TRANSVERSE VIBRATION ANALYSIS OF ROTATING SHAFTS

Victor Fernando Deorsola Sacramento
Instituto Tecnológico de Aeronáutica – ITA
Div. de Engenharia Mecânica-Aeronáutica, 12228-900, São José dos Campos – SP Brazil
victorsacramento@uol.com.br

João Carlos Menezes
Instituto Tecnológico de Aeronáutica - ITA
Div. de Eng. Mecânica - Aeronáutica, 12228-901, São José dos Campos - SP – Brazil
menezes@mec.ita.br

Abstract: The aim of this work is to study transverse vibration of rotating shafts, with an approach that makes possible, during design, to prevent unstable vibration at working speed. The Finite Element Method is used and the motion equations are obtained from Lagrange equations which describe the motion in two transverse planes. A local rotating coordinate system is used and for the shaft's motion matrix equation a gyroscopic matrix has to be taken in account. The beam elements have two nodes and six degrees of freedom in each node, three displacements and three rotations. As a result, each element produces twelve differential equations. The Mathematica® software was used to obtain the mass, the gyroscopic, and the stiffness matrices, by integration of the kinetic and the potential energies, and by differentiation of the energies according to Lagrange equations. With the matrices obtained, a computer software written in Fortran accomplishes the assembly of the global matrices of the shaft, and simulates the shaft's response, according to established dimensions, material, angular velocity. This software has been successfully tested taking a classical case as a reference. Comparison of the results obtained with the analytical model, that doesn't include unbalancing effects, reveasl a very good agreement. These results are presented in this work.
Keywords. Mechanical Vibrations, Rotating Shafts, Gyroscopic effect

1. INTRODUCTION

Design of rotating equipment operating smoothly under various conditions of speed, has been a challenge to designers, due to the fact that operating speeds are increasing every day and equipments have to work satisfactorily in low and high speeds.

Designers have to obtain the critical speed of a system, and compare with the operating range, to make sure that the motion will be stable and low-level amplitude of vibration prevails.

Under certain conditions, a rotating shaft can perform a motion with very high amplitude of vibration. This motion, can lead to the destruction of the system. The aim of this work is to study transverse vibration of rotating shafts, with an approach that makes possible, during design, to preview unstable vibration at working speed. The subject is most interesting in the design of aeronautic turbines shafts.

The Finite Element Method is used, adopting beam elements with two nodal points and six degree of freedom for each node. The motion equations are obtained from Lagrange's equations, and describe the motion in two transverse planes. A local rotating coordinate system is used and shaft's matrix equation of motion presents a characteristic gyroscopic matrix.

Each node of the finite element has six degrees of freedom, three displacements and three rotations. As a result, for each element, twelve differential equations have to be accounted for. To obtain the mass, gyroscopic and stiffness matrix, the software Mathematica® was used for the integrations of the kinetic and potential energy, and for the differentiation required by Lagrange's equations. A Fortran computer software was written, to assembly the shaft element matrices into global matrices, and to simulate the shaft response, for several dimensions, materials, angular velocities and unbalanced loads.

To obtain the time domain response, the integration procedure of Newmark was adopted. This software was tested, and numerical results were compared to a classical case reported by the literature. Results reveal a good agreement with the analytical solution of the problem.

2. MATHEMATICAL DEVELOPMENT

Consider a vibrating shaft rotating about axis X with constant angular velocity Ω relative to the inertial axes XYZ. A set of axes xyz rotating with the shaft is adopted according to Figure (1). The displacement vector can be written in the vector form as

$$\vec{u} = uy \cdot e\vec{y} + uz \cdot e\vec{z} \tag{1}$$

where $e\vec{y}$ and $e\vec{z}$ are unit vectors according to the rotating xyz axes and uy and uz are the corresponding displacement, according to Figure 2.

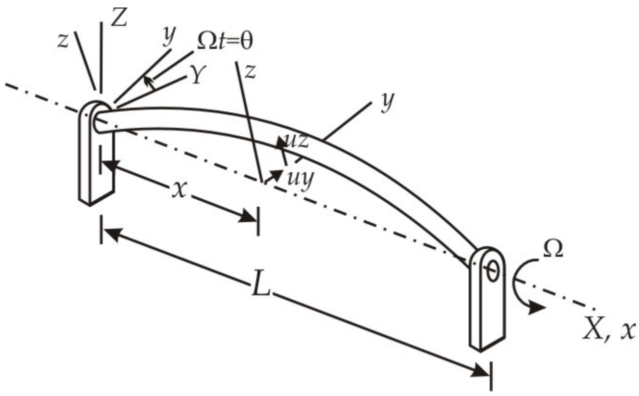

Figure 1 – Rotating elastic shaft in bending simply supported at both ends

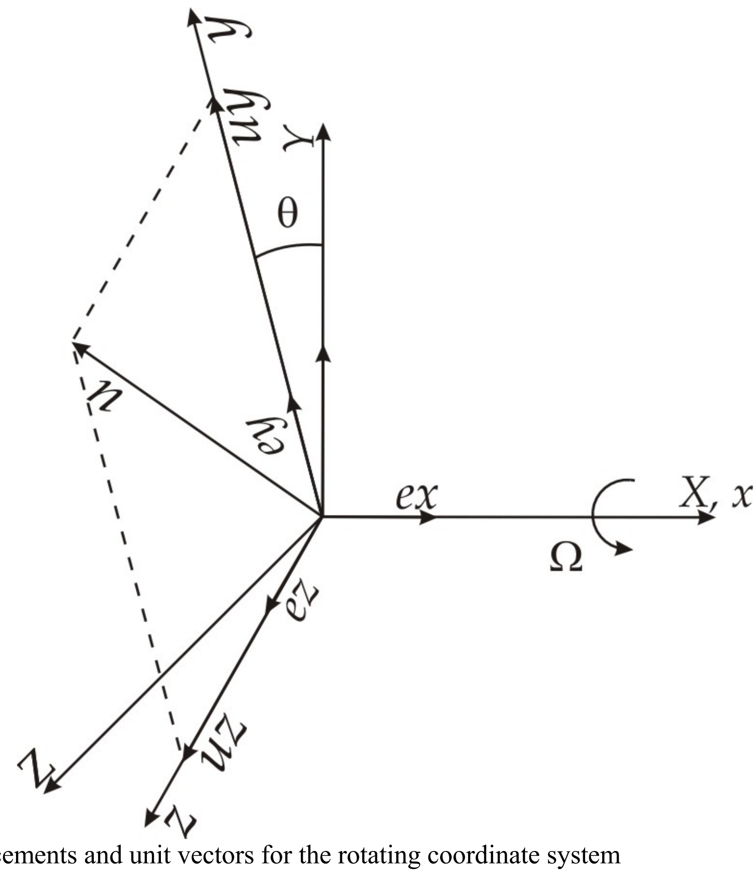

Figure 2 – Displacements and unit vectors for the rotating coordinate system

Differentiating the preceding equation with respect to time, one may obtain the velocity vector of a typical point on the shaft as

$$\frac{\partial \vec{u}}{\partial t} = \left(\frac{\partial uy}{\partial t} - uz \cdot \Omega \right) \vec{ey} + \left(\frac{\partial uz}{\partial t} + uy \cdot \Omega \right) \cdot \vec{ez} \quad (2)$$

The kinetic energy T of the shaft can be stated as (Meirovitch, 1986 and 1997)

$$T = T\ transversal + T\ longitudinal + T\ rotational \quad (3)$$

where

$$Tlongitudinal = \frac{1}{2} \cdot \int_0^l m \cdot \left(\frac{\partial ux}{\partial t} \right)^2 dx \quad (4)$$

where m= mass per unit length of the shaft and

$$ym = y + e \cdot \cos(\alpha) \quad (5)$$

$$zm = z + e \cdot sin(\alpha) \quad (6)$$

where y and z are geometric center coordinates of the cross sectional area of the shaft, e is the distance from geometric center to gravity center, an α is the initial angular position of the gravity center

$$Tlongitudinal = \frac{1}{2} \cdot \int_0^l m \cdot \left(\frac{\partial ux}{\partial t} \right)^2 dx \quad (7)$$

where ux=longitudinal displacement

$$Trotacional = \frac{1}{2} \cdot \int_0^l Ip \cdot \left(\frac{\partial \theta}{\partial t} \right)^2 dx \quad (8)$$

where Ip = mass polar moment of inertia of the cross-sectional área of the shaft.

The potential energy V is considered as

$$V = V\ transversal + V\ longitudinal + V\ rotational \quad (9)$$

where

$$Vtransversal = \frac{1}{2} \cdot \int_0^l E \cdot Ia \cdot \left(\frac{\partial^2 u}{\partial x^2} \right)^2 dx \quad (10)$$

where E = elasticity modulus and Ia = area momentof inertia of the cross - sectional area of the shaft

$$Vlongitudinal = \frac{1}{2} \cdot \int_0^l E \cdot A \cdot \left(\frac{\partial ux}{\partial x} \right)^2 dx \quad (11)$$

where A = Cross - sectional area

$$Vrotational = \frac{1}{2} \cdot \int_0^l G \cdot Jp \cdot \left(\frac{\partial \theta}{\partial x} \right)^2 dx \quad (12)$$

where G = shear modulus and J p = polar moment of inertia of the cross - sectional areaof the shaft

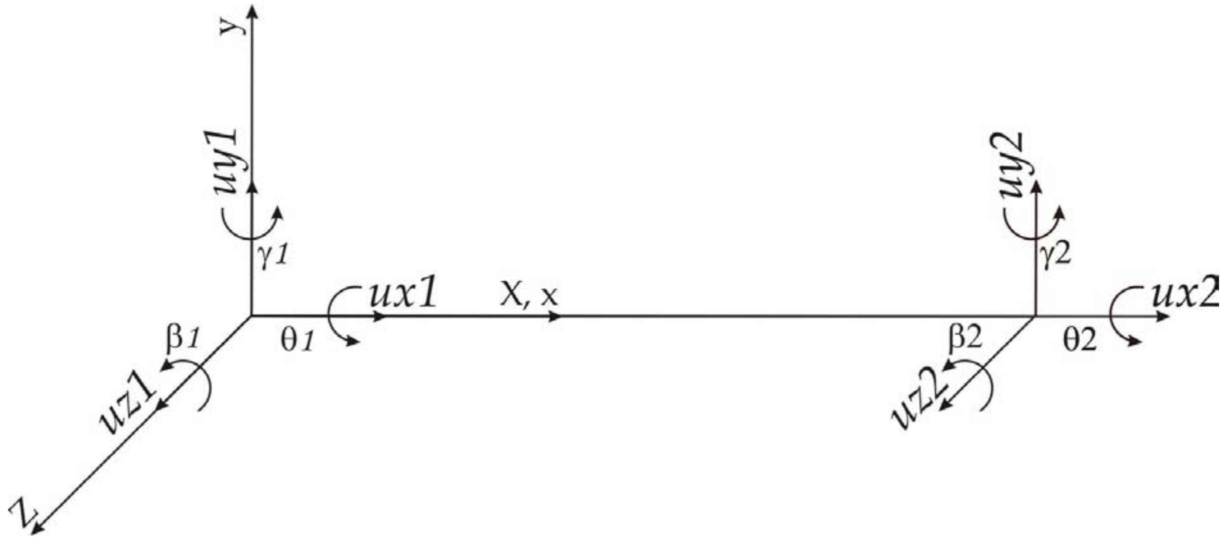

Figure 3 – Nodal displacements for a two noded beam finite element

A two noded beam element, with six degrees of freedom for each node, was adopted for the finite element model, as seen in Figure 3.

To obtain the displacemets along the element, the following shape funtions should be used (Meirovitch, 1986):

$$uy(x) = \left(1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}\right) \cdot uy1 + \left(\frac{x}{l} - \frac{2x^2}{l^2} + \frac{x^3}{l^3}\right) \cdot l \cdot \beta 1 + \left(\frac{3x^2}{l^2} - \frac{2x^3}{l^3}\right) \cdot uy2 - \left(\frac{x^2}{l^2} - \frac{x^3}{l^3}\right) \cdot l \cdot \beta 2 \quad (13)$$

$$uz(x) = \left(1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}\right) \cdot uz1 - \left(\frac{x}{l} - \frac{2x^2}{l^2} + \frac{x^3}{l^3}\right) \cdot l \cdot \gamma 1 + \left(\frac{3x^2}{l^2} - \frac{2x^3}{l^3}\right) \cdot uz2 + \left(\frac{x^2}{l^2} - \frac{x^3}{l^3}\right) \cdot l \cdot \gamma 2 \quad (14)$$

$$ux(x) = \left(1 - \frac{x}{l}\right) \cdot ux1 + \left(\frac{x}{l}\right) \cdot ux2 \quad (15)$$

$$\theta(x) = \left(1 - \frac{x}{l}\right) \cdot \theta 1 + \left(\frac{x}{l}\right) \cdot \theta 2 \quad (16)$$

Replacing Eqs. (13), (14), (15) and (16) into Eqs. (4), (7), (8), (10), (11) and (12), one may obtain the total kinetic and potencial energies for the element, according to Eqs. (3) and (9). Equations (3) and (9) can, finally, be expressed as a function of the nodal displacements $ux1, \theta 1, uy1, \gamma 1, uz1, \beta 1, ux2, \theta 2, uy2, \gamma 2, uz2$ and $\beta 2$. To accomplish the integrations involved in the equations of kinetic and potential energies dependent on the variable x, software Mathematica® was used. In order to obtain the equations of motion for the finite element, Lagrange's equations were adopted (Meirovitch, 1986 and 1997):

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}i}\right) - \frac{\partial T}{\partial qi} + \frac{\partial V}{\partial qi} = Qi \quad (17)$$

Where

T = element's kinetic energy
V = element's potential energy
qi = generalized coordinates
Qi = generalized forces

To accomplish the differentiations involved in Lagrange's equations, the software Mathematica® was utilized. Taking the nodal displacements $ux1, \theta 1, uy1, \gamma 1, uz1, \beta 1, ux2, \theta 2, uy2, \gamma 2, uz2$ and $\beta 2$ as generalized coordinates in Lagrange's equations, the equation of motion can be derived and expressed in the matrix form:

$$[m] \cdot \{\ddot{q}\} + [c] \cdot \{\dot{q}\} + [k] \cdot \{q\} = \{Q\} \quad (18)$$

Where

$[\mathrm{m}]$ = mass matrix
$\{\ddot{\mathrm{q}}\}$ = acceleration vector
$[\mathrm{c}]$ = gyroscopic matrix
$\{\dot{\mathrm{q}}\}$ = velocity vector
$[\mathrm{k}]$ = stiffness matrix
$\{\mathrm{q}\}$ = displacement vector
$\{\mathrm{Q}\}$ = unbalancing forces vector

For the matrix equation of motion, the following matrices were obtained:

$$[m]=\begin{bmatrix}
\frac{lm}{3} & 0 & 0 & 0 & 0 & 0 & \frac{lm}{6} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{Il}{3} & 0 & 0 & 0 & 0 & 0 & \frac{Il}{6} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{13lm}{35} & \frac{11l^2m}{210} & 0 & 0 & 0 & 0 & \frac{9lm}{70} & \frac{-13l^2m}{420} & 0 & 0 \\
0 & 0 & \frac{11l^2m}{210} & \frac{l^3m}{105} & 0 & 0 & 0 & 0 & \frac{13l^2m}{420} & \frac{-l^3m}{140} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{13lm}{35} & \frac{-11l^2m}{210} & 0 & 0 & 0 & 0 & \frac{9lm}{70} & \frac{13l^2m}{420} \\
0 & 0 & 0 & 0 & \frac{-11l^2m}{210} & \frac{l^3m}{105} & 0 & 0 & 0 & 0 & \frac{-13l^2m}{420} & \frac{l^3m}{140} \\
\frac{lm}{6} & 0 & 0 & 0 & 0 & 0 & \frac{lm}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{Il}{6} & 0 & 0 & 0 & 0 & 0 & \frac{Il}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{9lm}{70} & \frac{13l^2m}{420} & 0 & 0 & 0 & 0 & \frac{13lm}{35} & \frac{-11l^2m}{210} & 0 & 0 \\
0 & 0 & \frac{-13l^2m}{420} & \frac{-l^3m}{140} & 0 & 0 & 0 & 0 & \frac{-11l^2m}{210} & \frac{l^3m}{105} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{9lm}{70} & \frac{-13l^2m}{420} & 0 & 0 & 0 & 0 & \frac{13lm}{35} & \frac{11l^2m}{210} \\
0 & 0 & 0 & 0 & \frac{13l^2m}{420} & \frac{-l^3m}{140} & 0 & 0 & 0 & 0 & \frac{11l^2m}{210} & \frac{l^3m}{105}
\end{bmatrix} \qquad (19)$$

$$
[c]=\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{-26lm\Omega}{35} & \frac{11l^2m\Omega}{105} & 0 & 0 & 0 & 0 & \frac{-9lm\Omega}{35} & \frac{-13l^2m\Omega}{210} \\
0 & 0 & 0 & 0 & \frac{-11l^2m\Omega}{105} & \frac{2l^3m\Omega}{105} & 0 & 0 & 0 & 0 & \frac{-13l^2m\Omega}{210} & \frac{-1l^3m\Omega}{70} \\
0 & 0 & \frac{26lm\Omega}{35} & \frac{11l^2m\Omega}{105} & 0 & 0 & 0 & 0 & \frac{9lm\Omega}{35} & \frac{-13l^2m\Omega}{210} & 0 & 0 \\
0 & 0 & \frac{-11l^2m\Omega}{105} & \frac{-2l^3m\Omega}{105} & 0 & 0 & 0 & 0 & \frac{-13l^2m\Omega}{210} & \frac{1l^3m\Omega}{70} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{-9lm\Omega}{35} & \frac{13l^2m\Omega}{210} & 0 & 0 & 0 & 0 & \frac{-26lm\Omega}{35} & \frac{-11l^2m\Omega}{105} \\
0 & 0 & 0 & 0 & \frac{13l^2m\Omega}{210} & \frac{-1l^3m\Omega}{70} & 0 & 0 & 0 & 0 & \frac{11l^2m\Omega}{105} & \frac{2l^3m\Omega}{105} \\
0 & 0 & \frac{9lm\Omega}{35} & \frac{13l^2m\Omega}{210} & 0 & 0 & 0 & 0 & \frac{26lm\Omega}{35} & \frac{-11l^2m\Omega}{105} & 0 & 0 \\
0 & 0 & \frac{13l^2m\Omega}{210} & \frac{1l^3m\Omega}{70} & 0 & 0 & 0 & 0 & \frac{11l^2m\Omega}{105} & \frac{-2l^3m\Omega}{105} & 0 & 0
\end{bmatrix} \tag{20}
$$

$$
[k]=[k1]+[k2] \tag{21}
$$

$$
[k1]=\begin{bmatrix}
\frac{AE}{l} & 0 & 0 & 0 & 0 & 0 & \frac{-AE}{l} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & \frac{-GJ}{L} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{5040IE}{420l^3} & \frac{2520IEl}{420l^3} & 0 & 0 & 0 & 0 & \frac{-5040IE}{420l^3} & \frac{2520IEl}{420l^3} & 0 & 0 \\
0 & 0 & \frac{2520IE}{420l^2} & \frac{1680IEl}{420l^2} & 0 & 0 & 0 & 0 & \frac{-2520IE}{420l^2} & \frac{280IEl}{140l^2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{5040IE}{420l^3} & \frac{-2520IEl}{420l^3} & 0 & 0 & 0 & 0 & \frac{-5040IE}{420l^3} & \frac{-2520IEl}{420l^3} \\
0 & 0 & 0 & 0 & \frac{-2520IE}{420l^2} & \frac{1680IEl}{420l^2} & 0 & 0 & 0 & 0 & \frac{2520IE}{420l^2} & \frac{280IEl}{140l^3} \\
\frac{-AE}{l} & 0 & 0 & 0 & 0 & 0 & \frac{AE}{l} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-GJ}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-5040IE}{420l^3} & \frac{-2520IEl}{420l^3} & 0 & 0 & 0 & 0 & \frac{5040IE}{420l^3} & \frac{-2520IEl}{420l^3} & 0 & 0 \\
0 & 0 & \frac{2520IE}{420l^2} & \frac{280IEl}{140l^2} & 0 & 0 & 0 & 0 & \frac{-2520IE}{420l^2} & \frac{1680IEl}{420l^2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{-5040IE}{420l^3} & \frac{2520IEl}{420l^3} & 0 & 0 & 0 & 0 & \frac{5040IE}{420l^3} & \frac{2520IEl}{420l^3} \\
0 & 0 & 0 & 0 & \frac{-2520IE}{420l^3} & \frac{280IEl}{140l^3} & 0 & 0 & 0 & 0 & \frac{2520IE}{420l^2} & \frac{1680IEl}{420l^2}
\end{bmatrix} \tag{22}
$$

$$[k1]=\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-156l^4m\Omega^2}{420l^3} & \frac{-22l^5m\Omega^2}{420l^3} & 0 & 0 & 0 & 0 & \frac{-54l^4m\Omega^2}{420l^3} & \frac{13l^5m\Omega^2}{420l^3} & 0 & 0 \\
0 & 0 & \frac{-22l^4m\Omega^2}{420l^2} & \frac{-4l^5m\Omega^2}{420l^2} & 0 & 0 & 0 & 0 & \frac{-13l^4m\Omega^2}{420l^2} & \frac{l^5m\Omega^2}{140l^2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{-156l^4m\Omega^2}{420l^3} & \frac{22l^5m\Omega^2}{420l^3} & 0 & 0 & 0 & 0 & \frac{-54l^4m\Omega^2}{420l^3} & \frac{-13l^5m\Omega^2}{420l^3} \\
0 & 0 & 0 & 0 & \frac{22l^4m\Omega^2}{420l^2} & \frac{-4l^5m\Omega^2}{420l^2} & 0 & 0 & 0 & 0 & \frac{13l^4m\Omega^2}{420l^2} & \frac{l^5m\Omega^2}{140l^2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-54l^4m\Omega^2}{420l^3} & \frac{-13l^5m\Omega^2}{420l^3} & 0 & 0 & 0 & 0 & \frac{-156l^4m\Omega^2}{420l^3} & \frac{22l^5m\Omega^2}{420l^3} & 0 & 0 \\
0 & 0 & \frac{13l^4m\Omega^2}{420l^2} & \frac{l^5m\Omega^2}{140l^2} & 0 & 0 & 0 & 0 & \frac{22l^4m\Omega^2}{420l^2} & \frac{-4l^5m\Omega^2}{420l^2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{-54l^4m\Omega^2}{420l^3} & \frac{13l^5m\Omega^2}{420l^3} & 0 & 0 & 0 & 0 & \frac{-156l^4m\Omega^2}{420l^3} & \frac{-22l^5m\Omega^2}{420l^3} \\
0 & 0 & 0 & 0 & \frac{-13l^4m\Omega^2}{420l^2} & \frac{l^5m\Omega^2}{140l^2} & 0 & 0 & 0 & 0 & \frac{-22l^4m\Omega^2}{420l^2} & \frac{-4l^5m\Omega^2}{420l^2}
\end{bmatrix} \quad (22)$$

The displacement vector in the matrix equation is

$$\{q\}=[ux1 \quad \theta 1 \quad uy1 \quad \beta 1 \quad uz1 \quad \gamma 1 \quad ux2 \quad \theta 2 \quad uy2 \quad \beta 2 \quad uz2 \quad \gamma 2]^T \quad (23)$$

Then unbalancing force vector is

$$\{q\}=[0 \quad 0 \quad Qy1 \quad Q\beta 1 \quad Qz1 \quad Q\gamma 1 \quad 0 \quad 0 \quad Qy2 \quad Q\beta 2 \quad Qz2 \quad Q\gamma 2]^T \quad (24)$$

Where, considering the eccentricity e constant along the shaft

$$Qy1 = elm\Omega^2 \cos(\alpha)/2 \quad (25)$$

$$Q\beta 1 = el^2 m\Omega^2 \cos(\alpha)/12 \quad (26)$$

$$Qz1 = elm\Omega^2 \sin(\alpha)/2 \quad (27)$$

$$Q\gamma 1 = -el^2 m\Omega^2 \cos(\alpha)/12 \quad (28)$$

$$Qy2 = elm\Omega^2 \cos(\alpha)/2 \quad (29)$$

$$Q\beta 2 = -el^2 m\Omega^2 \cos(\alpha)/12 \quad (30)$$

$$Qz2 = elm\Omega^2 \sin(\alpha)/2 \quad (31)$$

$$Q\gamma 2 = el^2 m\Omega^2 \cos(\alpha)/12 \quad (32)$$

A Fortran computer software was written, to assembly the shaft element matrices into global matrices, and simulates the shaft response, for several dimensions, materials, angular velocities and loads. To obtain the time domain response, the integration procedure of Newmark was adopted.

This software has been successfully tested taking a classical case as a reference. Comparison of the results obtained with the analytical model and the numerical results reveal a very good agreement.

3. RESULTS

An analytical solution of a rotating elastic shaft symply supported at both ends has been presented by Meirovitch, 1997. The free vibration solution was obtained in the time domain for the local rotating set of axes. In this work, Meirovitch's solution is used as a reference for the finite element model test. Therefore, some conditions were estabilished for comparison. Since Meirovitch didn´t included unbalancing efects, the present results are for free vibration.

A constant cross-section steel shaft 400mm long and 25mm diameter has been analysed. For the shaft simply supported at both ends and the finite element model having 20 elements, it has been found that the first natural lateral frequency of the non-rotating shaft is 1986.1 rad/s. For both models, results were obtained for the rotating speed at 50%, 100%, 200%, and 400% of the first natural frequency. These results are presented in local coordinates for the central node of the shaft, acording to Figures (4), (5), (6) and (7).

Analytical Result **Numerical Result**

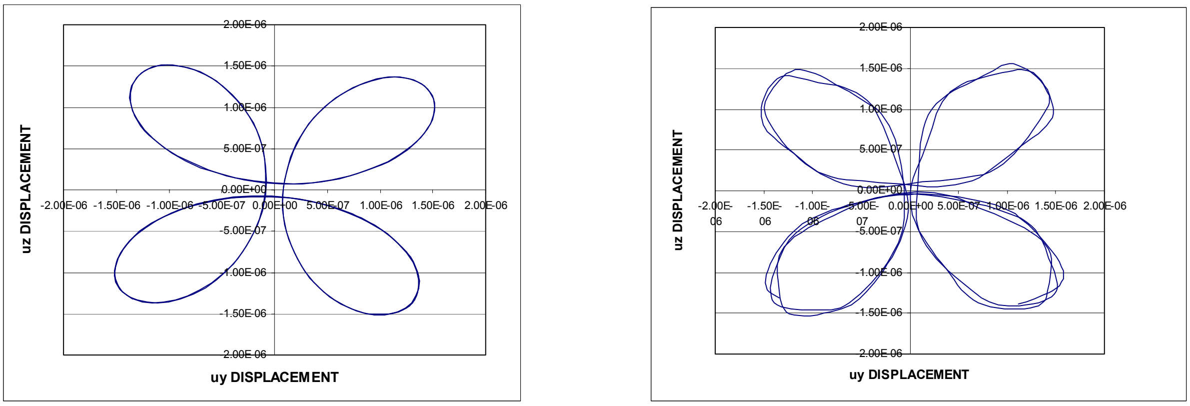

Figure 4 – Local trajectory of the shaft for the rotating speed equal to 50% of the first frequency

Analytical Result **Numerical Result**

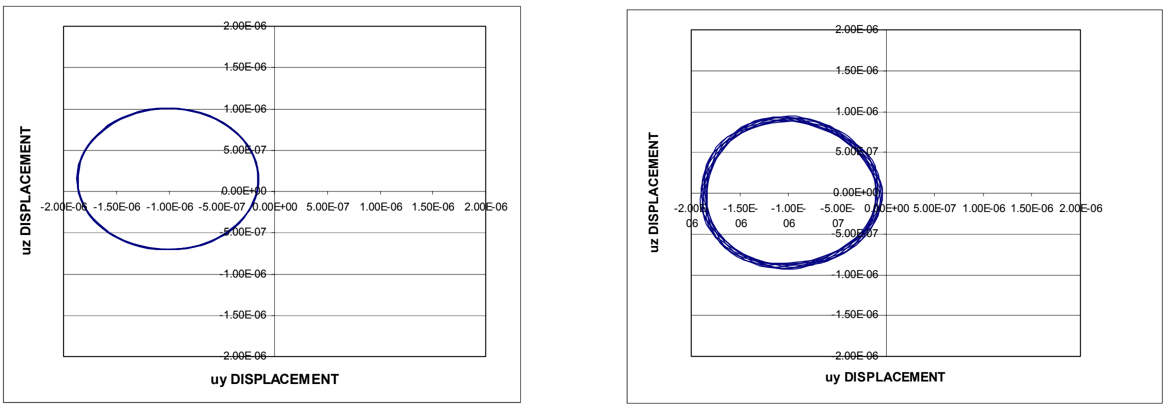

Figure 5 – Local trajectory of the shaft for the rotating speed equal to 100% of the first frequency

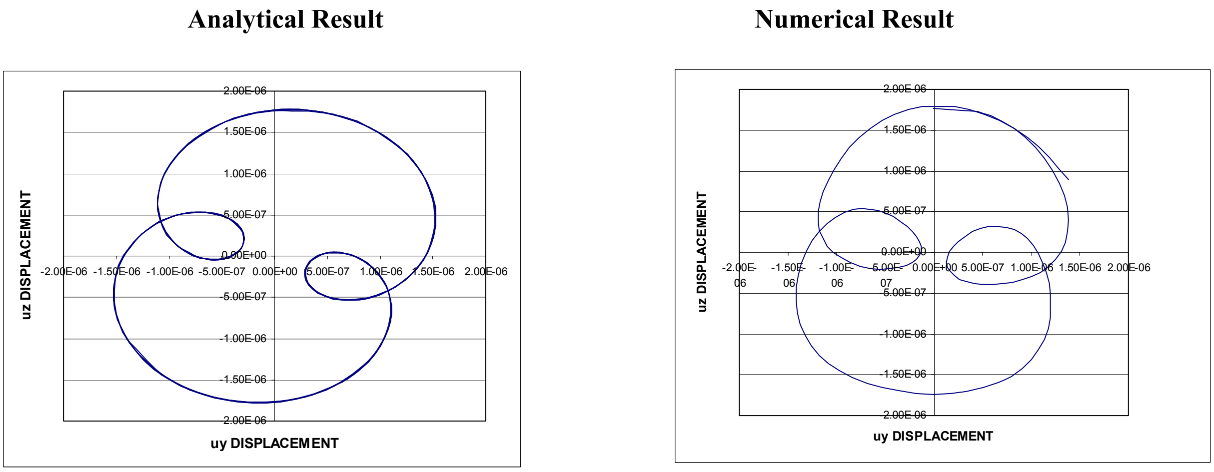

Figure 6 – Local trajectory of the shaft for the rotating speed equal to 200% of the first frequency

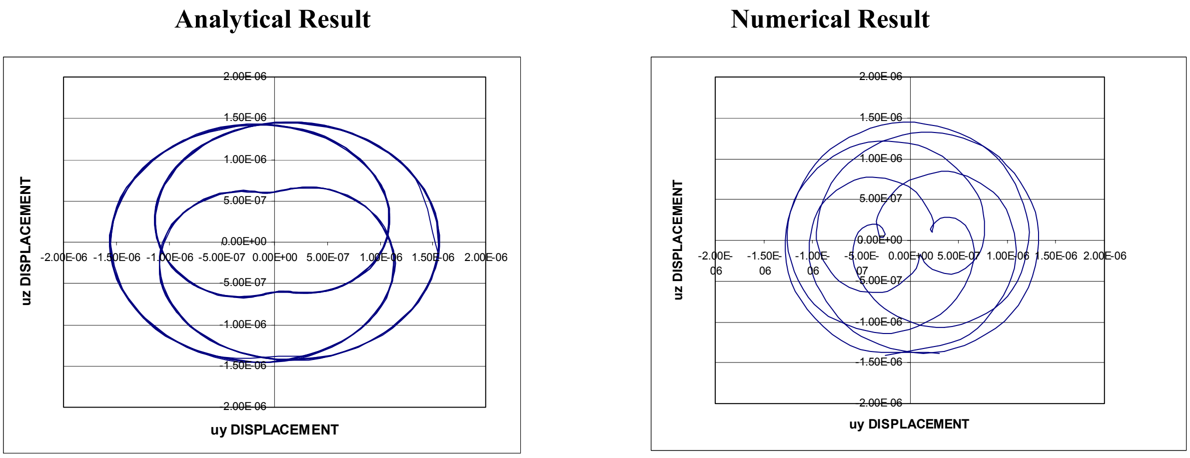

Figure 7 – Local trajectory of the shaft for the rotating speed equal to 400% of the first frequency

4. CONCLUSION

The present work presents a finite element model which allows a vibration analysis of a rotating shaft. One can notice that the mass, stiffness and gyroscopic matrices for the rotating velocity iqual to zero, are those classical matrices for the nonrotating case. The present results show additional matrix elements in the stiffness matrix and the occurance of the gyroscopic matrix when rotation is considered. It also can be observed that the angular velocity Ω does not affect the mass matrix. One particular aspect of the matrix equation of motion is the anti-symmetric characteristic of the giroscopic matrix, which has to be taken into account for the correct solution of the matrix equation.

For the purpose of corroborating the model, numerical results for some cases were compared to results obtained from a analytical solution proposed by Meirovitch, 1997, which predicts the response of a constant cross section of a rotating shaft symply supported at both ends. For comparision, four cases were devised. For the same shaft, results were obtained for angular velocities Ω iqual to 50%, 100%, 200% and 400% of the first natural frequency of the same nonrotating shaft. Local trajectory of the shaft for the four rotating speeds are presented by Figures (4), (5), (6) and (7), considering both models. These results show a very good agreement for the first three cases and a poorer match for the 400 % case.

One has to consider also the qualitative aspects of the results, which does not show any tendency of resonance, as it should be, taking into account that the free motion of the shaft has been analysed.

A more general model for the rotating shaft coud also be considered, which is not the purpose of this work. One can also considerer the shaft suporting discs of considerable inertia and the effect of a umbalancing forces produced by these parts. At present, the model allows the consideration of a concentrated and considerable mass in any point of the shaft, neglecting the rotating inertia. It can be done by simply altering the geometrical parameters of the finite element at the point where the concentrated mass is to be considered.

5. ACNOWLEDGMENTS

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