A GENERALIZED SOLUTION FOR THE LAMINAR VERTICAL PLATE FREE CONVECTION FLOWS

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Abstract. The classical boundary layer similar solution for vertical flat plates natural convection flows, is valid in the regions far from the leading edge, where the flow pattern can be considered as fully developed. Thus a simplified solution, which can describe accurately the flow in the leading edge vicinity, is of great scientific interest. In this paper, a new general formulation of free convection boundary-layer flow is obtained using the generalized boundary layer formulation and the similarity analysis approach. The partial differential equations are reduced to an ordinary differential equation through an adapted change of coordinates. The resulting ordinary differential equation will be numerically solved and the velocity and temperature profiles obtained will be used to describe the near leading edge behavior of the heat and the momentum transfer.

Keywords. Natural convection, asymptotic analysis, boundary layer

1. Introduction

The laminar natural convection boundary-layer along a vertical semi-infinite flat plate is a very important problem in the study of heat transfer in external surfaces. In spite of its importance, the complete solution of this problem is still not available. The principal unanswered questions are related to the behavior of the velocity and the temperature fields near the flat plate leading edge. It is well know that the classical boundary layer similar solution is valid for the regions far from the leading edge, where the Grashof number is high and the flow pattern can be considered as fully developed. In a effort to obtain a better approximation for moderate Grashof number, Yang and Jerger (1964) developed a second order boundary layer correction solution using the method of matched asymptotic expansion. The obtained solution contributes with a negative value for the higher order correction of Nusselt number, indicating that the value of the integrated heat transfer diminishes as the leading edge approaches. Other important work was developed by Messiter and Liñán (1976) who, using the so-called “triple deck” approach, calculated the leading effects into the integrated heat transfer. In this case, the contribution of the correction is positive and furnished a increment for the integrated heat transfer.

An alternative formulation for the laminar natural convection phenomenon on a vertical flat plate was proposed by Silva (2003). In his work, the concept of principal equation proposed by Kaplun (1967) was applied to the Navier Stokes equations in cartesian system to obtain a generalized boundary layer formulation. This formulation requires no division between inner and outer solution, which makes unnecessary any type of matching procedure or some kind of viscous-inviscid interaction. In this work, a quasi-similar ordinary differential equation obtained form the generalized boundary layer formulation described in Cruz (2002) is studied in detail. It is shown that the quasi-similar equation can describe the flow near the leading edge. A comparison of the results of the present formulation with the theories of Yang and Jerger (1964) and Messiter and Liñán (1976) will be made and discussed in detail.

2. The Generalized Boundary Layer Equation

The concept of distinguished limits (Kaplun, 1967) will now be used to determinate the asymptotic behavior of the Navier Stokes equation as Re→∞. The mathematical framework necessary to obtain the high Reynolds number asymptotic behavior of the Navier Stokes is exhaustively discussed in Cruz (2002) and Silva Freire (1999). Here, just some of the principal steps will be presented. For a laminar, incompressible, stationary and two-dimensional flow of a Newtonian fluid the continuity and the momentum equations can be written as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial P}{\partial x}+\frac{1}{Re}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right] \quad (2)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial P}{\partial y}+\frac{1}{Re}\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right] \quad (3)$$

In the above equations the variables are made non-dimensional using a characteristic length and a characteristic velocity of the flow. The parameters u and v represents the non-dimensional velocities on the x and y directions respectively and P is the non-dimensional pressure (see figure 1).

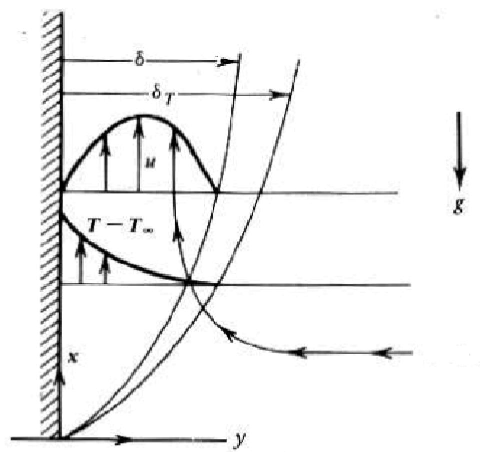

Figure 1. Natural convection phenomena

The parameter Re represents the Reynolds number that is assumed to be large (i.e. $1/Re \ll 1$)
The intermediate variables are defined as

$$\hat{y}=\frac{y}{\eta(\varepsilon)} \quad (4)$$

$$\hat{v}=\frac{v}{\eta(\varepsilon)} \quad (5)$$

where $\varepsilon=\frac{1}{Re}$.
The insertion of Eqs. (4) and (5) into equations Eqs. (1), (2) and (3) results

$$\frac{\partial u}{\partial x}+\frac{\partial \hat{v}}{\partial \hat{y}}=0 \quad (6)$$

$$u\frac{\partial u}{\partial x}+\hat{v}\frac{\partial u}{\partial \hat{y}}=-\frac{\partial P}{\partial x}+\frac{1}{Re}\left[\frac{\partial^2 u}{\partial x^2}+\frac{1}{\eta(\varepsilon)^2}\frac{\partial^2 u}{\partial \hat{y}^2}\right] \quad (7)$$

$$\eta(\varepsilon)u\frac{\partial \hat{v}}{\partial x}+\eta(\varepsilon)\hat{v}\frac{\partial \hat{v}}{\partial y}=-\frac{1}{\eta(\varepsilon)}\frac{\partial P}{\partial \hat{y}}+\frac{1}{Re}\left[\eta(\varepsilon)\frac{\partial^2 \hat{v}}{\partial x^2}+\frac{1}{\eta(\varepsilon)}\frac{\partial^2 \hat{v}}{\partial \hat{y}^2}\right] \quad (8)$$

Applying the η-limit onto Eqs. (7) and (8) one gets
a) For the momentum equation on x direction:

$$O(\eta)=O(1):\ u\frac{\partial u}{\partial x}+\hat{v}\frac{\partial u}{\partial \hat{y}}=-\frac{\partial P}{\partial x} \quad (9)$$

$$O(1) > O(\eta) > O(\sqrt{\varepsilon}) : u \frac{\partial u}{\partial x} + \hat{v} \frac{\partial u}{\partial \hat{y}} = - \frac{\partial P}{\partial x} \tag{10}$$

$$O(\eta) = O(\sqrt{\varepsilon}) : u \frac{\partial u}{\partial x} + \hat{v} \frac{\partial u}{\partial \hat{y}} = - \frac{\partial P}{\partial x} + \frac{\partial^2 u}{\partial \hat{y}^2} \tag{11}$$

$$O(\eta) < O(\sqrt{\varepsilon}) : \frac{\partial^2 u}{\partial \hat{y}^2} = 0 \tag{12}$$

b) For the momentum equation on y direction:

$$O(\eta) = O(1) : u \frac{\partial \hat{v}}{\partial x} + \hat{v} \frac{\partial \hat{v}}{\partial y} = - \frac{\partial P}{\partial \hat{y}} \tag{13}$$

$$O(\eta) < O(1) : \frac{\partial P}{\partial \hat{y}} = 0 \tag{14}$$

In each of the above two sets of differential equations (Eq. (9) to Eq. (12) and Eq. (13) to Eq. (14)) there is only one principal equation according to Kaplun's definition. Equation (11) represents the principal equation for the x momentum equation and Eq. (13) is the principal equation for the y momentum equation. It should be noted that the terminology “principal” is related to the fact that the above mentioned equations exhibit some specific characteristics. In both cases the Eq. (11) and Eq. (13) contains the other equations and are not contained by any other of the remaining expressions (Cruz, 2002). This fact indicates that in the limit as $Re \to \infty$ the behavior Navier Stokes equations is adequately described by the following set of equations

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{\partial P}{\partial x} + \frac{1}{Re} \frac{\partial^2 u}{\partial y^2} \tag{15}$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{\partial P}{\partial y} \tag{16}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{17}$$

The Eqs. (15) to (17) represents a generalized form of the boundary layer theory which combines the Euler inviscid flow equations and the Prandtl classical boundary layer formulation into a single and more general formulation.

To analyze the natural convection problem, a buoyancy term must be introduced into Eq. (15) as well as the boundary layer approximation of the energy equation resulting into the following set of partial differential equations

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{\partial P}{\partial x} + \frac{1}{Re} \frac{\partial^2 u}{\partial y^2} + g \beta (T - T_\infty) \tag{18}$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{\partial P}{\partial y} \tag{19}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{20}$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha_\infty \frac{\partial T^2}{\partial y^2} \tag{21}$$

3. The near leading edge flat plate flow and the quasi-similar equation

The classical approach to describe the laminar solution for the vertical flat plate natural convection phenomena (figure 1) was presented by Ostrach (1953) who proposed the following set of similarity transformations

$$\Psi = 4\nu_\infty \, c \, x^{3/4} f(\eta) \tag{22}$$

$$\eta = y \frac{c}{x^{1/4}} \tag{23}$$

$$c^4 = \frac{g(|\rho_\infty - \rho_w|)}{4\nu_\infty^2 \rho_\infty} \tag{24}$$

$$Gr = \frac{g \; |T_w - T_\infty| \beta x^3}{\nu_\infty^2} \tag{25}$$

$$\theta = \frac{(T - T_\infty)}{(T_w - T_\infty)} \tag{26}$$

In the above equations Ψ represents the stream function, η is the similarity coordinate and G_r is the Grashof number. Inserting Eqs. (22) to (26) into Eqs. (18) to (21), the following set of ordinary differential equation is obtained

$$f^{IV} + 3\, f\, f''' - ff'' + \frac{1}{8\, G_r^{1/2}} \left[27ff' + 5\eta f'^2 - 7\eta^2 f' f'' - 2\eta^3 f' f''' - 3 \;\; \eta f\, f'' + 3\eta^2 f\, f''' \right] + \theta' = 0 \tag{27}$$

$$\theta'' + 3 \Pr f\, \theta' = 0 \tag{28}$$

The Eq. (27) represents a quasi-similar ordinary differential equation in the sense that not all of its coefficients are constants. This is an important characteristic since although this quasi-similar equation is an ordinary one or in other words, the derivatives that are present in the equation are related to η, it still carry some information relative to the non-similar developing region near the leading edge of the flow. Some questions may arise concerning the derivatives related to some function of the tangent to the plate coordinate alone (x). Although those derivatives exists, they are only important at the leading edge of the plat, i.e., for very low values of the Grashof number. For moderate and very high Grashof values, the derivatives are very small and can be disregarded (Cruz, 2002). A more detailed analysis of that affirmation will be presented latter.

4. Results and Discussion

The Eq. (27) was numerically solved using the FORTRAN-IMSL framework and the DIVPAG subroutine.

Figures 2, 3, 4 and 5 show the tangential velocity and temperatures profiles for various values of the Grashof number for values of the Prandt of 0.72 and 10 respectively. The developing behavior of tangential velocity and temperature profiles is clearly seen, indicating that for low values of Grashof number the peak of the transformed tangential velocity profile is also small. This can be explained by the fact that at moderate distances form the leading edge the streamwise fluid acceleration from the quiescent situation is high, thus high values of the perpendicular to the plate velocity are expected. In this situation, one the fundamental assumptions of the classical boundary layer theory is not valid. Therefore at least, the convective terms of the y momentum equation must be retained in order to describe the flow adequately.

The introduction of the v velocity convective terms is represented in Eq. (27) by the terms inside the brackets, and is divided by $Gr^{1/2}$. Therefore, the influence of those terms must vanish as Gr approaches infinity and in this case, the Eq.(27) reduces to the classical formulation as it should.

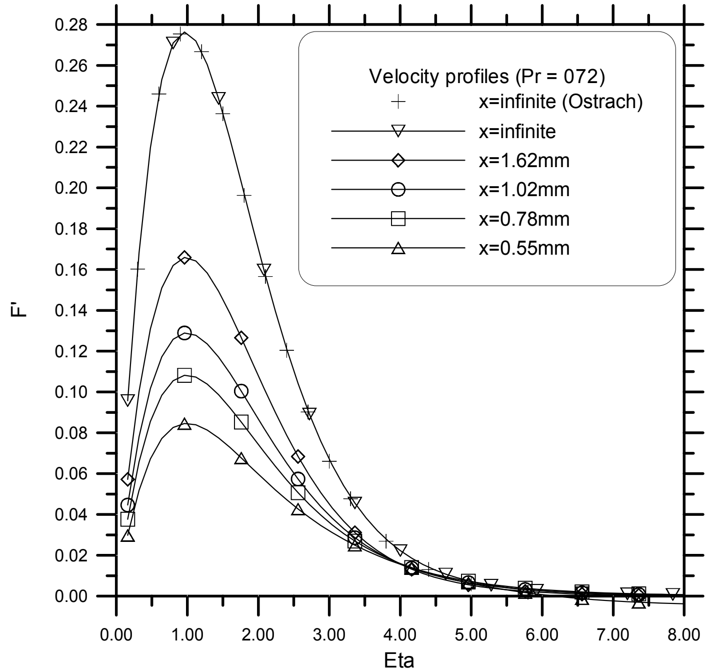

Figure 2. Velocity profiles for various distances form de leading edge

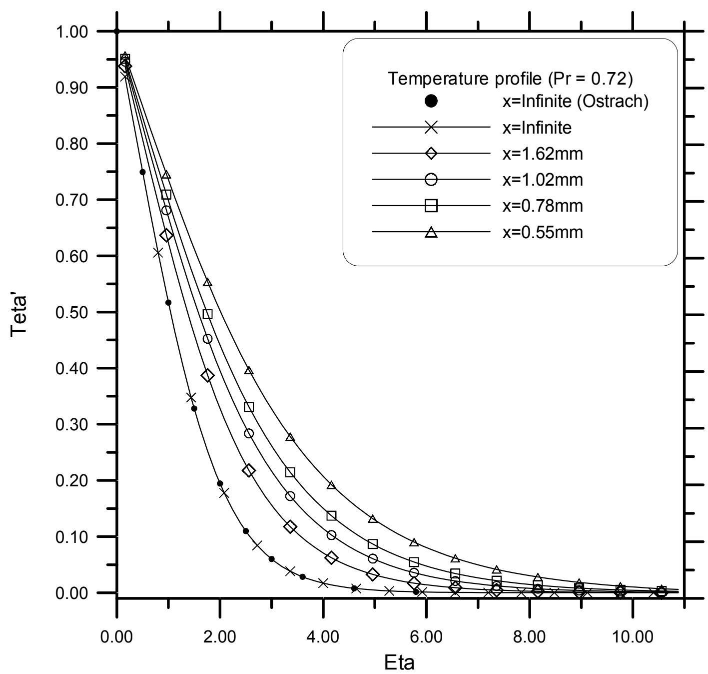

Figure 3. Tempereture profiles for various distances form de leading edge Pr = 0.72

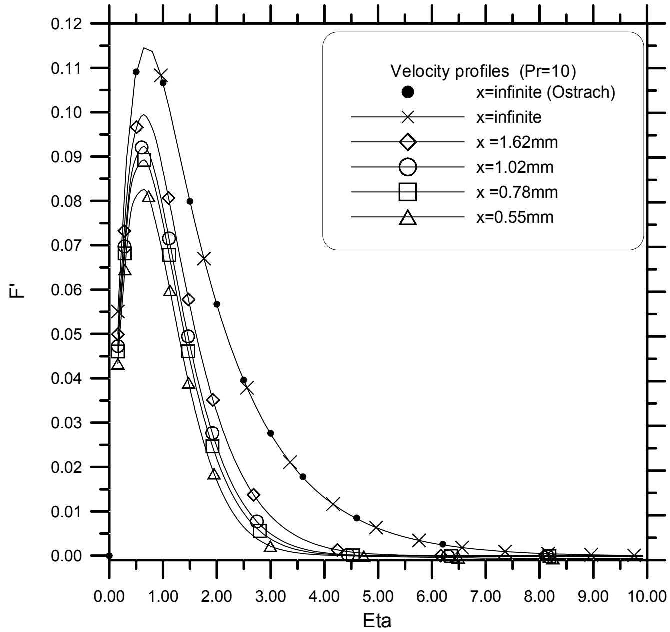

Figure 4. Velocity profiles for various distances form de leading edge locations Pr = 10

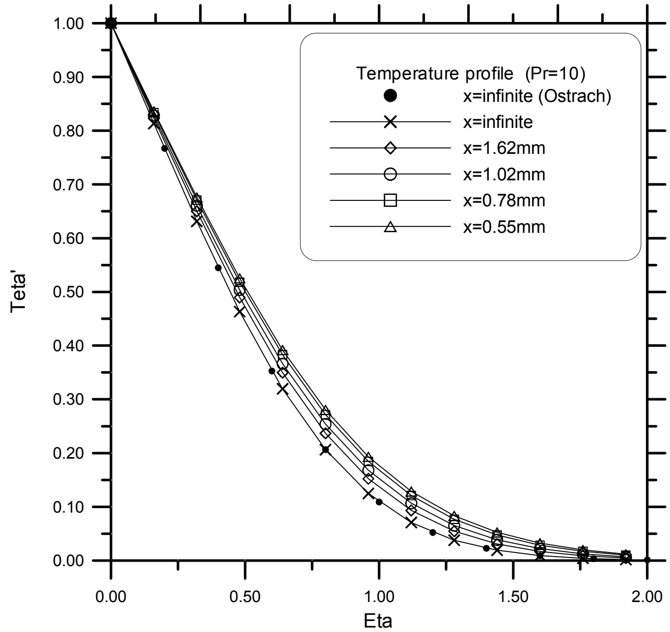

Figure 5. Tempereture profiles for various distances form de leading edge Pr = 0.72

One important parameter that can be used to compare various higher order boundary layer formulations for the vertical flat plate natural convection phenomenon is the average Nusselt number. Figures 6 e 7 shows the comparison of the present theory results and the of the higher order corrections contributions of Yang and Jerger (1964) and Messiter and Liñán (1976) which are described through the equations in Table 1.

Table 1 - Higher order corrections contributions of Yang and Jerger (1964) and Messiter and Liñán (1976)

Pr = 0.72	Yang Jerger	$0.475\,Gr^{1/4} - 0.312$
	Messiter and Liñán	$0.475\,Gr^{1/4} + 0.623$
Pr = 10	Yang Jerger	$1.102\,Gr^{1/4} - 0.216$
	Messiter and Liñán	$1.102\,Gr^{1/4} + 0.457$

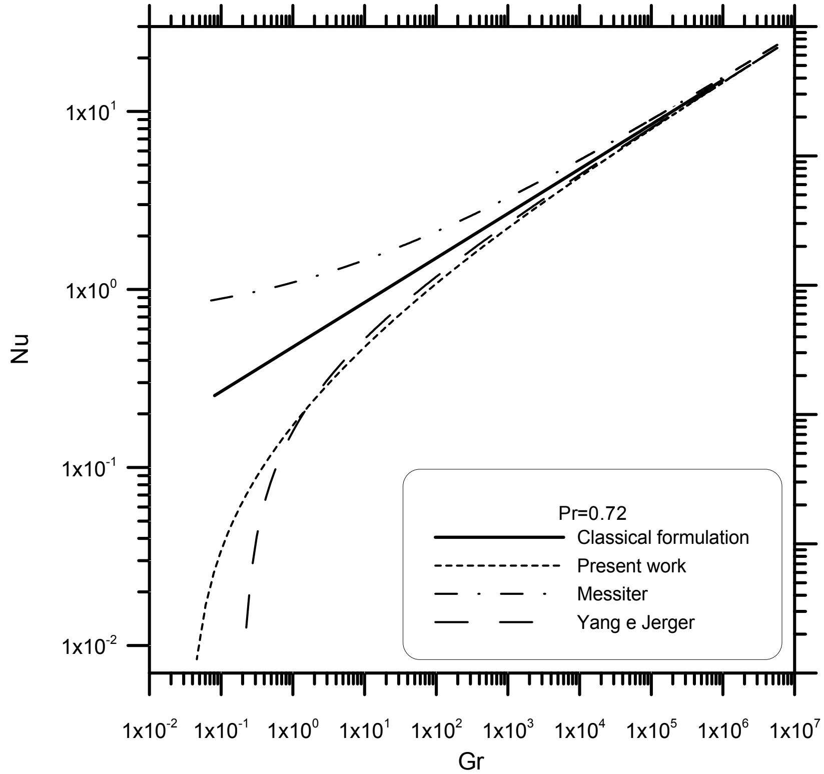

Figure 6. Comparison of average Nusselt number

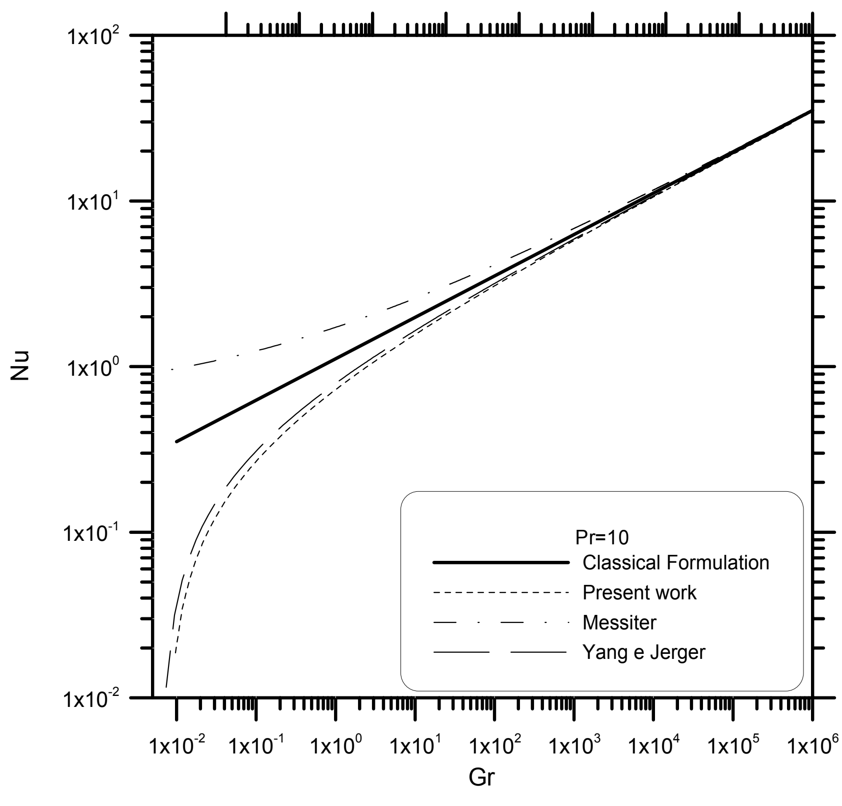

Figure 7. Comparison of average Nusselt number

The influence of the displacement thickness into the velocity and the temperature fields were studied by Yang and Jerger (1964). Using the matched asymptotic method expansions, the author was able to calculate the displacement thickness higher order correction influence on the average Nusselt number. A comparison of the results of Yang and Jerger (1964) with the present theory for values of the Prandtl number of 0.72 and 10 show a remarkable agreement between both theories. This fact suggests that Eq. (27) represents a generalized version of the natural convection flat plate problem. Despite the fact that Eq. (27) is an ordinary differential equation, its quasi-similar character allows the formulation to retain some information from the moderate Grashof number non-similar region. It is also important to note that this additional non-similar higher order information was obtained from a single differential equation, which was obtained from a first order analysis.

The results of the present work are also compared with the higher order correction of Messiter and Liñán (1976) who considered an asymptotic expansion in an effort to introduce the leading edge effects into the analysis. Their approach does not take into account the influence of the boundary layer into the inviscid flow, but instead creates an near leading edge correction which must be added to the classical theory solution. It is clear from Figs. 6 and 7 that the Messiter and Liñán (1976) results are not in agreement with the present theory neither with the Yang and Jerger (1964) higher order equation. The reason for that discrepancy is probably connected to the fact that for moderate values of the Grashof number the influence of the perpendicular to the plate velocity is not small and must be considered. In other words some type of interactive viscid-inviscid solution or an generalized formulation as Eqs. (27) and (28) must be applied in order to describe the flow at moderate Grashof numbers.

5. Conclusion

In this work a generalized boundary layer formulation was used to analyze the vertical flat plate natural convection phenomena. A quasi-similar equation was proposed which describes the flow for moderate values of the Grashof number. One major advantage of this quasi-similar equation is that, although its a ordinary differential equation, some information of the non-similar moderate Grashof number region is kept through one of the Eq. (27) coefficients which explicitly depends on the Grashof number.

The results of the present formulation were compared with the two natural convection boundary layer higher order theories of Yang and Jerger (1964) and Messiter and Liñán (1976). The results indicate that the perpendicular to the plate velocity influence should be considered for moderate values of the Grashof number.

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