illustrated by numerical problems for two-dimensional cases.

Keywords: electromagnetics, incompressible flows, magnetohydrodynamic

1. Introduction

Magnetohydrodynamics (MHD) is the branch of continuum mechanics that studies the flow of electrically conducting fluids in the presence of magnetic fields. Conduction occurs when there are free or quasi-free electrons, which can move under the action of applied fields. For a fluid, the fields act on both electrons and ionized atoms to produce dynamical effects, including bulk motion of the medium itself. This mass motion in turn produces modifications in the electromagnetic fields. Consequently we must deal with a coupled system of matter and fields (Jackson, 1975). The resulting coupled problem is of major interest for its numerous practical applications. Typical examples are the motion of liquid metals (mercury, liquid sodium, etc.) and plasma physics.

Two basic coupling effects can be identified in the motion of an electrically conducting fluid in a magnetic field. First, the motion induces an electromotive force that will modify the existent electromagnetic field. Secondly, the presence of an electric current, together with the magnetic field, will exert a mechanical force on the fluid particles that tends to accelerate them in the normal direction to the magnetic field and the electric current. Theses effects are described by Lorentz's forces. The physical system exhibits, therefore, a characteristic two-way coupling. The non-relativistic motion of Newtonian fluids is governed by the Navier-Stokes equations. Electromagnetic effects are governed by Maxwell's equations. A number of physical assumptions valid for the problems of interest reduce these two general systems to the final MHD equations. These assumptions characterize the magnetohydrodynamic approximation. These requirements represent the underlying assumptions of magnetohydrodymamics. Several applications usually involve a incompressible fluid flow leading to the incompressible MHD equations.

Salah *et al.* (2001) develop a finite element method for the solution of 3D incompressible magnetohydrodynamic flows. They select a conservative formulation in the sense that the local divergence-free condition of the magnetic field is accounted for in the variational sense. It is also proposed an algorithm for the solution of the couple problem (Navier-Stokes and magnetic equations). Armero and Simo (1996) also investigate incompressible MHD equations. They examine the long-term dissipativity and unconditional non-linear stability of time integration algorithms for mixed finite element approximations. The present study investigates two-dimensional incompressible magnetohydrodynamic fluid flows. Numerical results using a finite-difference technique are presented.

2. Magnetohydrodynamic approximation

The electromagnetic effects are modelled in the domain Ω by the Maxwell's equations (see e.g. Jackson, 1975, Landau and Lifshitz 1984):

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad (1)$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}, \quad (2)$$

$$\nabla \cdot \boldsymbol{D} = \rho_c , \tag{3}$$
$$\nabla \cdot \boldsymbol{B} = 0 , \tag{4}$$

where ρ_c is the electric charge density, $\boldsymbol{E}$ is the electric field, $\boldsymbol{H}$ is the magnetic field, $\boldsymbol{J}$ is the electric current density, and $\boldsymbol{D}$ is the electric displacement. Equations (1)-(3) are known as Faraday's law, Ampère-Maxwell's law, and Gauss 'law, respectively. Equation (4) represents the divergence-free constraint on the magnetic induction $\boldsymbol{B}$. The constitutive equations characterizing the electromagnetic media can be expressed by the linear relations

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} , \; \boldsymbol{H} = \frac{1}{\mu_0} \boldsymbol{B} - \boldsymbol{M} \tag{5}$$

valid for the free space with ε_0 the electric permittivity and μ_0 the magnetic permeability. For liquid metals, like mercury or liquid sodium, it can be assumed the following physical hypotheses know as MHD approximation: (a) non-relativistic motion; (b) phenomena involving low frequency; (c) quasi-neutrality; and (d) non-magnetizable and non-polarizable medium.

3. Incompressible magnetohydrodynamic equations

The MHD equations characterize the flow of a conducting fluid in the presence of a magnetic field. They represent the coupling of the fluid dynamical equations with Maxwell's equations of electrodynamics.

The motion of an incompressible viscous fluid is described by the Navier-Stokes equations valid for a domain Ω of $\mathbf{R}^3$ (see e.g. White, 1991):

$$\nabla \cdot \boldsymbol{V} = 0 , \tag{6}$$
$$\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V} \cdot \nabla \boldsymbol{V} = -\frac{1}{\rho} \nabla p^* + \nu \nabla^2 \boldsymbol{V} + \boldsymbol{f}_{ext} , \tag{7}$$

where $\boldsymbol{V}$ is the velocity field, ν is the constant kinematic viscosity, ρ is the constant density, p^* is the pressure field, and $\boldsymbol{f}_{ext}$ is the external force.

The electric current density is related, in a conductor, to the electric field by Ohm's law. We can write an expression for $\boldsymbol{J}$ as

$$\boldsymbol{J} = \sigma \left(\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B} \right), \tag{8}$$

where σ is the electric conductivity that is assumed scalar (isotropy).

Considering the MHD approximation, an incompressible MHD flow is characterized by the system below in terms of the non-dimensional space variable $\boldsymbol{x} / L$, time variable tU / L, still denoted by $\boldsymbol{x}$ and t, as well as the non-dimensional fields $\boldsymbol{u} \equiv \boldsymbol{V} / U$, $\boldsymbol{b} \equiv \boldsymbol{B} / B$, $p \equiv p^* / \rho U^2$ and $\boldsymbol{f} \equiv \boldsymbol{f}_{ext} L / U^2$:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} . \nabla \boldsymbol{u} = -\nabla p + \frac{1}{Re} \nabla^2 \boldsymbol{u} + Al (\nabla \times \boldsymbol{b}) \times \boldsymbol{b} + \boldsymbol{f} , \tag{9}$$
$$\frac{\partial \boldsymbol{b}}{\partial t} - \nabla \times (\boldsymbol{u} \times \boldsymbol{b}) = \frac{1}{Re_m} \nabla^2 \boldsymbol{b} , \tag{10}$$
$$\nabla \cdot \boldsymbol{u} = 0 , \tag{11}$$
$$\nabla \cdot \boldsymbol{b} = 0 , \tag{12}$$

where U, L and B are characteristic velocity, length, and magnetic induction, respectively, $Re = UL / \nu$ is the Reynolds number, $Re_m = \sigma \mu_0 UL$ is the magnetic Reynolds number, and $Al = B^2 / \mu_0 \rho U^2$ is the Alfvén number. The Stuart number (or interaction parameter) is defined as $N = Re_m Al$. A non-dimensional Hartmann number that is a measure of the imposed magnetic field independent of the characteristic velocity is specified as $Ha = BL(\sigma / \nu\rho)^{1/2} = (N\, Re)^{1/2}$.

The magnetic induction satisfies the divergence-free constraint expressed by Eq. (12). To enforce this condition we can introduce a vector potential

$$\boldsymbol{b} = \nabla \times \boldsymbol{a} , \tag{13}$$

with non-dimensional field $\boldsymbol{a} \equiv \boldsymbol{A} / BL$.

Using Eqs. (2) and (5), and the MHD approximation, the non-dimensional vector potential $\boldsymbol{a}$ can be related to the non-dimensional electric current density $\boldsymbol{j}$ by the expression

$$\boldsymbol{j} = \frac{1}{Re_m} \nabla \times \boldsymbol{b} \,. \tag{14}$$

Applying the curl to (10) and using Eq. (14), the non-dimensional current density can be computed by

$$\frac{\partial \boldsymbol{j}}{\partial t} + \nabla \times (\boldsymbol{u} \times \boldsymbol{j}) - \frac{1}{Re_m} \nabla \times \left(\boldsymbol{b} \times \omega^* \right) = \frac{1}{Re_m} \nabla^2 \boldsymbol{j} \,, \tag{15}$$

where $\boldsymbol{\omega}^* \equiv \boldsymbol{\omega} L / U$ is the non-dimensional vorticity field expressed by

$$\omega^* = \nabla \times \boldsymbol{u} \,. \tag{16}$$

For a perfectly conducting boundary, i.e. the electric current flows through the boundary, we can define the boundary conditions on Γ:

$$\boldsymbol{b} \cdot \boldsymbol{n} = 0 \,, \tag{17}$$
$$\boldsymbol{n} \times (\nabla \times \boldsymbol{b}) = \boldsymbol{0} \,, \tag{18}$$

where $\boldsymbol{n}$ is the a unit normal to the surface. In the two-dimensional case, the fields are expressed as $\boldsymbol{u} = (u_x, u_y, 0)^T = (\partial \psi / \partial y, -\partial \psi / \partial x, 0)^T$, $\boldsymbol{b} = (b_x, b_y, 0)^T = (\partial a / \partial y, -\partial a / \partial x, 0)^T$, $\boldsymbol{a} = (0,0,a)^T$, and $\boldsymbol{j} = (0,0,j)^T$, where ψ and a are stream functions.

4. Generalized Peaceman-Rachford ADI scheme for vorticity equations

The numerical method used in this work is based on Dai's (1997) generalized scheme for parabolic equation. We can write the generalized alternating-direction implicit (ADI) scheme for the two-dimensional vorticity equation as follows

$$\left(1 - \varepsilon_x \frac{\Delta t}{2Reh_x^2} \delta_x^2\right) \frac{\omega_{ij}^{n+1/2} - \omega_{ij}^n}{\Delta t / 2} = -\frac{u_{ij}^n}{2h_x} \delta_x \omega_{ij}^{n+1/2} - \frac{v_{ij}^n}{2h_y} \delta_y \omega_{ij}^n + \frac{1}{Reh_x^2} \delta_x^2 \omega_{ij}^{n+1/2} + \frac{1}{Reh_y^2} \delta_y^2 \omega_{ij}^n + N\left(\frac{b_{xij}^{\ n}}{2h_x} \delta_x j_{ij}^n + \frac{b_{yij}^{\ n}}{2h_y} \delta_y j_{ij}^n \right) \tag{19a}$$

$$\left(1 - \varepsilon_y \frac{\Delta t}{2Reh_y^2} \delta_y^2\right) \frac{\omega_{ij}^{n+1} - \omega_{ij}^{n+1/2}}{\Delta t / 2} = -\frac{u_{ij}^n}{2h_x} \delta_x \omega_{ij}^{n+1/2} - \frac{v_{ij}^n}{2h_y} \delta_y \omega_{ij}^{n+1} + \frac{1}{Reh_x^2} \delta_x^2 \omega_{ij}^{n+1/2} + \frac{1}{Reh_y^2} \delta_y^2 \omega_{ij}^{n+1} + N\left(\frac{b_{xij}^{\ n}}{2h_x} \delta_x j_{ij}^n + \frac{b_{yij}^{\ n}}{2h_y} \delta_y j_{ij}^n \right) \tag{19b}$$

where h_x and h_y are the grid spacing, $\Delta t = t^{n+1} - t^n$ is the time step, ε_x and ε_y are positive constants, and δ_x, δ_y, δ^2_x and δ^2_y are the usual central difference operators. The components of the velocity field u and v, the components of the magnetic induction b_x and b_y, and the electric current density j are fixed in each time step. Equation (19a) expresses the vorticity at the point (x_i, y_j, $t^{n+1/2}$) and Eq. (19b) is related to the vorticity at the point (x_i, y_j, t^{n+1}). When $\varepsilon_x = \varepsilon_y = 0$, Eq. (19) becomes the Peaceman-Rachford ADI scheme (Peaceman and Rachford, 1995). The present work also applies the same scheme to electric current density equation. Navarro and Ferreira (2002) described a generalized ADI scheme for incompressible viscous flow problems.

5. Numerical results

A driven cavity problem is represented by a rectangular domain $\boldsymbol{\Omega} = (0, x_{max}) \times (0, y_{max})$ and the boundary set $\Gamma = \{(x=0,\ 0 \le y \le y_{max}),\ (y=0,\ 0 \le x \le x_{max}),\ (x=x_{max},\ 0 \le y \le y_{max}),\ (y=y_{max},\ 0 \le x \le x_{max})\}$. The Cartesian coordinate system is positioned at the origin O(x=0, y=0). On the solid wall, the non-slip boundary condition $\boldsymbol{u}=\mathbf{0}$ is assumed for the velocity field. On the upper boundary, the velocity $\mathbf{u} = (u,v) = (4x^2(1-x^2),0)$ is imposed and the maximum of u is $u_{max}=1$ at $x=1/2^{1/2}$. We consider the case of an imposed in-plane magnetic induction transversal to the flow. We take $\boldsymbol{b}=(0,1)$ on the boundary. The time increment is constant $\Delta t=10^{-4}$ and the cavity is the square, $x_{max} = 1$, $y_{max} = 1$. The constant space increments are computed as $h_x = x_{max} / n_x$ and $h_y = y_{max} / n_y$. We shall consider the Reynolds number Re=100 and the grid with $n_x = 20$, $n_y = 20$. Computations have been carried out for different magnetic Reynolds numbers. Figure 1 shows the solutions of the magnetic induction for Re_m=0.1, 100, and the evolution from diffusive transport to convective is displayed. The Hartmann number Ha=10 is used.

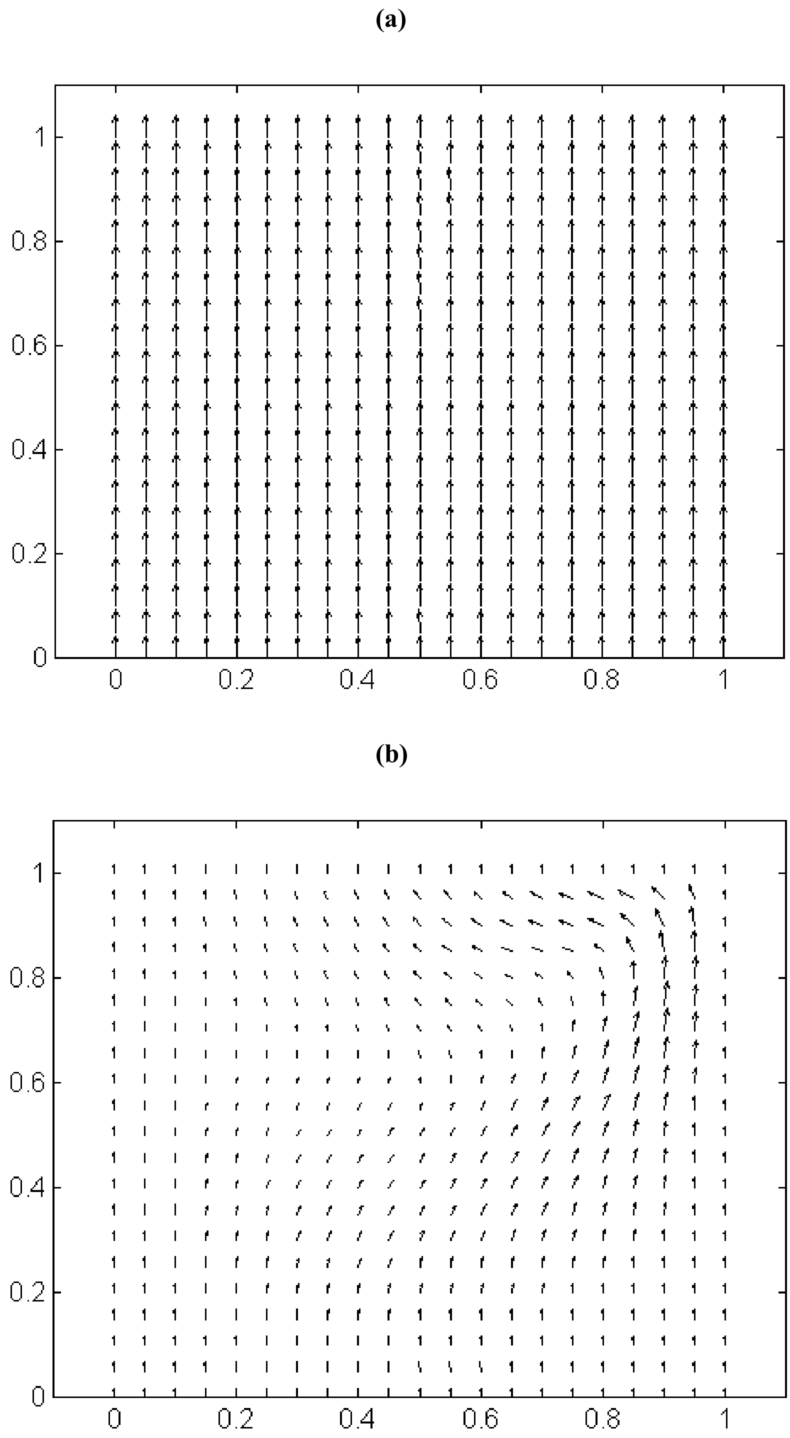
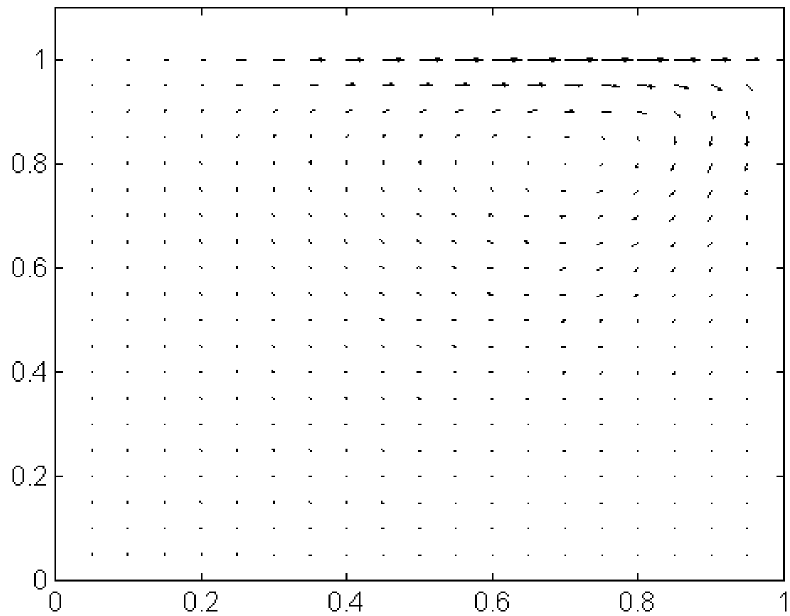


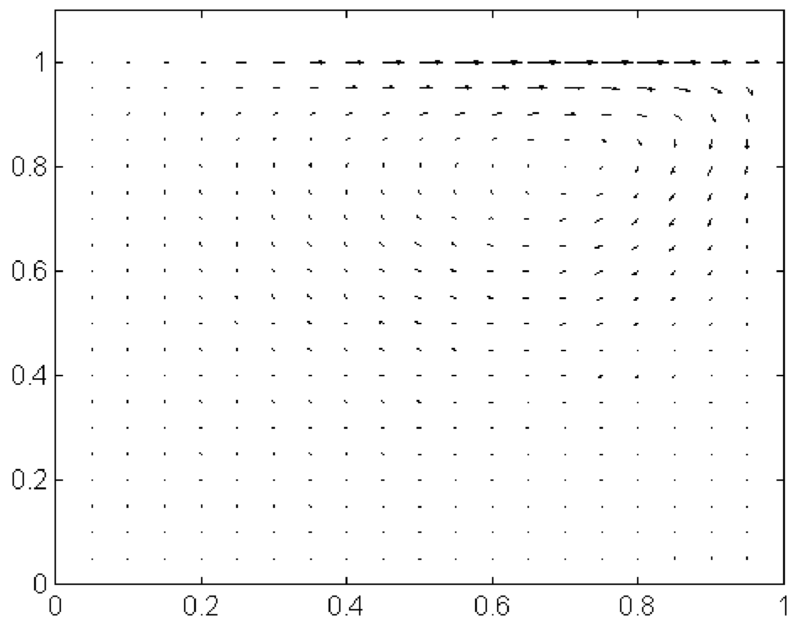
Figure. 1. Magnetic induction, Ha=10: (a) Re_m=0.1; (b) Re_m=100.

We now investigate the influence of the Hartmann *Ha* at a low magnetic Reynolds number where the magnetic diffusion dominates the magnetic convection. We set the magnetic Reynolds number to $Re_m = 1$. The Hartmann number is set to $Ha = 0, 5, 10, 12$. Figure 2 shows the evolution of the velocity field where we can note the change of the structure of the vortices. In the case of no external magnetic field applied (Ha=0), there is a large central vortex in the clockwise rotation. As the Hartmann number increases, the central vortex becomes more horizontal and breaks into two vortices. The small vortex in the bottom becomes larger (counter clockwise rotation). The increasing effect of the electromagnetic forces weaken the vortices strength leading to the breaking of vortices. The same effects present in figures 1 and 2 were also observed by Salah *et al.* (2001).

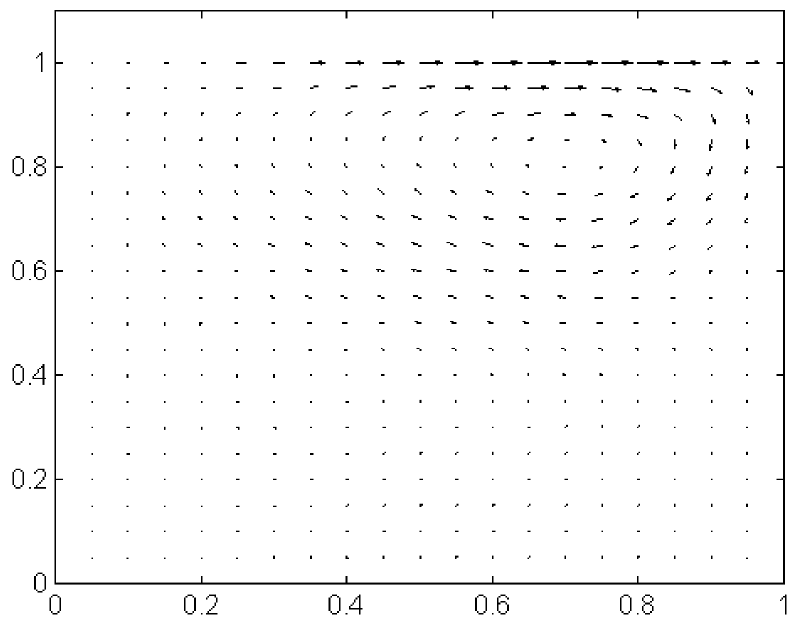
(a)



(b)



(c)



(d)

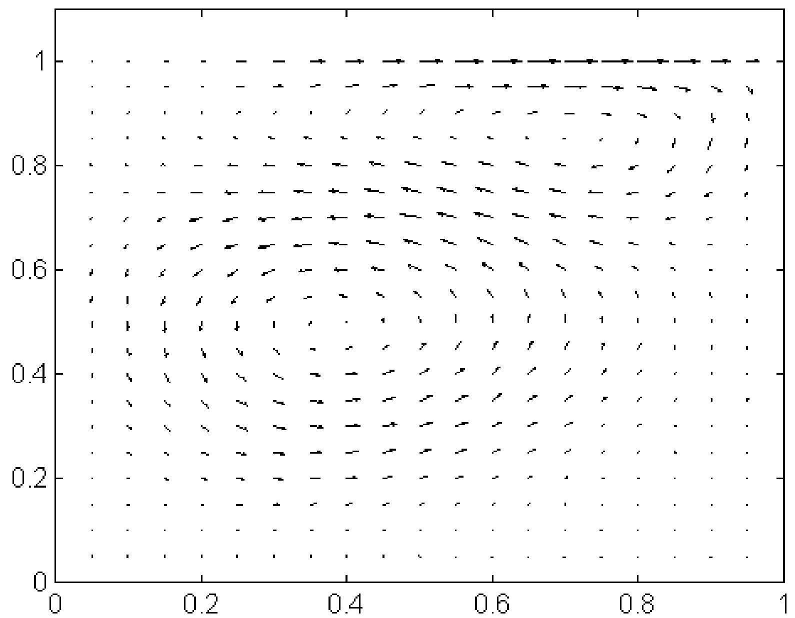


Figure 2. Velocity field Re_m=1: (a) Ha=0; (b) Ha=5; (c) Ha=10; (d) Ha=12.

6. Conclusions

This paper investigated incompressible magnetohydrodynamic flow for two-dimensional problems. The stream functions (ψ and a)- vorticity - current density formulation has been used to solve the incompressible MHD equations. We used this approach for both high and low magnetic Reynolds numbers. In the present simulations, this method captured the steady-state solution forced by the time-independent boundary conditions. We obtain good results for magnetic induction and velocity field with the same pattern presented by Salah *et al.* (2001).

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7. References

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