A SIMPLE AND ACCURATE PROCEDURE FOR SHAPE SENSITIVITY ANALYSIS

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Abstract. *This work presents a simple and accurate procedure for shape sensitivity computation of linear and nonlinear structures. This procedure is based on the combination of the Laplacian smoothing with the Refined Semi-Analytical Method, where the Laplacian smoothing generates the design velocity field (derivatives of nodal coordinates with respect to the shape variables) and the Refined Semi-Analytical Method computes the shape sensitivities using the nodal velocities as input data. The Refined Semi-Analytical Method has been succesfully applied with nodal velocities computed by the boundary node approch, which is more efficient than the Laplacian smoothing. However,it is well known that the boundary node approach can lead to inconsistent velocity fields.On the other hand, the examples presented here show that the Laplacian smoothing not only leads to a consistent velocity field, but also decreases the errors in the sensitivities computed by the Refined Semi-Analytical Method .*

Keywords*. Shape optimization, sensitivity analysis, laplacian smoothing, semi-analytical method*

1. Introduction

The aim of structural shape optimization is to find the shape of a given structure that minimizes a chosen cost function and satisfies a set of defined constraints. Using CAD tools, the shape of the structure can be defined by its boundary curves (or surfaces) whose geometry depends on a set of design variables. This geometric (or design) model is used for optimization purposes, while a finite element model is used to compute the structural responses, as displacements and stresses. This analysis model can be automatically constructed using an appropriate mesh generation algorithm.

The shape optimization problem is highly nonlinear, demanding robust and efficient optimization algorithms, as the Sequential Quadratic Programming (SQP) algorithms. These algorithms, and other gradient-based methods, require the computation of the derivatives (sensitivities) of the structural responses in order to find the search direction of the optimization process. Therefore, the accuracy of the computed sensitivities is of fundamental importance to the convergence of the optimization process.

The Semi-Analytical Method (SAM) combines the efficiency of the analytical approach with the generality and simplicity associated with the use of finite differences. Therefore, this method became widely used in shape optimization. However, this approach is not reliable, since the computed sensitivities present large errors when the displacement field of the individual elements is dominated by rigid body rotations. Using some orthogonality conditions, the Refined Semi-Analytical Method (RSAM) strongly reduces the errors presented by the conventional semi-analytical sensitivities, but maintains the efficiency, generality, and simplicity of the semi-analytical approach.

The shape sensitivities are generally computed perturbing only the nodes on the boundary curves associated with a given design variable (boundary nodes). This approach is simple and efficient, since only the elements connected to the boundary nodes contribute to displacement sensitivities. However, as will be discussed in this work, the boundary node approach can lead to sensitivities inconsistent with the associated geometric model and the adopted mesh update scheme. Therefore, other methods to compute the nodal velocities (derivatives of nodal coordinates w.r.t. the design variables) will be addressed.

In this work, the Laplacian smoothing was adopted for computation of nodal velocities owing to its generality and simplicity. The example presented here shows that this procedure not only generate a design velocity field compatible with the adopted geometric model, but also improves the quality of the computed sensitivities. Therefore, the combination of the Laplacian smoothing with the Refined Semi-Analytical Method results in a powerful tool for shape sensitivity analysis.

2. Basic equations

In the finite element analysis, the strains and stresses depend on the nodal displacements. Thus, once the displacement sensitivities are known, the sensitivities of strains and stresses can be easily computed. Therefore, the first objective in sensitivity analysis is the computation of the sensitivities of the nodal displacement vector. Considering, without loss of generality, a structure defined by a single design variable b, the FE nonlinear equilibrium equation can be written as

$$\mathbf{g}(\mathbf{u}, b) - \lambda\, \mathbf{f}(b) \;=\; \mathbf{0} \tag{1}$$

where $\mathbf{u}$ is the nodal displacement vector, $\mathbf{g}$ is the internal force vector, λ is the load factor, and $\mathbf{f}$ is the external load vector. It should be noticed that $\mathbf{u}$ is an implicit function of the design variable b.

For a *fixed* load factor, the total derivative of Eq. (1) w.r.t. b, is

$$\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\frac{d\mathbf{u}}{db} + \frac{\partial \mathbf{g}}{\partial b} - \lambda\frac{d\mathbf{f}}{db} = \mathbf{0} \tag{2}$$

Since the derivative of internal forces w.r.t. nodal displacements is the tangent stiffness matrix (**K**), the sensitivity of nodal displacements can be computed from

$$\mathbf{K}\frac{d\mathbf{u}}{db} = \mathbf{p}, \qquad \textit{where} \quad \mathbf{p} = \lambda\frac{d\mathbf{f}}{db} - \frac{\partial \mathbf{g}}{\partial b} \tag{3}$$

According to this equation, the pseudo-load vector (**p**) is computed from the derivatives of **f** and **g** w.r.t. the design variable *b* while keeping **u** fixed. It can be noted that even for nonlinear problems, the computation of displacement sensitivities does not involve iterative procedures. Moreover, since the stiffness matrix was factored in the $\mathbf{LDL^t}$ form during the analysis phase, the computation of displacement sensitivities requires only vector reductions and back-substitutions. Finally, it is worth noting that the global pseudo-load vector is assembled from element vectors in the same way that the internal force vector.

Equation (3) is used both by the analytical and semi-analytical approaches for sensitivity computation. In the Analytical Method it is necessary to perform the analytical differentiation of the external and internal force vectors. This procedure leads to exact sensitivities for a given finite element mesh. However, its use requires the determination and coding of specific expressions for each finite element type, resulting in a cumbersome process. The Analytical Method has been successfully applied to truss and isoparametric elements (Parente and Vaz, 1999; Parente and Vaz, 2003), but is difficult to apply to more complex elements.

The Semi-Analytical Method also uses Eq. (3). However, the computation of the pseudo-load vector is performed by the numerical differentiation of the external and internal force vectors using finite differences. Therefore, using the conventional forward-difference scheme, the derivative of the internal force vector can be computed from

$$\frac{d\mathbf{g}}{db} = \frac{\mathbf{g}(b+\Delta b) - \mathbf{g}(b)}{\Delta b} \tag{4}$$

where Δb is the absolute perturbation on the variable *b*. This approach is generic and simple to code, since it does not depend on the formulation of a particular finite element. Thus, the same procedure can be applied to the different element types implemented in a FE program. Since its efficiency is practically the same of the analytical approach, the semi-analytical approach became widely used in shape optimization.

3. Nodal velocities

The generic expressions presented in the previous section are valid for shape and sizing optimization problems. However, for practical application in shape sensitivity analysis some additional considerations are necessary. It should be noted that the shape of the structure is defined by the geometry of its boundary curves (or surfaces). Therefore, the design variables are the parameters that control the geometry of the boundary curves. On the other hand, the geometry of the finite element mesh is defined only by the coordinates of the nodal points (**a**). Therefore, all finite element quantities are functions of the nodal coordinates. Thus, using the chain rule, the sensitivity of the internal force vector can be computed from

$$\frac{d\mathbf{g}}{db} = \frac{\partial \mathbf{g}}{\partial a_j}\frac{da_j}{db} \quad \Rightarrow \quad \frac{d\mathbf{g}}{db} = \frac{d\mathbf{g}}{d\mathbf{a}}\cdot\frac{d\mathbf{a}}{db} = \frac{d\mathbf{g}}{d\mathbf{a}}\cdot\mathbf{v} \tag{5}$$

The vector $\mathbf{v} = d\mathbf{a}/db$ represents the derivatives of nodal coordinates w.r.t. the design variable *b* and is also known as vector of nodal velocities. It is worth noting that in this equation and in the rest of this paper, the summation convention for repeated indices is used.

Equation (5) implies that the sensitivity of the internal force vector should be evaluated computing the sensitivities w.r.t. the nodal coordinates, then multiplying these sensitivities by the nodal velocities. However, since the vector **a** is a function of the design variable *b*, the semi-analytical sensitivities of the internal force vector can be directly computed from the expression

$$\frac{d\mathbf{g}}{db} = \frac{\mathbf{g}(\mathbf{a}+\Delta\mathbf{a}) - \mathbf{g}(\mathbf{a})}{\Delta b}, \qquad \textit{where} \qquad \Delta\mathbf{a} = \mathbf{v}\,\Delta b \tag{6}$$

It is important to note that for a certain design variable, there are infinitely many possible choices of velocity fields, since for a given geometry, the finite element nodes can be moved infinitely many ways within the domain (Lindby and Santos, 1997). However, to be used in shape sensitivity analysis, a design velocity field must satisfy theoretical and practical requirements (Choi and Chang, 1994). Theoretically, the design velocity field must have the same regularity as the displacement field and depend linearly on the variation of shape parameters. These conditions are easily satisfied by the different methods used for the computation of the nodal velocities.

From a practical standpoint, the velocity field should maintain the boundary nodes on the boundary curves for all shape changes. This can be easily obtained keeping fixed the parametric coordinates of the boundary nodes. A design velocity computation method should retain the topology of the original mesh, since a mesh with different number of elements or nodes cannot be used for finite difference computation. It is also important that it produces perturbed meshes without severe distortions, in order to avoid large sensitivity errors. Finally, this method also should be efficient, simple to implement, and easily linked to a CAD system.

Finally, it should be noted that the design velocity field can be used not only in the sensitivity analysis, but also in the update of the finite element mesh from the current to the next step in the iterative process of shape optimization. The use of nodal velocities in the mesh update avoids the need to generate a new mesh for each new shape generated by the optimization procedure, increasing the efficiency of the optimization process. However, sooner or later this procedure leads to distorted meshes, yielding misleading "optimum designs". Therefore, the current trend is to use the design velocities in the sensitivity computation and to use an appropriate algorithm to generate a new mesh for each new geometry.

3.1. Computation methods

There are different approaches to the computation of nodal velocities. In this item these methods will be discussed in the context of the shape sensitivity analysis. In the boundary layer (or boundary node) approach only the nodes on the boundary curves linked to a given design variable are perturbed. This means that the nodal velocities are nonzero only for the boundary nodes. This procedure is simple, efficient, and easily incorporated in a CAD system. It should be noted that this approach is not only efficient for the velocity computation, but also for the FE sensitivity computation, since only the elements linked to the boundary nodes contribute to the pseudo-load vector. Several numerical examples show that this approach leads to high quality sensitivities (Parente and Vaz, 2001; Parente and Vaz, 2003). Therefore, this approach has been extensively used in practical applications.

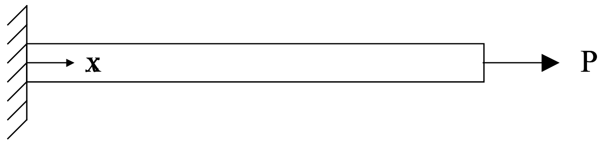

Figure 1. Primatic bar.

The boundary node approach can lead to a design velocity field that is not consistent with the geometric model adopted to describe the structure and with the mesh update scheme. For example, let us consider the prismatic bar of length L and axial stiffness EA depicted in Fig. (1). The displacement field of this bar is given by

$$u = \frac{Px}{EA} \tag{7}$$

A simple description of this structure is obtained considering L as the design variable and the following geometry parametrization

$$x = tL, \ \ 0 \le t \le 1 \qquad \Rightarrow \qquad \frac{dx}{dL} = t, \ \ \ 0 \le t \le 1 \tag{8}$$

where dx/dL is the design velocity field. The sensitivity of the displacement field can be computed from

$$\frac{du}{dL} = \frac{\partial u}{\partial L} + \frac{\partial u}{\partial x}\frac{dx}{dL} = 0 + \frac{P}{EA}t = \frac{Pt}{EA} \tag{9}$$

It is important to note the difference between the partial and total derivatives of the displacement w.r.t. the bar length. The former represents the displacement variation for a fixed Cartesian coordinate x, while the latter represents the displacement variation at a fixed parametric coordinate t.

On the other hand, the boundary node approach implicitly uses the following design velocity field

$$\frac{dx}{dL} = \begin{cases} 0, & \text{if } 0 \le t < 1 \\ 1, & \text{if } t = 1 \end{cases} \tag{10}$$

This velocity field is consistent with a mesh update where only the tip node moves and all other nodes remain in the same positions. Therefore, the sensitivity computed using the boundary node approach agrees with Eq. (9) for $t = 0$ and $t = 1$, which correspond to the boundaries of the structure. Thus, it can be concluded that for unidimensional problems the boundary node approach gives a good velocity field at the boundaries, but a rather crude approximation at the domain of the structure. For two and three-dimensional structures the problem is more complex, but the conclusions are similar.

The geometric approach is based on the division of the structure in a set of "design elements". The use of topologically triangular or quadrilateral design elements allows the generation of structured meshes and the definition of an explicit relation between the nodal coordinates and the design variables. The isoparametric and the transfinite mapping are the most used techniques to establish this relation. The differentiation of the explicit mathematical expression linking the nodal coordinates and the design variables leads to a simple and efficient computation of the design velocity field. Therefore, the geometric approach is simple and efficient. However, it requires the division of the structure in a set of triangular and quadrilateral regions, increasing the complexity of the geometric model and the difficulty of linking with CAD systems. This division is even more cumbersome for 3D structures where tetrahedral or hexahedral design elements should be used. Finally, during the optimization, the mesh can become severely distorted requiring the redefinition of the design elements.

The physical approach (Yao and Choi, 1989; Beckers, 1991) associates the geometric model of the structure with an elastic medium and compute the movements (design velocities) of the interior nodes as the nodal displacements due to prescribed displacements on the boundary. Thus, it is also known as the boundary displacement approach. Keeping fixed the parametric coordinates of the boundary nodes, the prescribed boundary displacements (design velocities) are easily determined by the analytical or numerical differentiation of the geometric description of the boundary curves. The computation of the design velocities of interior nodes is generally performed by a finite element analysis. The physical approach does not require the division of the geometric model in design elements. Moreover, it can be applied to structured or unstructured meshes yielding high quality design velocity fields. However, the stiffness matrix of the auxiliary model is different from the original one due to different boundary conditions. Thus, this method has a high computational cost since it requires the assembly and factorization of a new stiffness matrix. Finally, since it is based on finite element analysis, this method is not easily implemented in a CAD system.

The final approach considered here is the Laplacian smoothing. Initially, this method was used to update the position of nodal points at each optimization iteration (Yang, 1989), which works well only for small design changes due to element distortions. Thus, in order to obtain a more robust method, Kodiyalam et al. (1992) and Botkin (1992) used the Laplacian smoothing only for finding the velocities of interior nodes with a new mesh being generated at each optimization step. This approach is adopted in the present work. The Laplacian smoothing is a technique commonly used to improve the quality of unstructured meshes and consists of the iterative change of nodal coordinates of each interior node in order to place it at the center of gravity of the nodes connected with it

$$\mathbf{a}_i(b) = \frac{\sum_{i=1}^{n_i} \mathbf{a}_j(b)}{n_i} \tag{11}$$

where n_i is the number of adjacent nodes. The use of finite differences to compute the design velocities is based on the application of the Laplacian smoothing to the original and perturbed coordinates, thus

$$\mathbf{v}_i = \frac{\mathbf{a}_i(b+\Delta b) - \mathbf{a}_i(b)}{\Delta b} = \frac{1}{n_i} \frac{\sum_{j=1}^{n_i} \left(\mathbf{a}_j(b+\Delta b) - \mathbf{a}_j(b)\right)}{\Delta b} \tag{12}$$

or

$$\mathbf{v}_i = \frac{\sum_{j=1}^{n_i} \mathbf{v}_j}{n_i} \tag{13}$$

As in the physical approach, the velocities of boundary nodes used in the Laplacian smoothing are computed by the analytical or numerical differentiation of the geometric description of the boundary curves. The Laplacian smoothing can be used for structured and unstructured meshes, does not require the division of the model in design elements, and can be easily incorporated in a CAD system. On the other hand, it can require a large number of iterations to converge. Therefore, in order to achieve an efficient procedure, the number of iterations must be limited to a reasonable number. Finally, it is obvious from the definition of this method that it yields better results for uniform meshes, where Eq. (11) holds. It should be noted that this is not the case of meshes generated by adaptive schemes. Despite these problems, the Laplacian smoothing is the technique used in this work, since it is more simple and efficient than the physical approach, and more simple and generic than the geometric approach.

4. Refined Semi-Analytical Method

As discussed earlier, the semi-analytical approach combines the efficiency of the analytical approach with the simplicity and generality associated with the use of finite differences. Therefore, this approach became widely used in practical shape optimization. However, it as been verified that this approach can lead to large errors when applied to some structures. It has been shown that these errors occur when the displacement field of individual elements is dominated by rigid body rotations. Moreover, it was verified that the source of errors is the numerical differentiation of the internal force vector. The Refined Semi-Analytical approach (van Keulen and de Boer, 1998; de Boer and van Keulen, 2000; Parente and Vaz, 2001; Parente and Vaz; 2003) use some relations between the internal force vector and the rigid body modes in order to improve the quality of the semi-analytical sensitivities.

Regardless of the particular element formulation, the internal force vector of each element must satisfy the equilibrium conditions for a free body. The number of equilibrium equations is independent of the element formulation, but depends on the problem being considered. For 3D problems, there are three equations for the equilibrium of forces

$$\sum F_{\mathrm{x}} = 0, \quad \sum F_{\mathrm{y}} = 0, \quad \sum F_{\mathrm{z}} = 0 \tag{14}$$

and three for the equilibrium of moments

$$\sum M_{\mathrm{x}} = 0, \quad \sum M_{\mathrm{y}} = 0, \quad \sum M_{\mathrm{z}} = 0 \tag{15}$$

These equilibrium conditions can be easily written as

$$\mathbf{g}\cdot\mathbf{r_k} = 0 \tag{16}$$

where $\mathbf{r}_{\mathrm{k}}$ is a vector obtained from equilibrium conditions. Since a system of self-equilibrated forces (as $\mathbf{g}$) does not produces work through rigid body displacements, the vectors $\mathbf{r}_{\mathrm{k}}$ can be interpreted physically as the rigid body modes associated with the given element. These vectors form a base of a vector space. For the purposes of the Refined Semi-Analytical Method they should be mutually orthogonal ($\mathbf{r}_i \cdot \mathbf{r}_j = 0$, for $i \neq j$), but not unitary. Finally, as will be discussed later, these vectors are explicit functions of the nodal coordinates and can be exactly differentiated.

The differentiation of Eq. (16) with respect to the design variable b yields

$$\mathbf{g}'\cdot\mathbf{r_k} + \mathbf{g}\cdot\mathbf{r}'_{\mathbf{k}} = 0 \tag{17}$$

where ( )′ means d( )/db. Obviously, $\mathbf{g}'$ computed by the analytical approach satisfies this condition for the equilibrium of forces and moments. On the other hand, it was verified that the semi-analytical sensitivities satisfy this condition only for the equilibrium of forces and that the errors in the moment equations are very large for structures whose displacement field is characterized by large rigid body rotations (Parente and Vaz, 2001). The decomposition $\mathbf{g}'$ in a component in the space spanned by $\mathbf{r}_{\mathrm{k}}$ and another component orthogonal to this space yields

$$\mathbf{g}' = \frac{\mathbf{g}'\cdot\mathbf{r_k}}{\mathbf{r_k}\cdot\mathbf{r_k}}\mathbf{r_k} + \left(\mathbf{g}' - \frac{\mathbf{g}'\cdot\mathbf{r_k}}{\mathbf{r_k}\cdot\mathbf{r_k}}\mathbf{r_k}\right) \tag{18}$$

However, from Eq. (17)

$$\mathbf{g}'\cdot\mathbf{r_k} = -\,\mathbf{g}\cdot\mathbf{r}' \tag{19}$$

Therefore, using Eq. (18) and (19) the refined sensitivities can be written as

$$\mathbf{g}'_{\mathbf{ref}} = \mathbf{g}' - \left(\frac{\mathbf{g}'\cdot\mathbf{r_k}}{\mathbf{r_k}\cdot\mathbf{r_k}}\mathbf{r_k} + \frac{\mathbf{g}\cdot\mathbf{r}'_{\mathbf{k}}}{\mathbf{r_k}\cdot\mathbf{r_k}}\mathbf{r_k}\right) \tag{20}$$

or

$$\mathbf{g}'_{\mathbf{ref}} = \mathbf{g}' - \beta_k\mathbf{r_k}, \qquad \text{where} \qquad \beta_k = \frac{\mathbf{g}'\cdot\mathbf{r_k} + \mathbf{g}\cdot\mathbf{r}'_{\mathbf{k}}}{\mathbf{r_k}\cdot\mathbf{r_k}} \tag{21}$$

If $\mathbf{g}'$ in the equation above is computed by the analytical approach, the parameters β_k are equal to zero. However, when numerical differentiation is used to sensitivity computation, these parameters introduce a correction in the computed sensitivities in such way that the refined sensitivities always satisfy Eq. (17), even if the errors in conventional semi-analytical sensitivities are very large. Another interpretation of the Refined Semi-Analytical Method is that it replaces the components of $\mathbf{g}'$ in the direction of $\mathbf{r}_k$, which are inaccurately evaluated by finite differences, by its analytical derivative, which can be easily computed using Eq. (19).

Before to use Eq. (21) it is necessary to compute the orthogonal basis vectors $\mathbf{r}_k$ and the respective derivatives $\mathbf{r}_k'$. In fact, the Refined Semi-Analytical Method works because the computation of these vectors can be easily performed in an exact manner. This task is performed in two steps: initially a set of non-orthogonal rigid body modes $\bar{\mathbf{r}}_k$ and the respective derivatives $\bar{\mathbf{r}}_k'$ is determined, and after that an orthogonalization procedure is applied to these vectors yielding the desired vectors $\mathbf{r}_k$ and $\mathbf{r}_k'$.

The number and nature of the non-orthogonal basis vectors depend on the finite element type. It is important to note that they depend on the element degrees of freedom, but do not depend on the element formulation. Three-dimensional elements have three degrees of freedom per node, and all these d.o.f. are translations. As a consequence, the internal force vector contains only forces. For elements with rotational d.o.f., the internal force vector is composed by forces and moments. For a generic 3D element with *n* nodes, the first three basis vectors, which express the equilibrium of forces (or the rigid body translations) in each global direction (*x*, *y*, *z*), are given by

$$\begin{bmatrix} \bar{\mathbf{r}}_1 \\ \bar{\mathbf{r}}_2 \\ \bar{\mathbf{r}}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{F}_1 & \mathbf{F}_2 & \Lambda & \mathbf{F}_n \end{bmatrix}, \qquad \text{where } \mathbf{F}_i = \mathbf{I}_{3x3} \tag{22}$$

These vectors do not depend on the nodal coordinates. Therefore,

$$\bar{\mathbf{r}}_k' = \mathbf{0}, \qquad \text{for } k \leq 3 \tag{23}$$

The other basis vectors express the equilibrium of moments (or the rigid rotations) around each global axis (*x*, *y*, *z*)

$$\begin{bmatrix} \bar{\mathbf{r}}_4 \\ \bar{\mathbf{r}}_5 \\ \bar{\mathbf{r}}_6 \end{bmatrix} = \begin{bmatrix} \mathbf{M}_1 & \mathbf{M}_2 & \Lambda & \mathbf{M}_n \end{bmatrix}, \qquad \text{where } \mathbf{M}_i = \begin{bmatrix} 0 & -(z_i + w_i) & (y_i + v_i) \\ (z_i + w_i) & 0 & -(x_i + u_i) \\ -(y_i + v_i) & (x_i + u_i) & 0 \end{bmatrix} \tag{24}$$

In this expression (x_i, y_i, z_i) are the nodal coordinates and (u_i, v_i, w_i) the nodal displacements. The displacements need to be included in this expression because the equilibrium equation of geometrically nonlinear structure must be written in the deformed configuration. On the other hand, they should not be included when the analysis is linear. The differentiation of these vectors with respect to the nodal coordinates yields

$$\mathbf{M}_i' = \begin{bmatrix} 0 & -z_i' & y_i' \\ z_i' & 0 & -x_i' \\ -y_i' & x_i' & 0 \end{bmatrix} \tag{25}$$

where (x'_i, y'_i, z'_i) are the nodal velocities.

Therefore, the non-orthogonal rigid body modes can easily evaluated in an exact manner provided that the nodal velocities are known. The orthogonalization process will not be addressed here, since a detailed discussion can be find in the literature (van Keulen and de Boer, 1998; Parente and Vaz, 2001). Finally, the application of the Refined Semi-Analytical Method to other elements is straightforward, since the only additional implementation for each element is a function to compute the appropriate set of non-orthogonal basis vectors and the respective derivatives. Moreover, this implementation needs not to be performed for each individual element type, because these vectors are identical for all elements having the same degrees of freedom.

5. Numerical example

The cantilever beam depicted in Fig. (2) will be used in this work to assess the quality of displacement sensitivities computed using velocity fields evaluated by the boundary node and the Laplacian smoothing approaches. The beam has length *L = 10m*, load *P = 10 kN*, and bending stiffness *EI =* 10^6 *kNm*^{2}. A uniform mesh of 10 beam elements was used for the finite element analysis. The design variable considered here is the beam length.

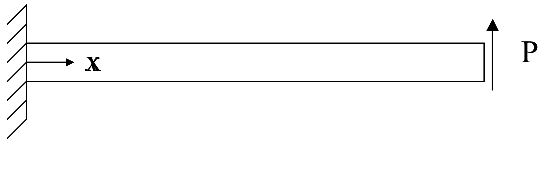

Figure 2. Cantilever beam.

For this simple structure, the transversal displacement (*v*) is given by

$$v = \frac{Px^2}{6EI}(3L - x) \tag{26}$$

In order to maintain a uniform mesh, the design velocity field should be exactly the same of Eq. (8). Therefore, the analytical sensitivity of the transversal displacement is given by

$$v' = \frac{3v}{L} \tag{27}$$

Two quantities should be properly defined in order to allow the comparison between the different methods. The first one is relative perturbation

$$\eta = \frac{\Delta L}{L} \tag{28}$$

and the other is the relative error

$$e = \left| \frac{v'_{approx} - v'_{exact}}{v'_{exact}} \right| \tag{29}$$

where the exact sensitivities are computed using Eq. (27).

Figure (3) shows the sensitivities of the transversal displacements along the beam length for $\eta = 10^{-4}$, where BN means the used of boundary node approach and LS the use of the Laplacian smoothing approach. These results show that the boundary node approach leads to poor results in the domain, but good results at the boundaries. This is an expected outcome since the boundary node approach is not consistent with the adopted mesh update scheme. On the other hand, the use of the Laplacian smoothing in this example not only leads to consistent results in the whole structure, but also greatly improves the quality of the conventional and refined semi-analytical sensitivities. Finally, it can be shown that the refined sensitivities are virtually equal to the analytical ones.

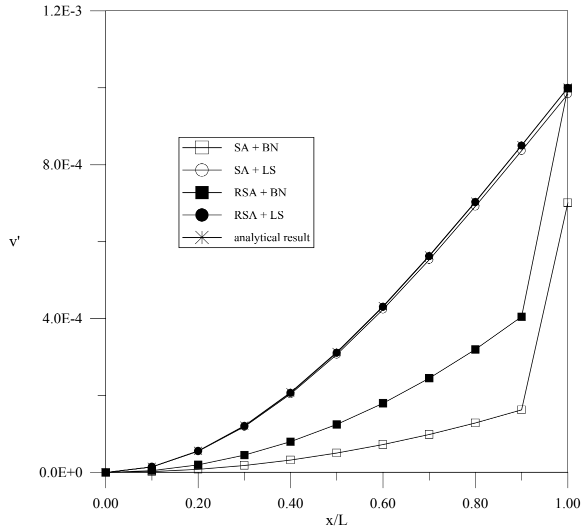

Figure 3. Displacement sensitivities along the beam.

To allow a better comparison of the methods discussed in this work, Fig. (4) displays the relative errors of the sensitivities of tip displacements for a wide range of relative perturbations. The four curves have a similar behavior, with a almost linear decrease (in the log scale) of the relative errors with the relative perturbations due to decreasing truncation errors until a given point where the errors start to increase due to growth of the rounding errors. The figure also shows that the Laplacian smoothing improves the quality of both conventional and refined semi-analytical sensitivities. Another important conclusion is that even for this mesh with few elements the refined sensitivities are more than an order of magnitude more accurate than the conventional ones for wide range of relative perturbations. However, it is important to note that for more refined meshes the advantage of the Refined Semi-Analytical Method increases (van Keulen and de Boer, 1998; Parente and Vaz, 2003).

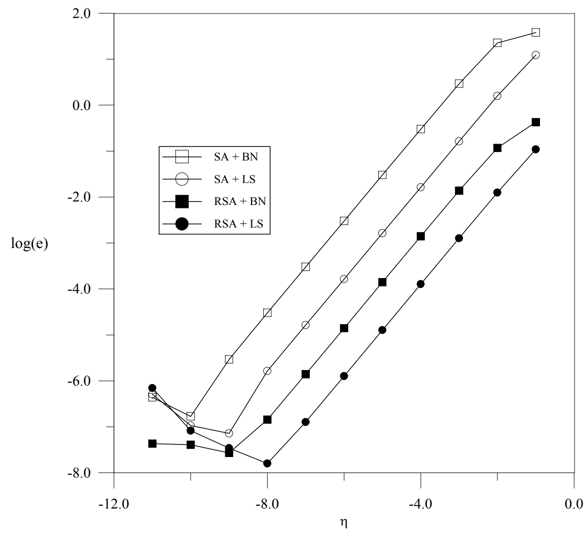

Figure 4. Relative errors of the sensitivities of tip displacements.

6. Conclusions

This work addressed two steps of the shape sensitivity analysis: the computation of the design velocity field and the evaluation of the displacement sensitivities in the context of the finite element method. From the existing approaches for computation of the design velocities, the boundary node and the Laplacian smoothing were chosen for a detailed comparison since both are simple, efficient, generic, and easily linked to CAD systems. Two methods for sensitivity computation, the Semi-Analytical and the Refined Semi-Analytical, were discussed in this work. The expressions for sensitivity improvement and the procedure for computation of the non-orthogonal rigid body modes and the respective derivatives were discussed in detail. It was shown that these derivatives can be easily evaluated in an exact manner provided that the design velocities are known.

The numerical example presented in this work shows that the use of Laplacian smoothing not only leads to consistent sensitivities for the entire domain, but also improves the quality of the conventional and refined semi-analytical sensitivities. It also demonstrates that, even for a coarse mesh, the refined semi-analytical sensitivities are much more accurate than the conventional ones. Therefore, the combination of the Laplacian smoothing with the Refined Semi-Analytical method forms a powerful tool for shape sensitivity analysis.

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