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A NOTE ON ERICKSEN'S PROBLEM IN THE CASE OF TWO DISTINCT CONSTANT STRETCHES

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Abstract. *In the case of an incompressible and elastic material, all universal deformations with two distinct constant stretches are known. In fact, Kafadar has shown that the strain field necessarily possesses a constant eigenvector and under a smoothness assumption he has shown that there is no new solution for this problem. In this paper we show, based on a new approach for this problem, that in fact the direction of such constant eigenvector can be explicitly determined from the field V, rendering unnecessary such strong smoothness hypothesis. In contrast with Kafadar's approach, the geometry of the problem sets the stage and renders easy all of its analytical aspect.*

Keywords. *Ericksen's problem, universal deformations.*

1. Introduction

In this work we start by considering a restricted kinematical problem of determining the class of all deformation fields with constant strain invariants. If we require that these deformations are to be such that, for a constitutive class of bodies, equilibrium is to be achieved in the absence of body forces, then the set of possible deformations will generally be further limited. A deformation in this set is called *universal*.

For an isotropic, incompressible, elastic body, all possible deformations with constant stretches $\lambda_1 = \lambda_2 \neq \lambda_3$ are known. The problem of determining all universal solutions with three distinct constant stretches is, as far as we know, open. Thus it is natural to guess that the purely kinematical problem for two distinct constant principal stretches could be also solved.

Let be given a smooth field of unit vectors $\mathbf{e}$ in a 3-dimensional Euclidean space and let $\mathbf{S} = \mathbf{I} + \lambda\mathbf{e} \otimes \mathbf{e}$ be a field of symmetric tensors with exactly two distinct eigenvalues: 1 and $1 + \lambda$. If we want to find a field $\mathbf{W}$ of skew tensors such that $\mathbf{S} + \mathbf{W}$ is a gradient, a classical problem in Linear Elasticity, curl curl $\mathbf{e} \otimes \mathbf{e} = \mathbf{0}$ is the necessary and sufficient condition for the existence of $\mathbf{W}$. Now if we want to find a field of rotations $\mathbf{R}$ such that $\mathbf{RS}$ is a gradient (the compatibility problem for finite strains), then:

(I) If curl $\mathbf{e} \bullet \mathbf{e} = 0$, then curl curl $\mathbf{e} \otimes \mathbf{e} = \mathbf{0}$ is the necessary and sufficient condition for the existence of $\mathbf{R}$.

(II) If curl $\mathbf{e} \bullet \mathbf{e}$ is constant and if $\mathbf{RS}$ is a gradient for some field $\mathbf{R}$, then curl $\mathbf{e} \bullet \mathbf{e} = 0$.

(III) If $\mathbf{RS}$ is a gradient and if $\mathbf{S}$ possesses a constant eigenvector, then curl $\mathbf{e} \bullet \mathbf{e} = 0$.

Thus these results, recently proved, remind us of the well known facts for universal solutions in the incompressible case: a deformation with two distinct constant principal stretches for which the field $\mathbf{e}$ is not normal to a family of surfaces (curl $\mathbf{e} \bullet \mathbf{e} \neq 0$, $\mathbf{e}$ is the eigenvector corresponding to the eigenvalue of multiplicity one), if any, has to be complicated.

Let's recall what is known for universal solutions for incompressible elastic bodies. Observe that if $\mathbf{F} = \mathbf{VR}$ is the polar decomposition of the gradient of a deformation, $\mathbf{F}^{-1} = \mathbf{R}^T\mathbf{V}^{-1}$ is the gradient of the inverse deformation and the classical problem (Ericksen's problem) of finding all universal deformations has been approached expressing all fields in the deformed configuration. This observation suffices to translate our results in a familiar setting. We want to know if the well known universal solutions with constant stretches are the only ones in this class.

Kafadar (1972) considered a field $\mathbf{U}$ with three constant stretches and supposed that $\mathbf{F} = \mathbf{RU}$ was a deformation gradient for an universal solution. He proved:

(K-I) If $\mathbf{U}$ possesses a Taylor series expansion and at least one constant eigenvector, then there exists no new solutions to Ericksen's problem.

(K-II) If $\mathbf{U}$ possesses at least two equal eigenvalues, then $\mathbf{U}$ has at least one constant eigenvector.

Thus, for two distinct eigenvalues, all universal solutions by (K-II) and (K-I) are known. Our result (III) shows that in fact curl $\mathbf{e} \bullet \mathbf{e} = 0$, and hence this universal solution belongs to the special class (I) of deformations with two distinct constant stretches of our purelly kinematical problem. In a naive way, (I) shows that infinitesimal and finite compatibility are the same whenever curl $\mathbf{e} \bullet \mathbf{e} = 0$.

The proof of (I), (II) and (III) is quite simple through the approach we have choose to deal with these questions. A hint for the dual role of curl curl $\mathbf{e} \otimes \mathbf{e} = \mathbf{0}$ can be seen in the following naive computation. Let's assume $\mathbf{F}_\lambda = \mathbf{R}_\lambda (\mathbf{I} + \lambda\, \mathbf{e} \otimes \mathbf{e})$ a smooth family of gradients stemming from a fixed field $\mathbf{e}$. The derivative $\mathbf{F}'$ of $\mathbf{F}_\lambda$ with respect to λ at $\lambda = 0$ gives:

$$\mathbf{F}' = \mathbf{R}' + \mathbf{R}\, \mathbf{e} \otimes \mathbf{e}.$$

Moreover, because at $\lambda = 0$ the corresponding factor $\mathbf{R}_\lambda$ is constant, let's assume $\mathbf{R} = \mathbf{I}$. Hence $\mathbf{R}' = \mathbf{W}$ is skew and

$$\mathbf{F}' = \mathbf{W} + \mathbf{e} \otimes \mathbf{e},$$

which implies curl curl $\mathbf{e} \otimes \mathbf{e} = \mathbf{0}$.

The content of this note is as follow. First we fix notations, which are all standard. We use some elementary facts on differential forms to deal with the compatibility issue. Then we show that if $\mathbf{F} = \mathbf{RU}$ the gradient of a deformation with constant stretches and $\mathbf{U}$ possesses a constant eigenvector, then $\mathbf{F} = \mathbf{R}_o\mathbf{G}$ for $\mathbf{R}_o$ constant and $\mathbf{G}$ the gradient of a plane deformation with constant transverse stretch. Thus we obtain (K-I) under less restricted hypothesis.

2. Notation

Let $\mathcal{E}$ be the euclidean 3-dimensional space, with translation space $\mathcal{V}$. We denote by $\mathbf{I}$ the identity of $\mathcal{Lin}(\mathcal{V})$, the set of linear transformations from $\mathcal{V}$ into $\mathcal{V}$. An orthogonal element $\mathbf{Q}$ of $\mathcal{Lin}(\mathcal{V})$ is a rotation if det $\mathbf{Q} = 1$. We use $\mathbf{A}^T$ to indicate the transposition of $\mathbf{A}$.

As our analysis is of local character, a deformation f is a smooth map from $\mathcal{E}$ into $\mathcal{E}$ such that its gradient $\mathbf{F}$ satisfies det $\mathbf{F} > 0$. By the polar decomposition theorem, $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$, where $\mathbf{R}$ is the rotation and $\mathbf{U}$, $\mathbf{V}$ are symmetric and positive.

We choose a fixed orthonormal frame in $\mathcal{E}$. If f is a deformation with constant stretches, $\mathbf{U} = \mathbf{Q}^T\mathbf{DQ}$ with $\mathbf{D}$ diagonal. Hence, absorbing the $\mathbf{Q}^T$ factor in the rotation term of the polar decomposition, $\mathbf{F}$ can be written as

$$\mathbf{F} = \mathbf{RDQ}. \tag{1}$$

As $\mathbf{F}$ is a gradient, its matrix has elements $F_{ij} = f_{i,j}$, i.e., $F_{ij} = \dfrac{\partial f_i}{\partial x_j}$. For the components f_i of the deformation f, its differential df_i is $df_i = f_{i,j}\, dx_j$, where we have used Einstein's convention. Moreover the coordinates of points in $\mathcal{E}$ are written (x_1, x_2, x_3) or, if convenient, also (x, y, z); also if $d\mathbf{X}$ denotes the column vector of forms (dx, dy, dz), we have $\mathbf{F}d\mathbf{X} = \mathbf{df}$ as a neat way to recall that $df_i = \dfrac{\partial f_i}{\partial x_j} dx_j$.

Let $\mathbf{A}$ be a matrix whose elements $A_{ij} = A_{ij}(x, y, z)$ are real valued functions defined on $\mathcal{E}$. We denote by $d\mathbf{A}$ the matrix with elements $(dA)_{ij}$ corresponding to the differential of A_{ij}:

$$(dA)_{ij} = d(A_{ij}) = \frac{\partial A_{ij}}{\partial x_m} dx_m = A_{ij,m}dx_m.$$

From (1) it follows that

$$d\mathbf{F} = d\mathbf{RDQ} + \mathbf{RD}d\mathbf{Q}, \tag{2}$$

or

$$\mathbf{R}^T d\mathbf{F} = \mathbf{R}^T d\mathbf{RDQ} + \mathbf{D}d\mathbf{Q},$$

and

$$\mathbf{R}^T d\mathbf{F}d\mathbf{X} = \mathbf{R}^T d\mathbf{RDQ}d\mathbf{X} + \mathbf{D}d\mathbf{Q}d\mathbf{X}. \tag{3}$$

We recall that $\mathbf{ddf} = \mathbf{0}$ implies $\mathrm{d}\mathbf{F}\mathrm{d}\mathbf{X} = \mathbf{0}$. On the other hand, if $\mathrm{d}\mathbf{A}\mathrm{d}\mathbf{X} = \mathbf{0}$ for a matrix-valued function $\mathbf{A}$ on $\mathcal{E}$, then the column of two forms $\mathrm{d}A_{ij}\mathrm{d}x_j$ is zero. Thus $\mathrm{d}(A_{ij}\mathrm{d}x_j) = 0$ and being $A_{ij}\mathrm{d}x_j$ closed, it is exact: there exists $f : \mathcal{E} \to \mathcal{E}$ such that $\mathrm{d}f_i = A_{ij}\mathrm{d}x_j$ and $\mathbf{A}$ is the gradient $\mathbf{F}$ of a deformation f if $\det \mathbf{A} > 0$.

3. The compatibility problem

If $\mathbf{R}$ is a field of rotations, $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ and $\mathrm{d}\mathbf{R}^T\mathbf{R} + \mathbf{R}^T\mathrm{d}\mathbf{R} = \mathbf{0}$ shows that the matrix of 1-forms $\mathbf{\Omega} = \mathbf{R}^T\mathrm{d}\mathbf{R}$ is skew. But $\mathrm{d}\Omega = \mathrm{d}\mathbf{R}^T\mathrm{d}\mathbf{R} = \mathrm{d}\mathbf{R}^T\mathbf{R}\mathbf{R}^T\mathrm{d}\mathbf{R} = -\,\Omega \wedge \Omega$. In fact, if

$$\Omega = \begin{bmatrix} \mathbf{0} & \mathbf{A} & \mathbf{B} \\ -\mathbf{A} & \mathbf{0} & \mathbf{C} \\ -\mathbf{B} & -\mathbf{C} & \mathbf{0} \end{bmatrix}$$

then formally multiplying $\mathbf{\Omega}$ by $\mathbf{\Omega}$ and taking care of the skew symmetry of the product of two forms, we see that $\mathrm{d}\mathbf{\Omega} = -\,\mathbf{\Omega} \wedge \mathbf{\Omega}$ is equivalent to

$$\mathrm{d}\mathbf{A} = \mathbf{B} \wedge \mathbf{C}, \quad \mathrm{d}\mathbf{B} = \mathbf{C} \wedge \mathbf{A}, \quad \text{and} \quad \mathrm{d}\mathbf{C} = \mathbf{A} \wedge \mathbf{B}. \tag{4}$$

Conversely, it can be proved that if $\mathbf{\Omega}$, a skew matrix of 1-forms, satisfies the relation $\mathrm{d}\mathbf{\Omega} = -\,\mathbf{\Omega} \wedge \mathbf{\Omega}$, then there exists a field of rotations $\mathbf{R}$ such that

$$\mathbf{\Omega} = \mathbf{R}^T\mathrm{d}\mathbf{R}. \tag{5}$$

Moreover if $\mathbf{R}$ solves (5) all solutions of (5) are of the form $\mathbf{R}_o\mathbf{R}$ for $\mathbf{R}_o$ constant, i.e., $\mathbf{\Omega}$ determines $\mathbf{R}$ up to a constant rotation.

Now we are ready to pose the compatibility problem we are addressing. Suppose *given **U**, a smooth, symmetric tensor field of constant eigenvalues, and with a constant eigenvector, such that **F** = **RU** is a gradient.*

We call such $\mathbf{U}$ a compatible field constant stretches.

We write $\mathbf{F}$ as $\mathbf{F} = \mathbf{RDQ}$ and choose a frame for which

$$\mathbf{Q} = \begin{bmatrix} \cos\alpha & \operatorname{sen}\alpha & 0 \\ -\operatorname{sen}\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $\alpha = \alpha(x, y, z)$. Thus the third axis corresponds to the fixed eigenvector. We assume λ, β, 1 being the three distinct constant stretches of $\mathbf{U}$, without loss of generality. Hence, by calling $\mathbf{\Omega} = \mathbf{R}^T\mathrm{d}\mathbf{R}$,

$$\mathbf{\Omega}\mathbf{D}\mathbf{Q}\mathrm{d}\mathbf{X} + \mathbf{D}\mathrm{d}\mathbf{Q}\mathrm{d}\mathbf{X} = \mathbf{0}, \tag{6}$$

holds and if we call $\mathbf{q}_1 = \cos\alpha\, \mathrm{d}x + \sin\alpha\, \mathrm{d}y$, $\mathbf{q}_2 = -\sin\alpha\, \mathrm{d}x + \cos\alpha\, \mathrm{d}y$, $\mathbf{q}_3 = \mathrm{d}z$ and if we express the 1-forms $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ for $\mathbf{\Omega}$ in the basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ writing for instance $\mathbf{A} = A_1\mathbf{q}_1 + A_2\mathbf{q}_2 + A_3\mathbf{q}_3$, then (6) is equivalent to

$$\begin{gathered} \beta\mathbf{A} \wedge \mathbf{q}_2 + \mathbf{B} \wedge \mathbf{q}_3 + \lambda\, \mathrm{d}\mathbf{q}_1 = \mathbf{0} \\ -\lambda\mathbf{A} \wedge \mathbf{q}_1 + \mathbf{C} \wedge \mathbf{q}_3 + \beta\mathrm{d}\mathbf{q}_2 = \mathbf{0} \\ -\lambda\mathbf{B} \wedge \mathbf{q}_1 - \beta\mathbf{C} \wedge \mathbf{q}_2 = \mathbf{0}. \end{gathered} \tag{7}$$

We find then the components of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ by multiplying each equation of (7) by $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ obtaining:

$$B_3 = C_3 = 0\,, \qquad \lambda B_2 - \beta C_1 = 0\,,$$

from $(7)_3$.

Observe now that a simple computation shows $d\mathbf{q}_1 \wedge \mathbf{q}_2 = d\mathbf{q}_2 \wedge \mathbf{q}_1 = 0$ and $d\mathbf{q}_1 \wedge \mathbf{q}_1 = d\mathbf{q}_2 \wedge \mathbf{q}_2 = -\alpha_{,z}\, dxdydz = -\frac{\partial \alpha}{\partial z} dxdydz$. As $\mathbf{q}_1 \wedge \mathbf{q}_2 \wedge \mathbf{q}_3 = dxdydz$, $(7)_1$ and $(7)_2$ imply

$$B_1 = 0, \quad C_2 = 0, \quad -\beta A_3 + B_2 - \lambda\alpha_{,z} = 0\,, \quad -\lambda A_3 - C_1 - \beta\alpha_{,z} = 0\,.$$

As $d\mathbf{B} = \mathbf{CA}$ and $d\mathbf{C} = \mathbf{AB}$ it follows that

$$\begin{aligned} B_2 d\mathbf{q}_2 \wedge \mathbf{q}_2 &= -C_1 A_3\, \mathbf{q}_1 \wedge \mathbf{q}_2 \wedge \mathbf{q}_3\,, \\ C_1 d\mathbf{q}_1 \wedge \mathbf{q}_1 &= -A_3 B_2\, \mathbf{q}_1 \wedge \mathbf{q}_2 \wedge \mathbf{q}_3\,; \end{aligned} \tag{8}$$

or

$$B_2\alpha_{,z} = C_1 A_3 \quad \text{and} \quad C_1\alpha_{,z} = A_3 B_2.$$

Hence $B_2^2\, \alpha_{,z} = C_1^2\, \alpha_{,z}$ and $(\beta/\lambda)^2 C_1^2\, \alpha_{,z} = C_1^2\, \alpha_{,z}$ follows from $\lambda B_2 - \beta C_1 = 0$. Thus $\alpha_{,z} = 0$ or $C_1 = 0$. But if $C_1 = 0$, $B_2 = 0$ follows and $-\beta A_3 - \lambda\alpha_{,z} = 0$ and $-\lambda A_3 - \beta\alpha_{,z} = 0$ imply $\alpha_{,z} = 0$.
On the other hand, $\alpha_{,z} = 0$ implies $A_3 = B_2 = C_1 = 0$.

Thus, under our hypothesis, if $\mathbf{F} = \mathbf{RU}$ is a gradient, the field $\mathbf{Q}$ giving the attitude of the eigenvectors of $\mathbf{U}$ has α as a function $\alpha = \alpha(x, y)$ and the corresponding $\mathbf{\Omega}$ reduces to

$$\mathbf{\Omega} = \begin{bmatrix} \mathbf{0} & \mathbf{A} & \mathbf{0} \\ -\mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

with $\mathbf{A} = A_1 dx + A_2 dy$. As $d\mathbf{A} = \mathbf{0}$ both A_1 and A_2 are independent of z and we can choose the field $\mathbf{R}$ as corresponding to a rotation of $\theta(x, y)$ along the z axis.

Finally observe that we could have assumed $\beta = 1$ without change in the conclusions.

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