SIMULATION OF WAVE PROPAGATION IN AN HETEROGENEOUS ELASTIC ROD

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Abstract. *A continuous mathematical model for describing the dynamical response of a heterogeneous linear-elastic rod, left in a nonequilibrium state is presented in this work. The problem is represented by a set of two hyperbolic partial differential equations that, in general, does not admit continuous solutions. In addition, the rod is assumed to be composed by N different materials, giving rise to N-1 stationary discontinuities in the strain field. The phenomenon is described in the reference configuration in a linear elasticity context, giving rise to N different propagation speeds. The (generalized) solution presents shock waves, even for cases involving continuous initial data. Simulations involving boundary conditions (not usual for hyperbolic problems) are considered in order to provide a way for describing the dynamics of finite rods.*

Keywords*. wave propagation, heterogeneous elastic rods, Riemann problem.*

1. Introduction

Some techniques for studying the mechanical response of materials are based on the propagation of stress waves. This provides information about the way solids behave when the forces acting on them are no longer in static equilibrium.

In this paper we shall discuss the dynamical response of a piecewise homogeneous (heterogeneous) elastic-linear rod left in a nonequilibrium state. In other words, our objective is to obtain the strain, the stress and the velocity fields starting from a given initial data and subjected (or not) to some boundary condition.

Mathematically, this one-dimensional phenomenon is represented, in the reference configuration, by a linear hyperbolic system of partial differential equations whose eigenvalues depend on the position. In fact, these eigenvalues are piecewise constant, since the rod is assumed to be piecewise homogeneous (composed by N different materials).

As it will be shown later, this hyperbolic system will not admit (in general) a solution in the Classical sense. So, it will be necessary to work with the jump conditions associated with the set of equations in order to deal with discontinuous functions (generalized solutions of the problem). A composition of these discontinuous functions will give rise to the complete solution of the problem.

2. Governing and constitutive equations

From Continuum Mechanics (Billington and Tate, 1981) we have that, for the one-dimensional phenomenon under study here,

$$\begin{aligned} &\frac{\partial e}{\partial t} - \frac{\partial v}{\partial X} = 0 \\ &r\,\frac{\partial v}{\partial t} - \frac{\partial s}{\partial X} = 0 \end{aligned} \tag{1}$$

where r is the mass density in the reference configuration (piecewise constant here), v is the velocity, s is the normal component of the Piola-Kirchhoff tensor and e is the strain. The first equation above represents a geometrical compatibility while, the second one, represents the linear momentum balance in the reference configuration. In both the equations t represents the time while X represents the position (in the reference configuration).

The strain field e is defined as

$$e = \frac{\partial x}{\partial X} - 1 \tag{2}$$

in which x represents the position in the current configuration (spatial position).

In this work it will be assumed (linear elasticity) that the Piola-Kirchhoff normal stress s is a piecewise linear function of the strain e . In other words,

$$s = c_i e, \text{ for } \quad X_i < X < X_{i+1} \tag{3}$$

where c_i is a positive constant. The mass density r will be assumed constant in $X_i < X < X_{i+1}$.

$$r = r_i = \text{constant, for } \quad X_i < X < X_{i+1} \tag{4}$$

3. The associated Riemann problem and its generalized solution

Let us consider now the following initial data problem (named Riemann problem)

$$\begin{aligned} &\frac{\partial e}{\partial t} - \frac{\partial v}{\partial X} = 0 \\ &r\frac{\partial v}{\partial t} - \frac{\partial s}{\partial X} = 0 \\ &(e,v) = (e_L, v_L) \quad \text{for} \quad X < X_0 \\ &(e,v) = (e_R, v_R) \quad \text{for} \quad X > X_0 \end{aligned} \tag{5}$$

in which e_L, e_R, v_L and v_R are known constants.

The solution of Eq. (5) consists of connecting the left state (e_L, v_L) to the right state (e_R, v_R) by means of rarefactions (continuous solutions) and/or shocks (discontinuities satisfying the entropy conditions). Two states are connected by a rarefaction if, and only if, between these states, the corresponding eigenvalue is an increasing function of the ratio $(X - X_0)/t$ (Smoller, 1983; Lax, 1971 and John, 1974).

The eigenvalues associated to the hyperbolic system are given, in crescent order, by

$$l_1 = -\left[\frac{s'}{r_i}\right]^{1/2} = -\left[\frac{c_i}{r_i}\right]^{1/2} \text{ and } l_2 = \left[\frac{s'}{r_i}\right]^{1/2} = \left[\frac{c_i}{r_i}\right]^{1/2} \text{ for } \quad X_i < X < X_{i+1} \tag{6}$$

So, if we assume that $X_i \to -\infty$ and that $X_{i+1} \to +\infty$, the solution of Eq. (5) will depend only on the ratio $(X - X_0)/t$ and, since the eigenvalues are constant, the solution (generalized) will be discontinuous. In other words, the left state (e_L, v_L) will be connected to an intermediate state (e^*, v^*) by a discontinuity (called 1-shock or back shock) while the right state (e_R, v_R) will be connected to this intermediate state by another discontinuity (called 2-shock or front shock). Since $l_1 < 0 < l_2$ we have, from the entropy (Keyfitz and Kranzer, 1978; Callen, 1960) conditions that the shock speed s_1 (back shock speed) is always negative while s_2 (front shock speed) is always positive.

The intermediate state (e^*, v^*) is obtained from the Rankine-Hugoniot jump conditions given, for this hyperbolic system, by (Slattery, 1972)

$$\frac{[v]}{[e]} = \frac{[s]}{[r v]} = -s \tag{7}$$

where s denotes the shock speed while the brackets denote the "jump".

From the equations represented in Eq. (7) we have that

$$\begin{aligned} &\frac{v_L - v^*}{e_L - e^*} = \frac{s_L - s^*}{r(v_L - v^*)} = -s_1 \\ &\frac{v^* - v_R}{e^* - e_R} = \frac{s^* - s_R}{r(v^* - v_R)} = -s_2 \end{aligned} \tag{8}$$

This set of equations gives rise to the following (generalized) solution

$$
e = \begin{cases} e_L & \text{for} \quad -\infty < (X - X_0)/t < s_1 \\ e^* & \text{for} \quad s_1 < (X - X_0)/t < s_2 \\ e_R & \text{for} \quad s_2 < (X - X_0)/t < \infty \end{cases}
\qquad
v = \begin{cases} v_L & \text{for} \quad -\infty < (X - X_0)/t < s_1 \\ v^* & \text{for} \quad s_1 < (X - X_0)/t < s_2 \\ v_R & \text{for} \quad s_2 < (X - X_0)/t < \infty \end{cases} \tag{9}
$$

where

$$
\begin{aligned}
e^* &= \frac{v_R - v_L}{2\sqrt{c_i / r_i}} + \frac{e_R + e_L}{2} \\
v^* &= \frac{v_R + v_L}{2} + \frac{e_R - e_L}{2}\sqrt{c_i / r_i} \\
s_1 &= -\sqrt{\frac{c_i}{r_i}} \\
s_2 &= \sqrt{\frac{c_i}{r_i}}
\end{aligned} \tag{10}
$$

It is to be noticed that the 1-shock and the 2-shock are contact discontinuities, since $s_1 = l_1$ and $s_2 = l_2$. So, there is no entropy generation associated with these shocks (Lax, 1971).

Figure (1) shows the above solution in the plane $X - t$. Figure (2) shows the strain and the velocity as functions of the ratio $(X - X_0)/t$.

Now, let us suppose that $X_i \to -\infty$, $X_{i+1} = X_0$ and $X_{i+2} \to \infty$. This case represents an infinite rod composed by two homogeneous parts.

In such a case the solution will present a stationary shock at the (reference) position X_{i+1}. The (generalized) solution of Eq. (5) will depend only on the ratio $(X - X_0)/t$ too. Nevertheless the 1-shock (left) and the 2-shock (right) speeds have different absolute values.

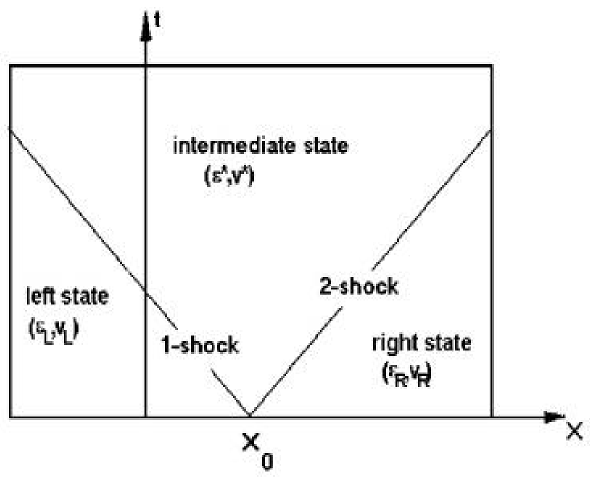


Figure 1. The solution of the Riemann problem (Eq. (5)) represented in the plane $X - t$ for the case in which $X_i \to -\infty$ and $X_{i+1} \to +\infty$, It is to be noticed that $|s_1| = |s_2|$.

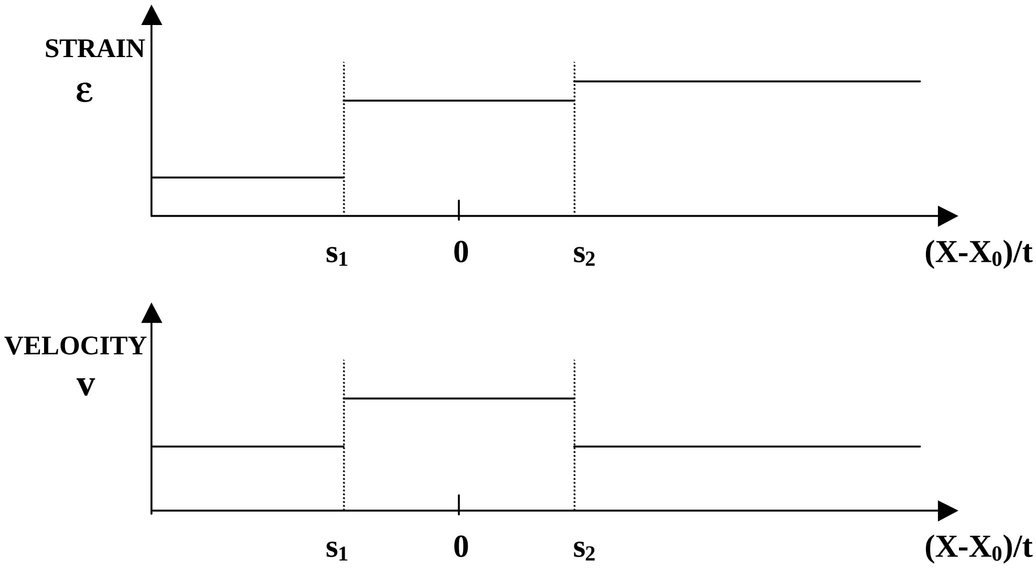


Figure 2. The solution of the Riemann problem (Eq. (5)), represented as a function of $(X - X_0)/t$, for a case in which $v_R = v_L$ and $e_R > e_L$. Since $s\ /e$ is a constant for $-\infty < X < \infty$, the stress behaves like the strain.

Since there exists an stationary shock at $X = X_{i+1} = X_0$, we conclude, from the jump conditions across this shock, that velocity and stress do not jump at this point. So, only the strain e jumps across the stationary shock and, since $[s\ ] = 0$, we have that (Keyfitz and Kranzer, 1978)

$$c_i e_-^* = c_{i+1} e_+^* \quad \text{with} \quad e_-^* = \lim_{X \to X_{i+1}, X < X_{i+1}} e \quad \text{and} \quad e_+^* = \lim_{X \to X_{i+1}, X > X_{i+1}} e \tag{11}$$

In this case, the jump conditions give rise to the following set of equations

$$\begin{aligned} \frac{v_L - v^*}{e_L - e_-^*} &= \frac{s_L - s^*}{r_i (v_L - v^*)} = -s_1 \\ \frac{v^* - v_R}{e_+^* - e_R} &= \frac{s^* - s_R}{r_{i+1} (v^* - v_R)} = -s_2 \end{aligned} \tag{12}$$

and the complete solution is given by

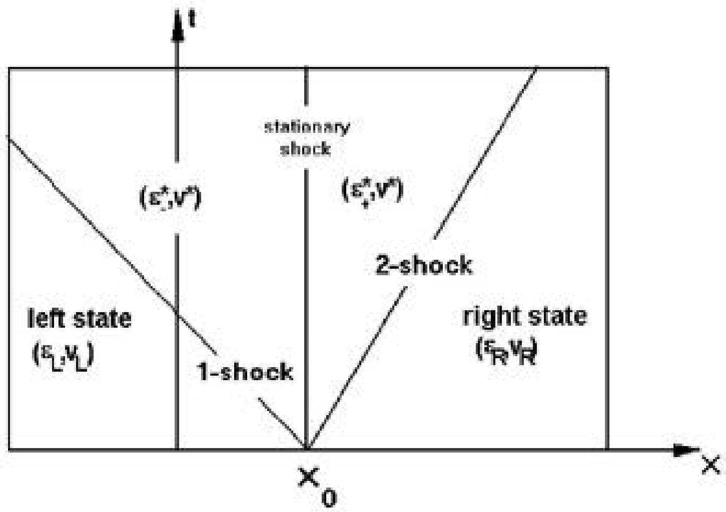
$$\begin{aligned} e &= \begin{cases} e_L & \text{for} \quad -\infty < (X - X_{i+1})/t < s_1 \\ e_-^* & \text{for} \quad s_1 < (X - X_{i+1})/t < 0 \\ e_+^* & \text{for} \quad 0 < (X - X_{i+1})/t < s_2 \\ e_R & \text{for} \quad s_2 < (X - X_{i+1})/t < \infty \end{cases} \\ v &= \begin{cases} v_L & \text{for} \quad -\infty < (X - X_{i+1})/t < s_1 \\ v^* & \text{for} \quad s_1 < (X - X_{i+1})/t < s_2 \\ v_R & \text{for} \quad s_2 < (X - X_{i+1})/t < \infty \end{cases} \end{aligned} \tag{13}$$

where

$$
\begin{aligned}
e_-^* &= \frac{c_{i+1}(v_R - v_L)}{c_{i+1}\sqrt{c_i / r_i} + c_i\sqrt{c_{i+1} / r_{i+1}}} + \frac{c_{i+1}(e_L\sqrt{c_i / r_i} + e_R\sqrt{c_{i+1} / r_{i+1}})}{c_{i+1}\sqrt{c_i / r_i} + c_i\sqrt{c_{i+1} / r_{i+1}}} \\
e_+^* &= \frac{c_i(v_R - v_L)}{c_{i+1}\sqrt{c_i / r_i} + c_i\sqrt{c_{i+1} / r_{i+1}}} + \frac{c_i(e_L\sqrt{c_i / r_i} + e_R\sqrt{c_{i+1} / r_{i+1}})}{c_{i+1}\sqrt{c_i / r_i} + c_i\sqrt{c_{i+1} / r_{i+1}}} \\
v^* &= \frac{c_{i+1}e_R - c_i e_L}{\sqrt{c_i r_i} + \sqrt{c_{i+1} r_{i+1}}} + \frac{v_L\sqrt{c_i r_i} + v_R\sqrt{c_{i+1} r_{i+1}}}{\sqrt{c_i r_i} + \sqrt{c_{i+1} r_{i+1}}} \\
s_1 &= -\sqrt{\frac{c_i}{r_i}} \\
s_2 &= \sqrt{\frac{c_{i+1}}{r_{i+1}}}
\end{aligned}
\tag{14}
$$

It is remarkable that Eq. (10) consists of a particular case of Eq. (14), obtained when $c_i = c_{i+1}$ and $r_i = r_{i+1}$. In this case there is no stationary jump at X_{i+1} and $e_-^* = e_+^* = e^*$ even for $(X - X_{i+1})/t = 0$.

Figure (3) presents the solution, obtained with the aid of Eq. (13), in the plane $X - t$, for a case in which $X_i \to -\infty$, $X_{i+1} = X_0$ and $X_{i+2} \to \infty$. The representation in the plane $X - t$ does not depend on the initial data (e_L, v_L)



and (e_R, v_R), once the propagation speeds do not depend on the states (e, v).

Figure 3. The solution of the Riemann problem (Eq. (5)) represented in the plane $X - t$ for a case in which $X_i \to -\infty$, $X_{i+1} = X_0$ and $X_{i+2} \to \infty$.

4. The associated Riemann problem for cases in which $X_0 \neq X_i$ for any i

Now we shall consider the cases in which the interface between two different materials (placed at any position X_i) does not coincide with the jump in the initial data (placed at X_0). Here, the solution of the Riemann problem will no longer depend on $(X - X_0)/t$. In fact, the solution will depend on $(X - X_0)/t$ only until a shock (front or back) reaches a stationary shock. At this point, a new Riemann problem arises, centered at the position of the stationary shock, having as "initial time" the time in which the shock interaction has been.

So, let us consider the problem defined by Eq. (5) assuming that $X_i < X_0 < X_{i+1}$. Since X_0 is different from any X_i, we have the 1-shock and the 2-shock centered at X_0 and a stationary shock at each X_i. The solution will depend on the ratio $(X - X_0)/t$ while there is no shock interaction between shocks. When the 1-shock reaches the stationary shock at $X = X_i$, the solution changes its behavior. In any case, the intermediate state becomes a new initial data (which respect to the time in which the shock interaction occurred) giving rise to a new Riemann problem. The solution of this new Riemann problem has always the same structure of Eq. (13) and, therefore, enable us to solve Eq. (5) for any piecewise initial data.

In order to illustrate the solution procedure, let us suppose a particular case in which (infinite rod composed by three different homogeneous parts) $X_1 \to -\infty$, $X_2 = -0.7L$, $X_0 = 0$, $X_3 = 0.3L$ and $X_4 \to \infty$ with $\sqrt{c_1 / r_1} = 3\sqrt{c_2 / r_2}$ and $\sqrt{c_3 / r_3} = 0.3\sqrt{c_2 / r_2}$.

Starting from the initial data

$$
\begin{aligned}
(e, v) &= (e_L, v_L) \quad \text{for} \quad X < X_0 \\
(e, v) &= (e_R, v_R) \quad \text{for} \quad X > X_0
\end{aligned}
\tag{15}
$$

we have the intermediate state (e^*, v^*) (called here (e_L, v_L)) given by Eq. (10). At the point “a” the front shock (with speed $\sqrt{c_2 / r_2}$) reaches the stationary shock placed at X_3. So, a new Riemann problem, centered at “a” arises, having as its left state (e_1, v_1) and at its right state (e_R, v_R). For this Riemann problem, the intermediate states (e_-^*, v^*) and (e_+^*, v^*) are given by Eq. (14). Repeating this procedure, we construct Fig. (4) below in which the solution is presented in the plane $X - t$.

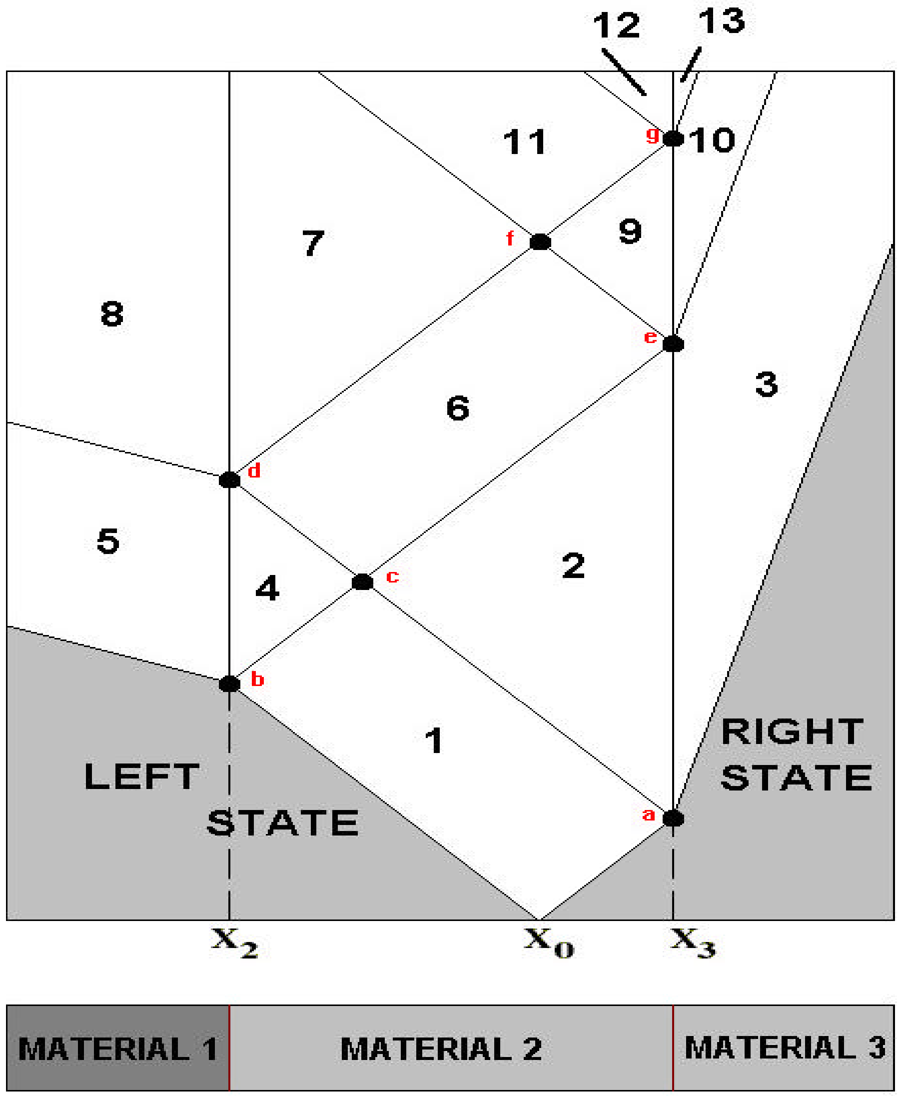


Figure 4. The solution of the Riemann problem represented in the plane $X - t$ for $X_1 \to \infty$, $X_2 = -0.7L$, $X_0 = 0$, $X_3 = 0.3L$ and $X_4 \to \infty$ with $\sqrt{c_1 / r_1} = 3\sqrt{c_2 / r_2}$ and $\sqrt{c_3 / r_3} = 0.3\sqrt{c_2 / r_2}$.

Table (1) shows the relationship between Eq. (14) and each one of the states presented in Fig. (4).

Table 1. The states 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 and 13 and their relation with Eq. (14).

Riemann problem centered at	**LEFT STATE**	**INTERMEDIATE STATE "-"** Eq.(14)	**INTERMEDIATE STATE "+"** Eq.(14)	**RIGHT STATE**
a	**1**	**2**	**3**	**R**
b	**L**	**5**	**4**	**1**
c	**4**	**6**	**6**	**2**
d	**5**	**8**	**7**	**6**
e	**6**	**9**	**10**	**3**
f	**7**	**11**	**11**	**9**
g	**11**	**12**	**13**	**10**

Since all the propagation speeds are previously known and constant, it is very easy to determine the time associated with each shock interaction (a, b, c, d, e, f and g). For instance, the point "a" is reached when $t = 0.3L\sqrt{r_2 / c_2}$. The point "e" is reached when $t = 1.7L\sqrt{r_2 / c_2}$. The point "c" is reached when $t = L\sqrt{r_2 / c_2}$ ...

5. Finite rods – problems involving boundary conditions

The tools presented up to this point are sufficient for describing wave propagation in rods in which an edge is assumed to be fixed $(v = 0)$ or free $(s = 0$ and $e = 0)$.

Such boundary conditions are automatically satisfied by means of the introduction of artificial states beyond the actual rod. In other words, for imposing a fixed edge at X_1, it is sufficient to assume the existence of a rod at the left ($X < X_1$), with a state such that $v^* = 0$. For imposing a fixed edge at X_{N+1}, it is sufficient to assume the existence of a rod at the right ($X_{N+1} > X$), with a state such that $v^* = 0$. On the other hand, for imposing a free edge boundary condition, it is sufficient to consider the artificial rod with a state such that $e^* = 0$. This can be done in an easy way too. The choice of the state in the artificial extension of the rod is done based on Eq. (10), once the material of the artificial extension can be the same of the actual rod.

For instance, let us consider a problem in which $X_1 = 0$, $X_2 = 4L$, $X_3 = 12L$, $X_0 = 7L$ and $\sqrt{c_1 / r_1} = 2\sqrt{c_2 / r_2}$. Two distinct situations will be simulated:

i. A rod fixed at both edges (with $r_2 = r_1$);

ii. A rod fixed at the left edge, with the right edge free (with $r_2 = r_1$).

Table (2) and Fig.(5) present some results associated with the above cases. The solution procedure is based on the employment of the Eq. (14) after each shock interaction. While Tab. (2) presents quantitative results for specified left and right states as well as given boundary conditions, Fig. (5) presents results, which do not depend on the initial data or boundary condition. This is a feature of this kind of hyperbolic system in which the eigenvalues (speeds of propagation) do not depend on the states.

In Fig. (5) the red dots (at the left) are associated with the center of Riemann problems constructed in order to satisfy the boundary condition at the left. The blue dots play the same role, at the right. The black dots indicate the shock interaction within a same material while, the green dots, indicate the interaction between a (front and back) shock and a stationary one.

Table 2. Some results obtained for cases i and ii, assuming the rod at rest for $t = 0$. Here $w = v\sqrt{r_2 / c_2}$.

case	e_L	e_R	e_1	w_1	e_2	w_2	e_3	w_3	e_4	w_4	e_5	w_5	e_6	w_6
i	**0.2**	**0.2**	**0.2**	**0.0**	**0.1**	**-0.2**	**0.4**	**-0.2**	**0.0**	**0.0**	**0.2**	**0.0**	**0.03**	**0.07**
ii	**0.2**	**0.2**	**0.2**	**0.0**	**0.1**	**-0.2**	**0.4**	**-0.2**	**0.0**	**0.0**	**0.0**	**-0.2**	**0.03**	**0.07**
i	**-0.4**	**-0.4**	**-0.4**	**-0.4**	**-0.2**	**0.4**	**-0.8**	**0.4**	**0.0**	**0.0**	**-0.4**	**0.0**	**-0.07**	**-0.13**
ii	**-0.4**	**-0.4**	**-0.4**	**-0.4**	**-0.2**	**0.4**	**-0.8**	**0.4**	**0.0**	**0.0**	**0.0**	**0.4**	**-0.07**	**-0.13**
i	**0.0**	**0.5**	**0.25**	**0.25**	**0.08**	**0.17**	**0.33**	**0.17**	**0.17**	**0.0**	**0.0**	**0.0**	**0.14**	**-0.06**
ii	**0.0**	**0.5**	**0.25**	**0.25**	**0.08**	**0.17**	**0.33**	**0.17**	**0.17**	**0.0**	**0.0**	**0.0**	**0.14**	**-0.06**
i	**0.0**	**-0.5**	**-0.25**	**-0.25**	**-0.08**	**-0.17**	**-0.33**	**-0.17**	**-0.17**	**0.0**	**0.0**	**0.0**	**-0.14**	**0.06**
ii	**0.0**	**-0.5**	**-0.25**	**-0.25**	**-0.08**	**-0.17**	**-0.33**	**-0.17**	**-0.17**	**0.0**	**0.0**	**0.0**	**-0.14**	**0.06**
i	**0.8**	**0.4**	**0.6**	**-0.2**	**0.33**	**-0.93**	**1.33**	**-0.93**	**-0.13**	**0.0**	**0.8**	**0.0**	**0.02**	**0.09**
ii	**0.4**	**0.6**	**0.5**	**0.1**	**0.23**	**-0.33**	**0.93**	**-0.33**	**0.07**	**0.0**	**0.0**	**-0.4**	**0.12**	**0.11**
ii	**-0.5**	**-0.1**	**-0.3**	**0.2**	**-0.18**	**0.63**	**-0.73**	**0.63**	**0.13**	**0.0**	**0.0**	**0.5**	**-0.21**	**-0.21**

6. Final Remarks

The tools presented in this paper allow, in a very simple way, the simulation of any initial data problem (even with boundary conditions) involving linear-elastic rods.

For cases in which the mass density and the ratio stress/strain are not piecewise constant functions (that were not considered here) the previously presented results are available too, provided these fields be approximated by piecewise constant ones. This can be done, for instance, choosing, between X_i and X_{i+1}, the mean value of the mass density and the mean value of the ratio stress/strain.

So, our original problem defined by

$$\begin{aligned} &\frac{\partial e}{\partial t} - \frac{\partial v}{\partial X} = 0 \\ &r\frac{\partial v}{\partial t} - \frac{\partial s}{\partial X} = 0, \quad \text{with} \quad s = \tilde{c}(X)e \quad \text{and} \quad r = \tilde{r}(X) \\ &(e,v) = (e_L, v_L) \quad \text{for} \quad X < X_0 \\ &(e,v) = (e_R, v_R) \quad \text{for} \quad X > X_0 \\ &\text{and boundary conditions} \end{aligned} \tag{16}$$

is replaced by the following one

$$\begin{aligned} &\frac{\partial e}{\partial t} - \frac{\partial v}{\partial X} = 0 \\ &r\frac{\partial v}{\partial t} - \frac{\partial s}{\partial X} = 0, \text{ with } \; s = c_i e \; \text{ and } \; r = r_i \text{ for } X_i < X < X_{i+1} \\ &(e,v) = (e_L, v_L) \quad \text{for} \quad X < X_0 \\ &(e,v) = (e_R, v_R) \quad \text{for} \quad X > X_0 \\ &\text{and boundary conditions} \end{aligned} \tag{17}$$

where

$$c_i = \frac{1}{X_{i+1} - X_i}\left(\int_{X_i}^{X_{i+1}} \tilde{c}(X)dX\right) \quad \text{and} \quad r_i = \frac{1}{X_{i+1} - X_i}\int_{X_i}^{X_{i+1}} \tilde{r}(X)dX \tag{18}$$

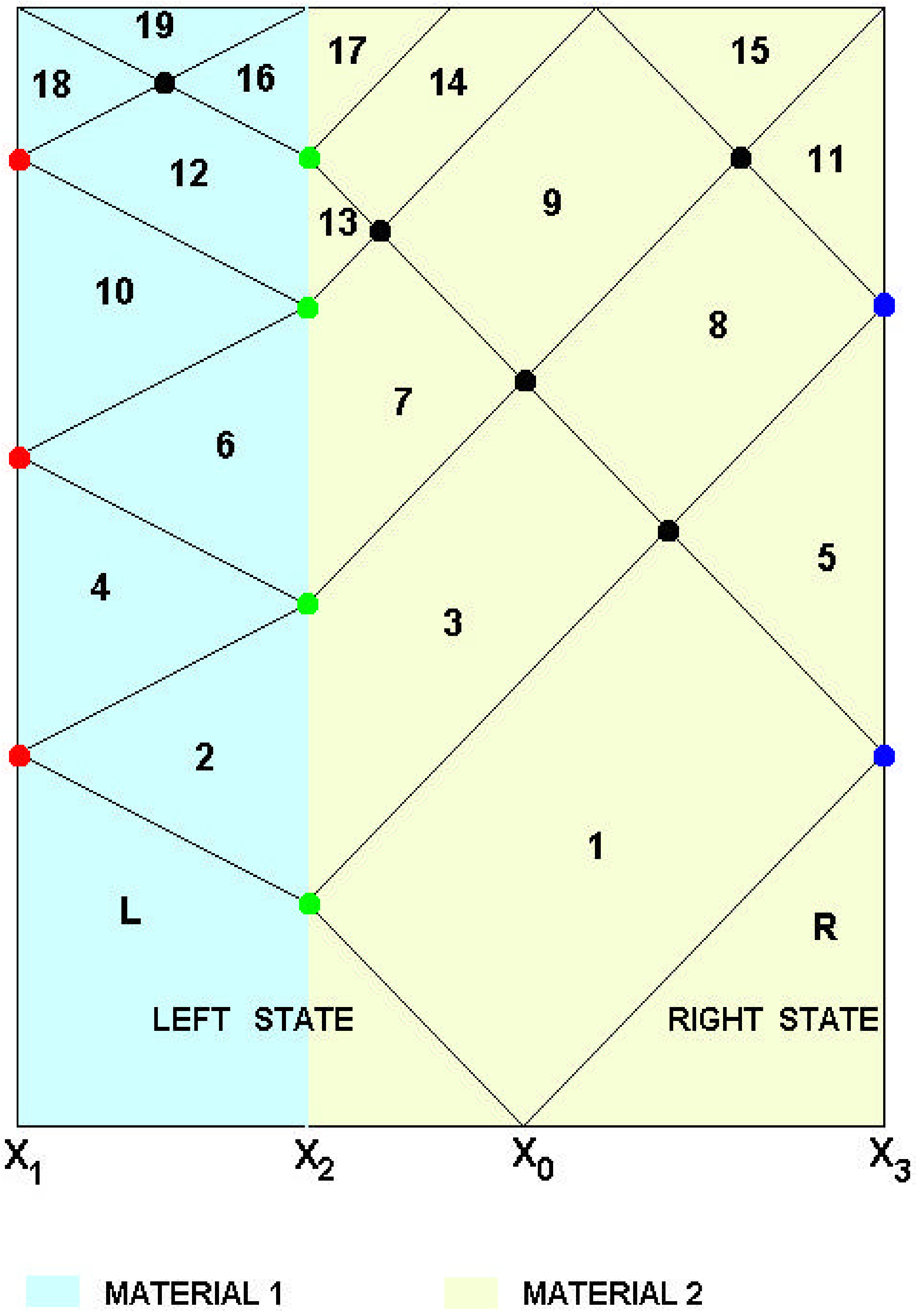


Figure 5. The solution (for cases i and ii) represented in the plane $X - t$, for a finite rod with length $12L$.

7. Acknowledgements

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