STUDY OF THE POTENTIAL OF IRREGULAR SHAPED BODIES

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Abstract. The conventional spherical harmonic representations of the gravitational potential of non-spherical bodies require expansions of high degree and order, which are difficult to obtain. The polyhedral method is well suited to evaluate the gravitational field of an irregularly shaped body such as asteroids, comet nucleus, and small planetary satellites. If complete coverage of the surface can be obtained, a polyhedral model of the body can be constructed. With a minimum effort, the method can incorporate important surface features, such as large craters and ridges. Expressions in closed forms are developed for the gravitational potential and for the acceleration due to the polyhedron with constant density. We make an analytical study for the potential and the components of the acceleration of some homogeneous bodies with well-defined geometric shapes such as the wire segment, the rectangle, the triangle and the disk. Results are developed in closed forms, instead of an infinite-series expansion, and involve only elementary functions (arc-tangent and logarithm). With these expressions, we can study orbits around such bodies. The results consist of sets of analytical equations that give the potential due to different geometrical forms that were implemented and tested and showed to be in agreement with the expectations.

Keywords. *gravitational potential, polyhedron, non-spherical bodies, astrodynamics, celestial mechanics* .

1. Introduction

The aim of this paper is to explore the closed-form expressions derived for the potential of some homogeneous bodies with well-defined geometric shapes. With the potential determined we obtained the components of the acceleration of these bodies to study the possible orbits of a spacecraft around them.

In this paper are derived the gravitational newtonian potential created by different mass distributions. We show some objects in one and two dimensions such as the wire segment, the rectangle, the triangle and the disk. All these potentials are expressed in closed forms and there is the presence of two kinds of terms: logarithms and arc tangents.

We describe in the next paragraphs some of the most important papers that we were able to find in the literature.

Werner (1994) models irregularly shaped objects such as asteroids, comet nuclei, and small satellites using polyhedron. With minor effort, this model can incorporate important surface features such the crater Stickney. The author develops closed-form expressions for the exterior gravitational potential and acceleration components due to a constant-density polyhedron. He illustrates an equipotential surface of the inner Martian satellite Phobos and compares the results obtained by the polyhedron's technique and by conventional spherical harmonics.

Veverka et al. (2001) study the images obtained during the landing of the NEAR-Shoemaker spacecraft on asteroid 433 Eros, on 12 February 2001, and about 70 images were obtained. The last sequence of images reveals a transition from the blocky surface to a smooth area, which the authors interpret as a 'pond'.

Romain et al. (2001) show ellipsoidal harmonic expansions to the modeling of the gravity field because small bodies of the solar system are targets of space exploration. Many of these bodies have elongated, non-spherical shapes, and the usual spherical harmonic expansions of their gravity fields are not well suited for the modeling of spacecraft orbits around these bodies. The authors study the future mission that the Rosetta module, Roland, should land on the surface of comet Wirtanen. Their main problem is to accurately compute spacecraft trajectories close to the surface of this body. They present a mathematical theory as well as the simulation of the landing on the surface of the comet. The result show that with an ellipsoidal harmonic expansion up to degree 5, the error on the landing position is at the meter level, while the corresponding error for the spherical harmonic expansion can reach tens of meter.

Broucke (2002) presents the Newton's law of gravity applied to round bodies, mainly spheres and shells. He also treats circular cylinders and disks with the same methods used for shells and it applies very well, almost with no modifications. The results are complete derivations for the potential and the force for the interior case as well as the exterior case.

The present work is a detailed revision of the works developed by Kellogg (1929) and Broucke (1995) and starts with the development of the potential generated by a segment of a straight line, because the logarithmic expression obtained plays an important role when the two and three-dimensional bodies are considered. The others are two-dimensional bodies (plates).

The approach will be as elementary as possible. The derivations are based on a table of elementary integrals given in the Appendix. All the expressions for the potentials and for the components of the acceleration are in closed forms.

According to Newton's gravitational law, the magnitude of the force between two particles, one of mass m_1, situated at point P, and another of mass m_2, situated at Q, is given by

$$F = G\frac{m_1 m_2}{r^2} \tag{1}$$

where r is the distance between P and Q points; G is the constant of the proportionality that depends solely on the units used.

By definition we call the body at the point P as the attracted particle and the other body as the attracting, then the components of the force will be determined at the point P.

2. The Potential of a Straight Wire Segment

Let a straight wire segment (a thin homogeneous massive rod), with linear density σ, extend in space from point P_1 (0, 0, 0) to point P_2 (0, 0, ℓ), and the attracted particle is in line with the wire, at the point P (0, 0, Z), $Z > \ell$ (Figure 1). Let us consider that the wire is divided into intervals by the points $\xi_0 = 0, \xi_1, \xi_2, \ldots, \xi_n = \ell$; then, the interval (ξ_k, ξ_{k+1}) has mass $\sigma\Delta\xi_k$, which is assumed to be concentrated at some point ξ'_k of the interval. So, the force due to the particle thus constructed will lie along the z-axis to the point considered, are (Kellogg, 1929):

$$Fz_k = -\frac{G\sigma\Delta\xi_k}{r_k^2}; \qquad Fx_k = Fy_k = 0. \tag{2}$$

where the distance between the particle and the point ξ'_k is given by: $r_k^2 = \left|Z - \xi'_k\right|^2$.

The force generated by the whole segment is the limit of the sum of the forces generated by the system of particles, or:

$$Fz = -G\sigma\int_0^{\ell}\frac{d\xi}{(Z-\xi)^2} \quad \rightarrow \quad Fz = -\frac{G\sigma\,\ell}{Z(Z-\ell)}$$

$$Fx = Fy = 0 \tag{3}$$

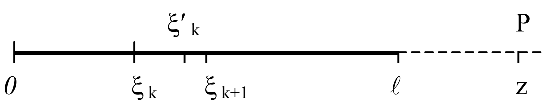


Figure 1. Straight wire segment along the z-axis.

Another approach to find the potential of the straight wire segment has been made by Werner (1994; 1996) who used the Gauss Divergence Theorem and derived the potential of a 3D polyhedron, with sub expressions equivalent to the potential of a 2D polygon and a 1D straight wire, which contain sub expressions involving a logarithm and a solid angle.

3. The Potential of the Rectangular Plate

Let us consider a homogeneous plane rectangular plate and an arbitrary point P, not on the rectangle (Figure 2). Take x and y-axes parallel to the sides of the rectangle, and their corners referred to these axes are $A(b, c)$, $B(b', c)$, $C(b', c')$, and $D(b, c')$. The distances from P (0, 0, Z) to these four points are, respectively:

$$
\begin{aligned}
d_1^2 &= b^2 + c^2 + Z^2 \\
d_2^2 &= b'^2 + c^2 + Z^2 \\
d_3^2 &= b'^2 + c'^2 + Z^2 \\
d_4^2 &= b^2 + c'^2 + Z^2
\end{aligned} \tag{4}
$$

Let ΔS_k denote a typical element of the surface, containing a point Q_k localized in the rectangular plate with coordinates (x_k, y_k). The distance r_k between the particle and the point Q_k is given by:

$$
r_k = \overline{PQ}_k \;\rightarrow\; r_k = \sqrt{(X - x_k)^2 + (Y - y_k)^2 + (Z - z_k)^2}
$$

but, $z_k = 0$ and $X = Y = 0$; then:

$$
r_k = \sqrt{x_k^2 + y_k^2 + Z^2} \tag{5}
$$

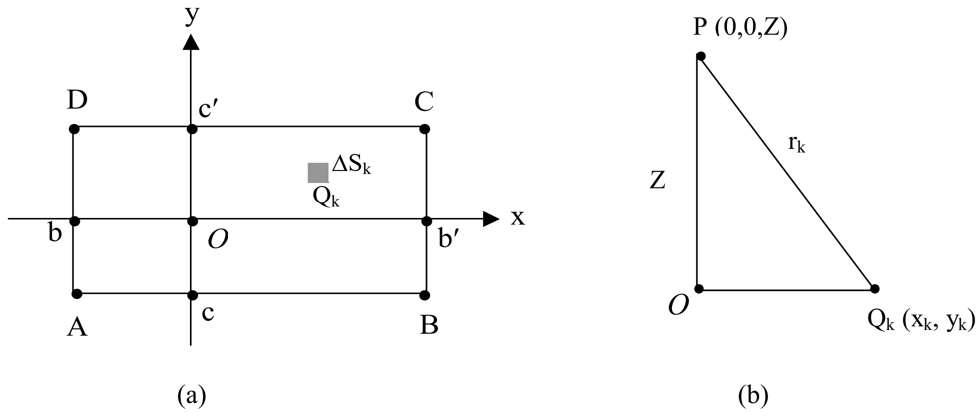


Figure 2. Rectangular plate – (a) rectangular plate is on the plane (x, y), (b) the distance between the element of the surface, containing a point Q_k, and the point P.

The potential of the rectangular plate can be given by the expression:

$$
U_k = \frac{G\sigma \Delta S_k}{r_k} \tag{6}
$$

or, $$U = G\sigma \int_c^{c'}\int_b^{b'} \frac{dxdy}{\sqrt{x^2 + y^2 + Z^2}} = G\sigma\left\{ \int_c^{c'} \ln\left(b' + \sqrt{b'^2 + y^2 + Z^2} \right) dy - \int_c^{c'} \ln\left(b + \sqrt{b^2 + y^2 + Z^2} \right) dy \right\}$$

In evaluating this integral we use the results 1 and 2 from the Appendix. We find:

$$
U = G\sigma \left[\begin{array}{l} c' \ln \dfrac{(b' + d_3)}{(b + d_4)} + c \ln \dfrac{(b + d_1)}{(b' + d_2)} + b' \ln \dfrac{(c' + d_3)}{(c + d_2)} + b \ln \dfrac{(c + d_1)}{(c' + d_4)} \\ + Z \left(\tan^{-1} \dfrac{b.c'}{Z.d_4} - \tan^{-1} \dfrac{b'.c'}{Z.d_3} + \tan^{-1} \dfrac{b'.c}{Z.d_2} - \tan^{-1} \dfrac{b.c}{Z.d_1} \right) \end{array} \right] \tag{7}
$$

and the magnitude of the element of the force at point P (0, 0, Z) is $F_k = \dfrac{G\sigma \Delta S_k}{r_k^2}$ (8)

This force has the direction cosines $\dfrac{x_k}{\sqrt{x_k^2 + y_k^2}}$; $\dfrac{y_k}{\sqrt{x_k^2 + y_k^2}}$; $-\dfrac{Z}{r_k}$

The components of the force are given by

$$Fx = G\sigma \int_{c}^{c'}\int_{b}^{b'} \frac{x\,dx\,dy}{\left(x^2+y^2+Z^2\right)\sqrt{x^2+y^2}}; \qquad Fy = G\sigma \int_{c}^{c'}\int_{b}^{b'} \frac{y\,dx\,dy}{\left(x^2+y^2+Z^2\right)\sqrt{x^2+y^2}};$$

$$Fz = G\sigma Z \int_{c}^{c'}\int_{b}^{b'} \frac{dxdy}{\left[x^2+y^2+Z^2\right]^{3/2}}$$

In evaluating these integrals (Kellogg, 1929), the result is:

$$Fx = G\sigma \ln\left[\frac{d_2+c}{d_1+c}\cdot\frac{d_4+c'}{d_3+c'}\right]; \quad Fy = G\sigma \ln\left[\frac{d_4+b}{d_1+b}\cdot\frac{d_2+b'}{d_3+b'}\right];$$

$$Fz = -G\sigma\left[\tan^{-1}\frac{b.c}{Z.d_1} - \tan^{-1}\frac{b'.c}{Z.d_2} + \tan^{-1}\frac{b'.c'}{Z.d_3} - \tan^{-1}\frac{b.c'}{Z.d_4}\right] \tag{9}$$

An important generalization of this formula can be done, to the case where the point P has arbitrary coordinates (X, Y, Z) instead of being on the z-axis. This task is rather easy, because the four vertices A, B, C, D of the rectangle have completely arbitrary locations (Broucke, 1995).

In order to have more general results is sufficient to replace (b, b', c, c') by $(b-X,\ b'-X,\ c-Y,\ c'-Y)$ in the formula of the potential (7). This generates an expression for the potential U as a function of the variables (X, Y, Z). Then, it is possible to compute the components of the acceleration by taking the gradient of the potential. It is important to note that in taking the partial derivatives of $U(X, Y, Z)$, the arguments of the logarithms and the arc tangents were treated as constants by Broucke (1995). This simplifies the work considerably.

The general expressions of the acceleration allow us to study orbits around a rectangular plate.

4. The Potential of the Triangular Plate

We will give the potential at a point P (0, 0, Z) on the z-axis created by the triangle shown in Figure 3 located in the xy-plane. The side $\overline{P_1P_2}$ is parallel to the x-axis. The coordinates of P_1 and P_2 are $\vec{r}_1(x_1, y_1)$ and $\vec{r}_2(x_2, y_2)$, but we have that $y_1 = y_2$ and $x_1 > x_2 > 0$. The distances are given by:

$$d_1^2 = x_1^2 + y_1^2 + Z^2; \qquad d_2^2 = x_2^2 + y_1^2 + Z^2 \tag{10}$$

where: d_1 is the distance from P (0, 0, Z) to the corner (x_1, y_1) of the triangle at point P_1;
d_2 is the distance from P (0, 0, Z) to the corner (x_2, y_1) of the triangle at point P_2.

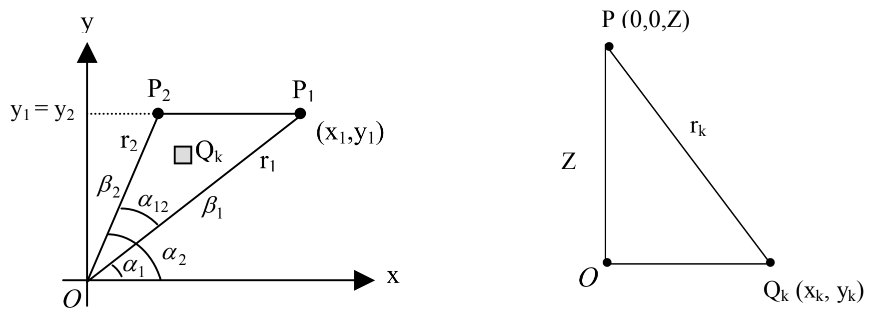


Figure 3. Triangular plate – (a) Triangle $\overline{OP_1P_2}$ located in the xy-plane(x, y), (b) the distance between the element of the surface, containing a point Q_k, and the point P.

The distance r_k between the particle and the point Q_k is given by:

$$r_k = \overline{PQ}_k \ \rightarrow\ r_k = \sqrt{(X-x_k)^2 + (Y-y_k)^2 + (Z-z_k)^2}$$

but, at the point P (0, 0, Z), the distance r_k remains: $r_k^2 = x_k^2 + y_k^2 + Z^2$ (11)

Using the definition of the potential (Eq. (6)) we have that the potential at P can be obtained by a double integration in x and y:

$$U = G\sigma \int_0^{y_1} \int_{\beta_2 y}^{\beta_1 y} \frac{dxdy}{r_k} = G\sigma \int_0^{y_1} \ln\left(\beta_1 y + \sqrt{(\beta_1^2 + 1) y^2 + Z^2}\right) dy - G\sigma \int_0^{y_1} \ln\left(\beta_2 y + \sqrt{(\beta_2^2 + 1) y^2 + Z^2}\right) dy \tag{12}$$

We used the integral 1 in x and the integral 3 in y from the Appendix; and we have that $\beta_1 = \frac{x_1}{y_1}$ and $\beta_2 = \frac{x_2}{y_2}$. Finally, the result is:

$$U = G\sigma \left\{ y_1 \ln\left[\frac{x_1 + d_1}{x_2 + d_2}\right] + Z \tan^{-1}\left(\frac{\beta_1 Z}{d_1}\right) - Z \tan^{-1}\left(\frac{\beta_2 Z}{d_2}\right) - |Z| \alpha_{12} \right\} \tag{13}$$

where $\alpha_{12} = \alpha_1 - \alpha_2$ represents the angle of the triangle at the origin O and it is showed in Figure 3.

The potential of this triangle at the point P (0, 0, Z) on the Z-axis must be invariant under an arbitrary rotation of the triangle around the same Z-axis. Therefore, Equation (13) should also be invariant under this rotation and their four terms are individually invariant, where they can be expressed in terms of invariant quantities, such as the sides and the angle of the triangle. Then, the new formula for the potential is

$$U = G\sigma \{ L_{12} + Z(S_1 + S_2) - |Z| \alpha_{12} \} \tag{14}$$

where the logarithmic term of Eq. (13) is given by:

$$L_{12} = \frac{C_{12}}{r_{12}} \ln\left[\frac{d_1 + d_2 + r_{12}}{d_1 + d_2 - r_{12}}\right], \tag{15}$$

and the two angles are of the form:

$$S_1 = \tan^{-1}\left(\frac{-S_z C_{12} d_1}{|Z|(r_1^2 - D_{12})}\right) \;\; ; \;\; S_2 = \tan^{-1}\left(\frac{-S_z C_{12} d_2}{|Z|(r_2^2 - D_{12})}\right) \tag{16}$$

Some expression in Equations (15) and (16) are defined as the dot-product $D_{12} = r_1 r_2 \cos\alpha_{12}$ and the cross-product $C_{12} = r_1 r_2 \sin\alpha_{12}$ by the vectors $\vec{r}_1, \vec{r}_2$, that are invariants. The symbol S_z is the sign of the variable Z, and r_{12} is the distance between P_1 and P_2. In order to obtain more details see Broucke (1995).

The potential of an arbitrary triangular plate $P_1P_2P_3$ can be obtained by the sum of three special triangles of the type used in this section. The potential at a point P (0, 0, Z) on the Z-axis is then obtained by adding three expressions of the type of Eq. (14). The result will have three logaritmic terms and six arc-tangent terms. The logarithmic terms are:

$$\begin{aligned} L_{12} &= \ln[(d_1 + d_2 + r_{12})/(d_1 + d_2 - r_{12})]; \\ L_{23} &= \ln[(d_2 + d_3 + r_{23})/(d_2 + d_3 - r_{23})]; \\ L_{31} &= \ln[(d_3 + d_1 + r_{31})/(d_3 + d_1 - r_{31})]. \end{aligned} \tag{17}$$

The six arc-tangents can be combined by pairs in three new terms. We first compute a numerator:

$$N = -Z(C_{12} + C_{23} + C_{31}) \tag{18}$$

and the denominators are:

$$\begin{aligned} D_1 &= Z^2(r_1^2 + D_{23} - D_{31} - D_{12}) - C_{12}C_{31} \;; \\ D_2 &= Z^2(r_2^2 + D_{31} - D_{12} - D_{23}) - C_{23}C_{12} \;; \\ D_3 &= Z^2(r_3^2 + D_{12} - D_{23} - D_{31}) - C_{31}C_{23}\,. \end{aligned} \tag{19}$$

So, the sum of four angles are:

$$\Sigma = \tan^{-1}\left(\frac{Nd_1}{D_1}\right) + \tan^{-1}\left(\frac{Nd_2}{D_2}\right) + \tan^{-1}\left(\frac{Nd_3}{D_3}\right) + S_z\pi \tag{20}$$

The potential function is given by:

$$U = G\sigma\left\{\frac{C_{12}}{r_{12}}L_{12} + \frac{C_{23}}{r_{23}}L_{23} + \frac{C_{31}}{r_{31}}L_{31} + Z\Sigma\right\} \tag{21}$$

This result is invariant with respect to an arbitrary rotation around the Z-axis.

In order to obtain the potential at a point $P(X,Y,Z)$, it is sufficient to replace all the vertex-coordinates (x_i, y_i) by $(x_i - X, y_i - Y)$ in the Equation (21). This gives us the new and more general expression of the potential U as a function of the three variables (X,Y,Z). As a consequence we can also compute the components of the acceleration as the gradient of U. It is important to note that, in taking the partial derivatives of $U(X,Y,Z)$, the arguments of the logarithms and the arc tangents were treated as constants. So, the components of the acceleration are given by:

$$\begin{aligned} Fx &= -G\sigma[(y_2 - y_1)L_{12}/r_{12} + (y_3 - y_2)L_{23}/r_{23} + (y_1 - y_3)L_{31}/r_{31}] \\ Fy &= G\sigma[(x_2 - x_1)L_{12}/r_{12} + (x_3 - x_2)L_{23}/r_{23} + (x_1 - x_3)L_{31}/r_{31}] \\ Fz &= G\sigma\Sigma \end{aligned} \tag{22}$$

5. The Potential of the Circular Disk

Let us take a homogeneous circular disk with radius 'a' and a particle at a point P (0, 0, Z) of its axis. Let the (x,y) – plane coincide with that of the disk, the origin being at the center (Figure 4). If σ denotes the superficial density, the element of the surface ΔS_k of the disk, containing the point $Q_k(\rho_k, \varphi_k)$ will have a mass $\sigma\Delta S_k$. Using Eq. (6), we can obtain the potential of the disk, with polar coordinates:

$$U = G\sigma\int_0^{2\pi}\int_0^{a}\frac{\rho d\rho d\varphi}{\sqrt{\rho^2 + Z^2}} = G\sigma\int_0^{2\pi}\left[\sqrt{a^2 + Z^2} - |Z|\right]d\varphi$$

or, $$U = G\sigma 2\pi\left[\sqrt{a^2 + Z^2} - |Z|\right] \tag{23}$$

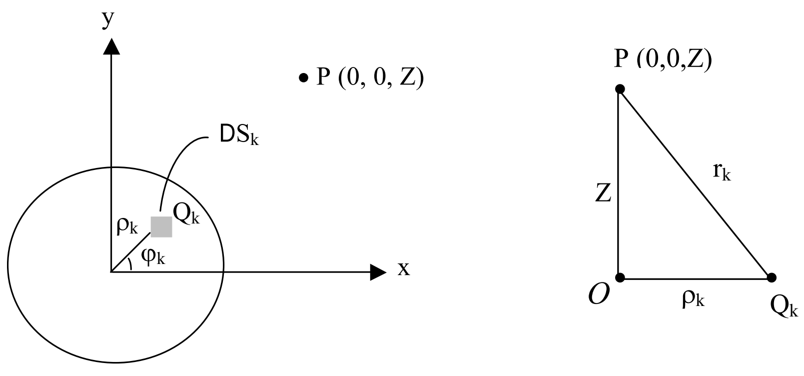
In Eq. (6), we use the distance r_k that is given by: $r_k = \overline{PQ}_k \rightarrow r_k = \sqrt{\rho_k^2 + Z^2}$ (24)

If this mass is sonsidered as concentrated at the point Q_k, it will exert on an unit particle P (0, 0, Z) a force whose magnitude is

$$F_k = \frac{G\sigma\Delta S_k}{r_k^2} \tag{25}$$

As the (x,y)-plane coincide with that of the disk, the components of the acceleration in x and y vanish by symmetry; and the direction cosine in the z-axis is $-Z/r_k$, the component of the force is given by:

$$Fz = -G\sigma Z\int_0^{2\pi}\int_0^{a}\frac{\rho d\rho d\varphi}{\left[\rho^2 + Z^2\right]^{3/2}} \Rightarrow Fz = -2\pi G\sigma Z\left[\frac{1}{|Z|} - \frac{1}{\sqrt{a^2 + Z^2}}\right] \tag{26}$$



(a) (b)

Figure 4. Circular disk – (a) disk is on the plane (x, y), (b) the distance between the element of the surface, containing a point Q_k, and the point P.

The derivations shown above give us the gravitational attraction of a homogeneous circular disk at a point $P(0,0,Z)$.

The objective to study the potential of simple forms, as well as the orbits around such bodies, is very important to obtain knowledge that will be necessary to study the cases of three-dimensional solids, such as polyhedra. Finally, we will study the potential and orbits around irregularly shaped bodies such as asteroids, comet nucleus, and small planetary satellites using the polyhedral method.

6. Results

The analytical development of expressions for the potential and the components of the acceleration of some geometric forms are performed in this work in order to obtain applied and theoretical knowledge with the objective to use it in more complex irregular forms, as the polyhedra. Besides, the expressions for the acceleration allow us to study orbits around those solids.

For some forms mentioned previously, some analyses were made, as the behavior of the potential and the components of the acceleration, and the validity of the approach (mentioned by Broucke, 1995) of the derivation of the potential.

We choose arbitrary values for the coordinates of the rectangle, the triangle, the disk, and for the proof point.

Rectangle: The coordinates of the rectangle were chosen arbitrarily in order to test the analytical expressions obtained. Those coordinates describe the vertexes of the rectangle and they are defined by: b = 10; b' = 20; c = 8; c' = 15. The coordinates of the particle, or proof point, are defined by: P (X,Y,Z) = P (0, 0, Z), with Z varying from 1 to 200. The constants used are Newton's gravitational constant and the density of the material are defined by: G = s = 1.

Figures (5) and (6) show the potential and the components of the acceleration for the rectangular plate with the data mentioned above.

In this work we can verify the validity of the approach. We obtained the expressions for the acceleration, according to the approach mentioned by Broucke, and the expressions of the acceleration taken into account the arguments of the logarithms and arc tangents. We compared the expressions in the two cases, and we verified that the error is of the order of 10^{-16}.

Figure (7) represents the potential of the rectangular plate and Figure (8) represents the component Fy of the acceleration with the points $P(X,Y,Z)$ with X = 0, Y varying from 0 to 20 and Z varying from 0 to 40. Figure (9) represents the component Fy of the acceleration and Figure (10) the component Fz of the acceleration with the points $P(X,Y,Z)$ with Y = 0, X varying from 0 to 20 and Z varying from 0 to 40. Those points were chosen arbitrarily. Another analysis of the potential and the attraction were made considering Z = 0 with X and Y varying. For the three cases we plot the components Fx, Fy, Fz; however, only some of the components were exemplified in that work.

Triangle: The coordinates of the triangle were chosen so that the triangle is equilateral and it is centered in the origin of the system. They are given by:

x_1 = -0.438691337650831; x_2 = -0.438691337650831; x_3 = 0.877382675301662;

y_1 = 0.759835685651593; y_1 = -0.759835685651593; y_1 = 0.000000000000000.

The coordinates of the proof point are: $P(X,Y,Z)$= P (0, 0, Z), with ' Z' varying from 1 to 200. The constants are: G = s = 1.

Figures (11) and (12) represent the potential and the component Fz of the acceleration for the triangular plate with the data mentioned above. Others components, Fx and Fy are zero. For the triangle we verify the validity of the approach mentioned by Broucke, and the error is of the order of 10^{-14}. Figures (13) and (14) show the potential and the component Fz of the acceleration for the triangular plate with the points $P(X, Y, Z)$ with X = 0, Y varying from 0 to 40 and Z varying from 0 to 40. Those points were chosen arbitrarily. Another analysis of the potential and the attraction were made considering a second case, Y = 0 with X and Z varying and, Z = 0 with X and Y varying.

Disk: For the disk, only its area should be defined and it was defined by: Area of the disk, a = 30. The coordinates of the proof point are: $P(X, Y, Z)$ = P (0, 0, Z), with ' Z' varying from 1 to 400. The constants are: G = s = 1.

Figures (15) and (16) show the potential and the components of the acceleration for the disk with the data mentioned above.

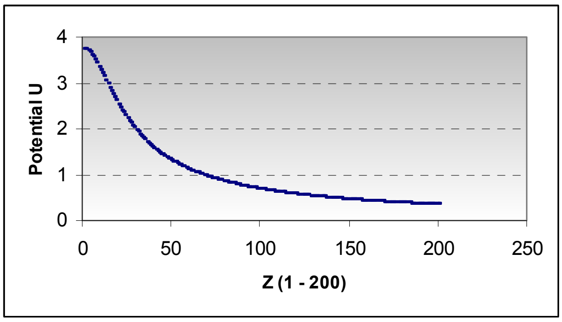


Figure 5 – Potential for the Rectangle – P (0, 0, Z)

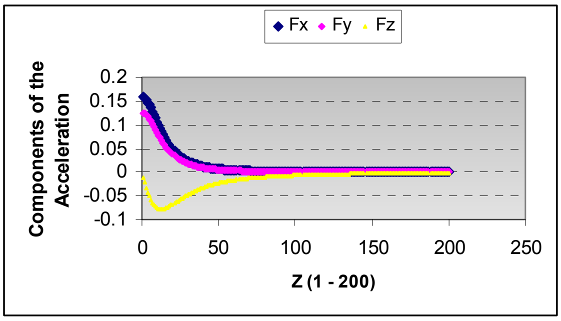


Figure 6 – Components of the Acceleration for the Rectangle – P (0, 0, Z)

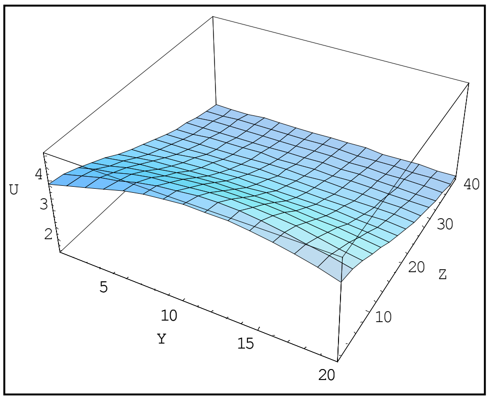


Figure 7 – Potential for the Rectangle – P (0, Y, Z), with Y varying from 0 to 20 and Z varying from 0 to 40

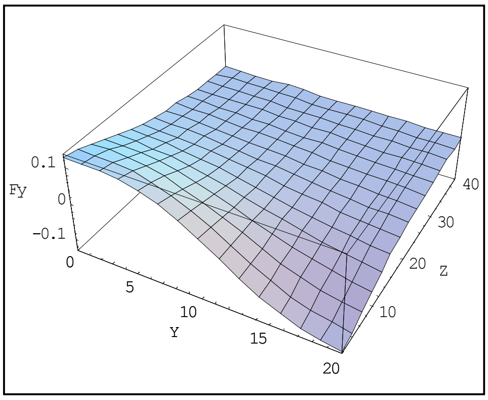


Figure 8 – Fy for the Rectangle – P (0, Y, Z), with Y varying from 0 to 20 and Z varying from 0 to 40

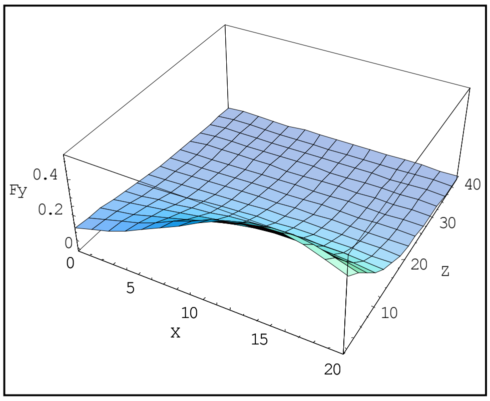


Figure 9 – Fy for the Rectangle – P (X, 0, Z), with X varying from 0 to 20 and Z varying from 0 to 40

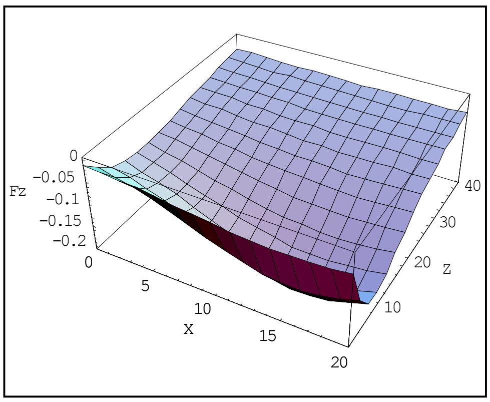


Figure 10 – Fz for the Rectangle – P (X, 0, Z), with X varying from 0 to 20 and Z varying from 0 to 40

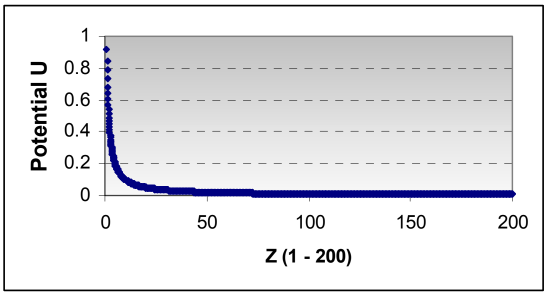


Figure 11 – Potential for the Triangle – P (0, 0, Z)

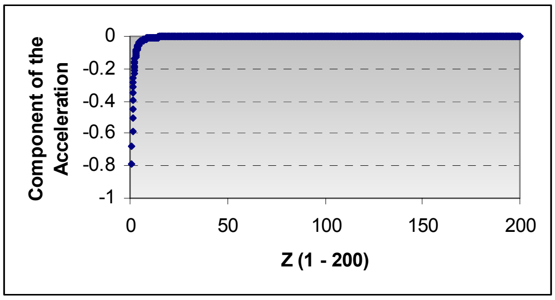


Figure 12 – Component Fz of the Acceleration for the Triangle – P (0, 0, Z)

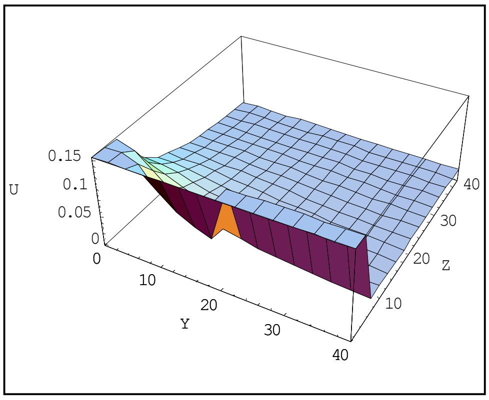


Figure 13 – Potential for the Triangle – P (0, Y, Z), with Y varying from 0 to 40 and Z varying from 0 to 40

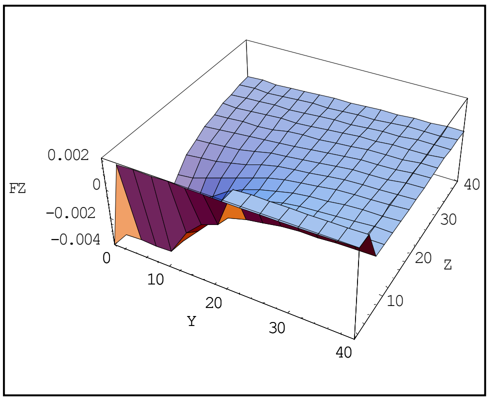


Figure 14 – Fz for the Triangle – P (0, Y, Z), with Y varying from 0 to 40 and Z varying from 0 to 40

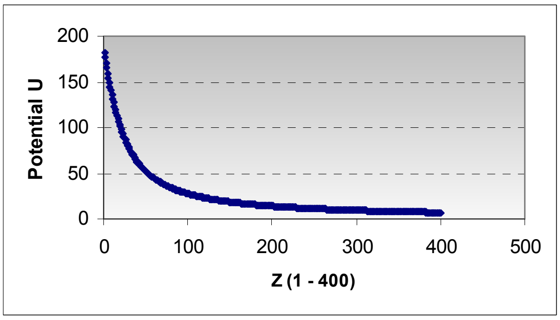


Figure 15 – Potential for the Disk – P (0, 0, Z)

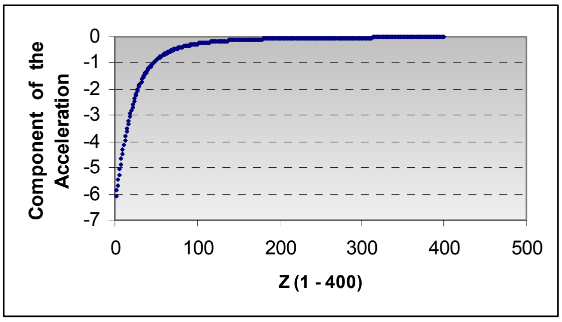


Figure 16 – Component Fz of the Acceleration for the Disk – P (0, 0, Z)

7. Conclusions

In this paper an analytical study was performed to obtain closed form solutions for the potential of some geometric forms. The development was applied to the wire segment, the rectangle, the triangle and the disk. The equations derived here open the way to obtain trajectories around those forms as well as to obtain the potential of three-dimensional bodies, such as the pyramid, the cube and so on. After obtaining the analytical equations, they were implemented and tested. The results showed to be very good and in agreement with the expectations.

8. Appendix – Table of Integrals

In order to obtain the potential and the components of the acceleration, some indefinite integrals have been used. The integrals used in the wire segment and in the disk have been done easily by simple substitution; others are taken from the table (Spiegel, 1981) or by the software Mathematica. The integral 2 is taken from Kellogg's book (1929, page 57). Some integrals are:

1) $\int \frac{dx}{\sqrt{x^2+y^2+z^2}} = \ln\left(x+\sqrt{x^2+y^2+z^2}\right)$

2) $\int \ln\left(y+\sqrt{x^2+y^2+z^2}\right)dx = x\ln\left(y+\sqrt{x^2+y^2+z^2}\right) + y\ln\left(x+\sqrt{x^2+y^2+z^2}\right) - x + z\tan^{-1}\left(\frac{x}{z}\right) -$

$z\tan^{-1}\left(\frac{xy}{z\sqrt{x^2+y^2+z^2}}\right)$

3) $\int \ln\left(Ax+\sqrt{Cx^2+z^2}\right)dx = x\ln\left(Ax+\sqrt{Cx^2+z^2}\right) - x + z\tan^{-1}\left(\frac{x}{z}\right) + z\tan^{-1}\left(\frac{Az}{\sqrt{Cx^2+z^2}}\right)$

The constants in the integral 3, used to obtain the potential of the triangle, are: $A=\beta$ and $C=\beta^2+1$. Note that Kellogg's integral has a misprint in the last of the five terms, checked by calculations and observed by Broucke (1995): the 'x' in the numerator should be changed into ξ.

9. Acknowledgement

The authors wish to express their appreciation for the supports provide by FAPESP (Foundation to Support Research in the São Paulo State), for the contracts 01/00080-7 and 00/14769-4, by CNPq (National Council for Scientific and Technological Development) - Brazil, for the contracts 300221/95-9 and 300347/01-4, and by CAPES, Brazil. They are very grateful to Roger A. Broucke for the informations and discussions.

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