A THEORY FOR LAMINATED PLATES WITH PIEZOELECTRIC LAYERS

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Abstract. *A theory for laminated composite plates with piezoelectric laminae is presented herein assuming a transverse distribution for the displacement field according to the Reissner-Mindlin hypothesis and also an electric potential which varies linearly across the thickness of each piezoelectric layer. The equations describing the plate behavior and the boundary conditions, in terms of the displacement components and the electric potential, are obtained in a consistent manner by means of the principle of virtual displacement. Navier type solutions are developed using different rectangular plate configurations with piezoelectric actuators or sensors included.*

Keywords. *Piezoelectricity, Exact Solution, Sensor, Actuator*.

1. Introduction

Piezoelectricity consists in converting mechanical energy into electrical energy, and vice-versa. In 1880, the brothers Pierre and Paul-Jacques Curie discovered the *direct piezoelectric effect* noticing electrical charge on the surface of a piezoelectric body due to mechanical deformation. One year later, Lippmann predicted from purely thermodynamical considerations the *converse piezoelectric effect,* i.e. the changing in the shape of a piezoelectric body submitted to an electrical field. Curie brothers observed experimentally the *converse piezoelectric effect* in the same year of 1881.

Nowadays, piezoelectric materials are wide used in electromechanical devices due to the possibility of creating structures and systems capable of adapting to or correcting for changing operating conditions. Inclusion of these types of material into the structure has the advantage of making the sensing and actuating mechanism part of the structure.

The study of piezoelectric materials embedded or bounded has been the subject of intensive studies in recent years. Accurate models for predict the electromechanical behavior is a key issue in the design and control of smart material devices. Mitchell and Reddy (1995) proposed a plate theory for composite laminates with piezoelectric laminae that utilizes third-order shear approximation for the displacement field and the so-called discrete-layer (or layerwise) assumption for the electric potential. Saravanos (1997) developed a shell theory considering a layerwise discretization for the electrical potential and a first-order shear approximation for the displacement field.

In this paper the authors present a piezolaminated plate theory utilizing a first-order shear approximation (*Reissner-Mindlin* hypothesis) and assuming an electric potential which varies linearly across the thickness of each piezoelectric layer. The theory corresponds to that one presented by Saravanos for shells, but emphasizes the potential difference instead of the potential itself in each piezoelectric layer. Three laminated plate problems, with piezoelectric actuators or sensors included, are analyzed by means of Navier solutions.

2. Governing Equations

Equations describing the electromechanical behavior of a deformable body may be classified in three groups: motion (equilibrium) equations, constitutive equations and strain-displacement relations. These equations are presented next.

2.1. Equilibrium Equations

The equilibrium equations of any point within a deformable solid submitted to the body forces $\{\overline{F}\}^T = \lfloor \overline{F}_x \quad \overline{F}_y \quad \overline{F}_z \rfloor$ may be described by

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \overline{F}_x = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \overline{F}_y = 0 \qquad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \overline{F}_z = 0 . \tag{1}$$

On the solid contour, where surface forces $\{\overline{T}\}^T = \lfloor \overline{T}_x \quad \overline{T}_y \quad \overline{T}_z \rfloor$ are present, the equilibrium condition is achieved if

$$\sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z = \overline{T}_x \qquad \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z = \overline{T}_y \qquad \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z = \overline{T}_z \tag{2}$$

in which $\{n\}^T = \lfloor n_x \quad n_y \quad n_z \rfloor$ is the outward normal unit vector.

If the solid exhibits a volumetric charge density $\overline{Q}_e$ Gauss's law states that the electric displacement $\{D\}^T = \lfloor D_x \quad D_y \quad D_z \rfloor$ must obey

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} - \overline{Q}_e = 0 \tag{3}$$

within the body and

$$D_x n_x + D_y n_y + D_z n_z = \overline{q}_e \tag{4}$$

on its surface, where $\overline{q}_e$ is the surface charge density.

2.2. Strain-Displacement Relations

If the displacement gradients are small, the following linear strain-displacement relations are valid

$$\begin{aligned}\{\varepsilon\}^T &= \lfloor \varepsilon_x \quad \varepsilon_y \quad \varepsilon_z \quad \gamma_{yz} \quad \gamma_{xz} \quad \gamma_{xy} \rfloor \\ &= \left\lfloor \frac{\partial u_x}{\partial x} \quad \frac{\partial u_y}{\partial y} \quad \frac{\partial u_z}{\partial z} \quad \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \quad \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \quad \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)\right\rfloor.\end{aligned} \tag{5}$$

Considering a conservative electric field $\{E\}$, a scalar potential function ϕ exists so that

$$\{E\}^T = \lfloor E_x \quad E_y \quad E_z \rfloor = \left\lfloor \left(-\frac{\partial \phi}{\partial x}\right) \quad \left(-\frac{\partial \phi}{\partial y}\right) \quad \left(-\frac{\partial \phi}{\partial z}\right)\right\rfloor. \tag{6}$$

2.3 Principle of Virtual Displacement

Equation (1) and Eq. (3) may be rewritten in the integral equivalent form

$$\begin{aligned}&\int_V \left[\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \overline{F}_x\right)\delta u_x + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \overline{F}_y\right)\delta u_y + \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \overline{F}_z\right)\delta u_z\right] dV \\ &+ \int_V \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} - \overline{Q}_e\right)\delta\phi \, dV = 0.\end{aligned} \tag{7}$$

Equation (7), after some mathematical manipulation and with the introduction of Eq. (2) and Eq. (4), leads to the following extension of the principle of virtual displacement

$$-\int_V \left(\{\sigma\}^T \{\delta\varepsilon\} - \{D\}^T \{\delta E\}\right) dV + \int_V \left(\{\overline{F}\}^T \{\delta u\} - \overline{Q}_e \delta\phi\right) dV + \int_{S_\sigma} \{\overline{T}\}^T \{\delta u\} dS + \int_{S_q} \overline{q}_e \delta\phi \, dS = 0 \tag{8}$$

where S_σ and S_q are, respectively, the surface portion with stress and electric charge prescribed and

$$\{\sigma\}^T = \lfloor \sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{yz} \quad \tau_{xz} \quad \tau_{xy} \rfloor \qquad \{u\}^T = \lfloor u_x \quad u_y \quad u_z \rfloor. \tag{9}$$

2.4. Constitutive Equations

In a piezoelectric material, the equations of elasticity and electrostatics interact through the constitutive equations. For a linear constitutive material (Reddy, 1997)

$$\begin{Bmatrix} \{\sigma\} \\ \{D\} \end{Bmatrix} = \begin{bmatrix} [c] & -[e]^T \\ [e] & [\xi] \end{bmatrix} \begin{Bmatrix} \{\varepsilon\} \\ \{E\} \end{Bmatrix} \tag{10}$$

where $[c]$ is the elastic stiffness tensor, $[e]$ is the piezoelectric tensor and $[\xi]$ is the dielectric tensor.

3. Plate Equations

The elasticity and electrostatics basic equations presented in the previous section will be simplified to expressions of a bi-dimensional problem by means of some simplifying hypothesis.

3.1 Field Variable Relations

Suppose that line segments normal to the midsurface surface before deformation remain straight but not necessarily normal to the deformed midsurface after deformation. Under this assumption the displacement field takes the form

$$\begin{aligned} u_x(x,y,z) &= u(x,y) + z\beta_x(x,y) \\ u_y(x,y,z) &= v(x,y) + z\beta_y(x,y) \\ u_z(x,y,z) &= w(x,y), \end{aligned} \tag{11}$$

where (u, v, w) are the displacement of a point on the midsurface of the laminate and (β_x, β_y) are the rotation angles of the line segments on the x-z and y-z planes, respectively. Substitution of Eq. (11) into Eq. (5) leads to

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \{\varepsilon_m\} + z\{\kappa\} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} + z \begin{Bmatrix} \dfrac{\partial \beta_x}{\partial x} \\ \dfrac{\partial \beta_y}{\partial y} \\ \dfrac{\partial \beta_x}{\partial y} + \dfrac{\partial \beta_y}{\partial x} \end{Bmatrix} \qquad \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \{\gamma\} = \begin{Bmatrix} \dfrac{\partial w}{\partial y} + \beta_y \\ \dfrac{\partial w}{\partial x} + \beta_x \end{Bmatrix} \qquad \varepsilon_z = 0. \tag{12}$$

If the electric potential ϕ varies linearly across the thickness of each piezoelectric layer

$$\phi(x,y,z) = \left(1 + \frac{z_k - z}{h_k}\right)\phi_k(x,y) \qquad \text{for } z_k \le z \le z_{k+1}, \tag{13}$$

where N is the number of laminated layers, ϕ_k is the electric potential at the bottom surface of layer k and h_k is the thickness of this layer. The electric field $\{E\}$ is thus obtained substituting Eq. (13) into Eq. (6):

$$\{E\} = \begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix} = \begin{Bmatrix} \left(\dfrac{z - z_k}{h_k} - 1\right)\dfrac{\partial \phi_k}{\partial x} \\ \left(\dfrac{z - z_k}{h_k} - 1\right)\dfrac{\partial \phi_k}{\partial y} \\ \dfrac{\phi_k}{h_k} \end{Bmatrix} \qquad \text{for } z_k \le z \le z_{k+1}. \tag{14}$$

3.2 Principle of Virtual Displacement

From Eq. (12) and Eq. (14), the first integral (internal virtual work) in Eq. (8) takes the form

$$\delta W_i = -\int_A \left(\{\delta\varepsilon_m\}^T\{N\} + \{\delta\kappa\}^T\{M\} + \{\delta\gamma\}^T\{Q\}\right)dxdy + \sum_{k=1}^{N}\int_A \left(L_x^{(k)}\frac{\partial \delta\phi_k}{\partial x} + L_y^{(k)}\frac{\partial \delta\phi_k}{\partial y} + J_k\delta\phi_k\right)dxdy, \tag{15}$$

with the stress resultants defined as

$$\{N\}=\begin{Bmatrix}N_x\\N_y\\N_{xy}\end{Bmatrix}=\int_{-h/2}^{h/2}\begin{Bmatrix}\sigma_x\\\sigma_y\\\tau_{xy}\end{Bmatrix}dz \qquad \{M\}=\begin{Bmatrix}M_x\\M_y\\M_{xy}\end{Bmatrix}=\int_{-h/2}^{h/2}\begin{Bmatrix}\sigma_x\\\sigma_y\\\tau_{xy}\end{Bmatrix}zdz \qquad \{Q\}=\begin{Bmatrix}Q_y\\Q_x\end{Bmatrix}=\int_{-h/2}^{h/2}\begin{Bmatrix}\tau_{yz}\\\tau_{xz}\end{Bmatrix}dz \tag{16}$$

and the piezoelectric resultants given in each layer by

$$\begin{Bmatrix}L_x\\L_y\end{Bmatrix}^{(k)}=\int_{z_k}^{z_{k+1}}\begin{Bmatrix}D_x\\D_y\end{Bmatrix}\left(\frac{z-z_k}{h_k}-1\right)dz \qquad J_k=\int_{z_k}^{z_{k+1}}\frac{D_z}{h_k}dz. \tag{17}$$

Similarly, based on Eq. (11) and Eq. (13) the second, third and fourth integrals (external virtual work) in Eq. (8) may be rewritten as

$$\delta W_e=\int_A\left(\bar{q}_x\delta u+\bar{q}_y\delta v+\bar{q}_z\delta w+\bar{m}_y\delta\beta_x-\bar{m}_x\delta\beta_y\right)dxdy+\sum_{k=1}^{N}\int_A\bar{q}_{ek}\delta\phi_k\,dxdy$$
$$+\int_{\Gamma_\sigma}\left(\bar{N}_n\delta u_{on}+\bar{N}_{ns}\delta u_{os}+\bar{Q}_n\delta w+\bar{M}_n\delta\beta_n+\bar{M}_{ns}\delta\beta_s\right)ds+\sum_{k=1}^{N}\int_{\Gamma_q^{(k)}}\bar{L}_n^{(k)}\delta\phi_k\,ds \tag{18}$$

where Γ_σ and $\Gamma_q^{(k)}$ are, respectively, the midsurface boundary with force prescribed and the k-layer bottom surface boundary with electric charge prescribed; δu_{on} and δu_{os} are, respectively, the normal and tangential displacement components along Γ_σ, $\bar{q}_{ek}$ is the surface charge density at the bottom surface of layer k.

The principle of virtual displacement, Eq. (8), may be rewritten after integrating Eq. (15) by parts to transfer all differentiations from the virtual displacement and potential to their coefficients. After collecting the coefficients of δu, δv, δw, $\delta\beta_x$, $\delta\beta_y$ and $\delta\phi_k$ the following expression is achieved

$$\delta W_i+\delta W_e=\int_A\left[\left(\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}+\bar{q}_x\right)\delta u+\left(\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}+\bar{q}_y\right)\delta v+\left(\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+\bar{q}_z\right)\delta w\right.$$
$$\left.+\left(\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x+\bar{m}_y\right)\delta\beta_x+\left(\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y-\bar{m}_x\right)\delta\beta_y\right]dxdy$$
$$-\sum_{k=1}^{N}\int_A\left(\frac{\partial L_x^{(k)}}{\partial x}+\frac{\partial L_y^{(k)}}{\partial y}-J_k-\bar{q}_{ek}\right)\delta\phi_k\,dxdy$$
$$-\int_{\Gamma_\sigma}\left[\left(N_n-\bar{N}_n\right)\delta u_{on}+\left(N_{ns}-\bar{N}_{ns}\right)\delta u_{os}+\left(Q_n-\bar{Q}_n\right)\delta w\right.$$
$$\left.+\left(M_n-\bar{M}_n\right)\delta\beta_n+\left(M_{ns}-\bar{M}_{ns}\right)\delta\beta_s\right]ds$$
$$+\sum_{k=1}^{N}\int_{\Gamma_q^{(k)}}\left(L_n^{(k)}-\bar{L}_n^{(k)}\right)\delta\phi_k\,ds=0, \tag{19}$$

where

$$\begin{Bmatrix}N_n\\N_{ns}\end{Bmatrix}=\begin{bmatrix}n_x^2 & n_y^2 & 2n_xn_y\\-n_xn_y & n_xn_y & n_x^2-n_y^2\end{bmatrix}\begin{Bmatrix}N_x\\N_y\\N_{xy}\end{Bmatrix} \qquad Q_n=Q_xn_x+Q_yn_y$$
$$\begin{Bmatrix}M_n\\M_{ns}\end{Bmatrix}=\begin{bmatrix}n_x^2 & n_y^2 & 2n_xn_y\\-n_xn_y & n_xn_y & n_x^2-n_y^2\end{bmatrix}\begin{Bmatrix}M_x\\M_y\\M_{xy}\end{Bmatrix} \qquad L_n^{(k)}=L_x^{(k)}n_x+L_y^{(k)}n_y\,. \tag{20}$$

The equilibrium equations are obtained by setting to zero the coefficients of δu, δv, δw, $\delta\beta_x$, $\delta\beta_y$ and $\delta\phi_k$ over the domain A of Eq. (19):

$$
\begin{aligned}
&\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}+\bar{q}_x=0\\
&\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}+\bar{q}_y=0\\
&\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+\bar{q}_z=0\\
&\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x+\bar{m}_y=0\\
&\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y-\bar{m}_x=0\\
&\frac{\partial L_x^{(k)}}{\partial x}+\frac{\partial L_y^{(k)}}{\partial y}-J_k-\bar{q}_{ek}=0.
\end{aligned}
\tag{21}
$$

3.3. Boundary Conditions

In view of Eq. (19), the boundary conditions are stated as

$$
\begin{array}{llll|llll}
u_{on}=\bar{u}_{on} & \text{or} & N_n=\bar{N}_n & & \beta_n=\bar{\beta}_n & \text{or} & M_n=\bar{M}_n \\
u_{os}=\bar{u}_{os} & \text{or} & N_{ns}=\bar{N}_{ns} & & \beta_s=\bar{\beta}_s & \text{or} & M_{ns}=\bar{M}_{ns} \\
w=\bar{w} & \text{or} & Q_n=\bar{Q}_n & & \phi_k=\bar{\phi}_k & \text{or} & L_n^{(k)}=\bar{L}_n^{(k)}.
\end{array}
\tag{22}
$$

3.4. Constitutive Equations

Substituting Eq. (10) into Eq. (16) and Eq. (17), one gets

$$
\begin{aligned}
&\begin{Bmatrix} N_x\\ N_y\\ N_{xy}\end{Bmatrix}=\begin{bmatrix} A_{11} & A_{12} & A_{16}\\ A_{12} & A_{22} & A_{26}\\ A_{16} & A_{26} & A_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_x^m\\ \varepsilon_y^m\\ \gamma_{xy}^m\end{Bmatrix}+\begin{bmatrix} B_{11} & B_{12} & B_{16}\\ B_{12} & B_{22} & B_{26}\\ B_{16} & B_{26} & B_{66}\end{bmatrix}\begin{Bmatrix}\kappa_x\\ \kappa_y\\ \kappa_{xy}\end{Bmatrix}-\sum_{k=1}^{N}\begin{Bmatrix}\bar{e}_{31}\\ \bar{e}_{32}\\ \bar{e}_{36}\end{Bmatrix}^{(k)}\phi_k\\
&\begin{Bmatrix} M_x\\ M_y\\ M_{xy}\end{Bmatrix}=\begin{bmatrix} B_{11} & B_{12} & B_{16}\\ B_{12} & B_{22} & B_{26}\\ B_{16} & B_{26} & B_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_x^m\\ \varepsilon_y^m\\ \gamma_{xy}^m\end{Bmatrix}+\begin{bmatrix} D_{11} & D_{12} & D_{16}\\ D_{12} & D_{22} & D_{26}\\ D_{16} & D_{26} & D_{66}\end{bmatrix}\begin{Bmatrix}\kappa_x\\ \kappa_y\\ \kappa_{xy}\end{Bmatrix}-\sum_{k=1}^{N}\begin{Bmatrix}\bar{e}_{31}\\ \bar{e}_{32}\\ \bar{e}_{36}\end{Bmatrix}^{(k)}\frac{z_k+z_{k+1}}{2}\phi_k\\
&\begin{Bmatrix} Q_y\\ Q_x\end{Bmatrix}=\begin{bmatrix} G_{44} & G_{45}\\ G_{45} & G_{55}\end{bmatrix}\begin{Bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{Bmatrix}+\sum_{k=1}^{N}\frac{h_k}{2}\begin{bmatrix}\bar{e}_{14} & \bar{e}_{24}\\ \bar{e}_{15} & \bar{e}_{25}\end{bmatrix}^{(k)}\begin{Bmatrix}\dfrac{\partial\phi_k}{\partial x}\\ \dfrac{\partial\phi_k}{\partial y}\end{Bmatrix}\\
&\begin{Bmatrix} L_x\\ L_y\end{Bmatrix}^{(k)}=-\frac{h_k}{2}\begin{bmatrix}\bar{e}_{14} & \bar{e}_{15}\\ \bar{e}_{24} & \bar{e}_{25}\end{bmatrix}^{(k)}\begin{Bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{Bmatrix}+\frac{h_k}{3}\begin{bmatrix}\bar{\xi}_{11} & \bar{\xi}_{12}\\ \bar{\xi}_{12} & \bar{\xi}_{22}\end{bmatrix}^{(k)}\begin{Bmatrix}\dfrac{\partial\phi_k}{\partial x}\\ \dfrac{\partial\phi_k}{\partial y}\end{Bmatrix}\\
&J_k=\begin{Bmatrix}\bar{e}_{31}\\ \bar{e}_{32}\\ \bar{e}_{36}\end{Bmatrix}^{(k)T}\begin{Bmatrix}\varepsilon_x^m\\ \varepsilon_y^m\\ \gamma_{xy}^m\end{Bmatrix}+\frac{z_k+z_{k+1}}{2}\begin{Bmatrix}\bar{e}_{31}\\ \bar{e}_{32}\\ \bar{e}_{36}\end{Bmatrix}^{(k)T}\begin{Bmatrix}\kappa_x\\ \kappa_y\\ \kappa_{xy}\end{Bmatrix}+\frac{\bar{\xi}_{33}^{(k)}}{h_k}\phi_k
\end{aligned}
\tag{23}
$$

where

$$\left(A_{ij}, B_{ij}, D_{ij}\right) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \overline{Q}_{ij}^{(k)} \left(1, z, z^2\right) dz \qquad i, j = 1,2,6$$

$$G_{ij} = k_{ij} \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \overline{Q}_{ij}^{(k)} dz \qquad i, j = 4,5 \tag{24}$$

and k_{ij} is the shear correction factor. Machado (2003) shows the relations between $\overline{Q}_{ij}, \overline{e}_{ij}, \overline{\xi}_{ij}$ and the constitutive coefficients of $[\,c\,]$, $[\,e\,]$ and $[\,\xi\,]$.

3.5. Equilibrium Equations in Terms of Displacements

The equilibrium equations can be expressed in terms of displacements by substituting Eq. (23) into Eq. (21):

$$A_{11} \frac{\partial^2 u}{\partial x^2} + 2A_{16} \frac{\partial^2 u}{\partial x \partial y} + A_{66} \frac{\partial^2 u}{\partial y^2} + A_{16} \frac{\partial^2 v}{\partial x^2} + \left(A_{12} + A_{66}\right) \frac{\partial^2 v}{\partial x \partial y} + A_{26} \frac{\partial^2 v}{\partial y^2}$$
$$+ B_{11} \frac{\partial^2 \beta_x}{\partial x^2} + 2B_{16} \frac{\partial^2 \beta_x}{\partial x \partial y} + B_{66} \frac{\partial^2 \beta_x}{\partial y^2} + B_{16} \frac{\partial^2 \beta_y}{\partial x^2} + \left(B_{12} + B_{66}\right) \frac{\partial^2 \beta_y}{\partial x \partial y} + B_{26} \frac{\partial^2 \beta_y}{\partial y^2}$$
$$- \sum_{k=1}^{N} \overline{e}_{31}^{(k)} \frac{\partial \phi_k}{\partial x} - \sum_{k=1}^{N} \overline{e}_{36}^{(k)} \frac{\partial \phi_k}{\partial y} = -\overline{q}_x$$

$$A_{16} \frac{\partial^2 u}{\partial x^2} + \left(A_{12} + A_{66}\right) \frac{\partial^2 u}{\partial x \partial y} + A_{26} \frac{\partial^2 u}{\partial y^2} + A_{66} \frac{\partial^2 v}{\partial x^2} + 2A_{26} \frac{\partial^2 v}{\partial x \partial y} + A_{22} \frac{\partial^2 v}{\partial y^2}$$
$$+ B_{16} \frac{\partial^2 \beta_x}{\partial x^2} + \left(B_{12} + B_{66}\right) \frac{\partial^2 \beta_x}{\partial x \partial y} + B_{26} \frac{\partial^2 \beta_x}{\partial y^2} + B_{66} \frac{\partial^2 \beta_y}{\partial x^2} + 2B_{26} \frac{\partial^2 \beta_y}{\partial x \partial y} + B_{22} \frac{\partial^2 \beta_y}{\partial y^2}$$
$$- \sum_{k=1}^{N} \overline{e}_{36}^{(k)} \frac{\partial \phi_k}{\partial x} - \sum_{k=1}^{N} \overline{e}_{32}^{(k)} \frac{\partial \phi_k}{\partial y} = -\overline{q}_y$$

$$G_{55} \frac{\partial^2 w}{\partial x^2} + 2G_{45} \frac{\partial^2 w}{\partial x \partial y} + G_{44} \frac{\partial^2 w}{\partial y^2} + G_{55} \frac{\partial \beta_x}{\partial x} + G_{45} \frac{\partial \beta_x}{\partial y} + G_{45} \frac{\partial \beta_y}{\partial x} + G_{44} \frac{\partial \beta_y}{\partial y}$$
$$+ \sum_{k=1}^{N} \frac{h_k \overline{e}_{15}^{(k)}}{2} \frac{\partial^2 \phi_k}{\partial x^2} + \sum_{k=1}^{N} \frac{h_k \left(\overline{e}_{14}^{(k)} + \overline{e}_{25}^{(k)}\right)}{2} \frac{\partial^2 \phi_k}{\partial x \partial y} + \sum_{k=1}^{N} \frac{h_k \overline{e}_{24}^{(k)}}{2} \frac{\partial^2 \phi_k}{\partial y^2} = -\overline{q}_z$$

$$B_{11} \frac{\partial^2 u}{\partial x^2} + 2B_{16} \frac{\partial^2 u}{\partial x \partial y} + B_{66} \frac{\partial^2 u}{\partial y^2} + B_{16} \frac{\partial^2 v}{\partial x^2} + \left(B_{12} + B_{66}\right) \frac{\partial^2 v}{\partial x \partial y} + B_{26} \frac{\partial^2 v}{\partial y^2}$$
$$- G_{55} \frac{\partial w}{\partial x} - G_{45} \frac{\partial w}{\partial y} - G_{55} \beta_x + D_{11} \frac{\partial^2 \beta_x}{\partial x^2} + 2D_{16} \frac{\partial^2 \beta_x}{\partial x \partial y} + D_{66} \frac{\partial^2 \beta_x}{\partial y^2}$$
$$- G_{45} \beta_y + D_{16} \frac{\partial^2 \beta_y}{\partial x^2} + \left(D_{12} + D_{66}\right) \frac{\partial^2 \beta_y}{\partial x \partial y} + D_{26} \frac{\partial^2 \beta_y}{\partial y^2}$$
$$- \sum_{k=1}^{N} \frac{h_k \overline{e}_{15}^{(k)} + \left(z_k + z_{k+1}\right) \overline{e}_{31}^{(k)}}{2} \frac{\partial \phi_k}{\partial x} - \sum_{k=1}^{N} \frac{h_k \overline{e}_{25}^{(k)} + \left(z_k + z_{k+1}\right) \overline{e}_{36}^{(k)}}{2} \frac{\partial \phi_k}{\partial y} = -\overline{m}_y$$

$$B_{16}\frac{\partial^2 u}{\partial x^2}+(B_{12}+B_{66})\frac{\partial^2 u}{\partial x\partial y}+B_{26}\frac{\partial^2 u}{\partial y^2}+B_{66}\frac{\partial^2 v}{\partial x^2}+2B_{26}\frac{\partial^2 v}{\partial x\partial y}+B_{22}\frac{\partial^2 v}{\partial y^2}$$
$$-G_{45}\frac{\partial w}{\partial x}-G_{44}\frac{\partial w}{\partial y}-G_{45}\beta_x+D_{16}\frac{\partial^2\beta_x}{\partial x^2}+(D_{12}+D_{66})\frac{\partial^2\beta_x}{\partial x\partial y}+D_{26}\frac{\partial^2\beta_x}{\partial y^2}$$
$$-G_{44}\beta_y+D_{66}\frac{\partial^2\beta_y}{\partial x^2}+2D_{26}\frac{\partial^2\beta_y}{\partial x\partial y}+D_{22}\frac{\partial^2\beta_y}{\partial y^2}$$
$$-\sum_{k=1}^{N}\frac{h_k\bar{e}_{14}^{(k)}+(z_k+z_{k+1})\bar{e}_{36}^{(k)}}{2}\frac{\partial\phi_k}{\partial x}-\sum_{k=1}^{N}\frac{h_k\bar{e}_{24}^{(k)}+(z_k+z_{k+1})\bar{e}_{32}^{(k)}}{2}\frac{\partial\phi_k}{\partial y}=\bar{m}_x$$

$$\bar{e}_{31}^{(k)}\frac{\partial u}{\partial x}+\bar{e}_{36}^{(k)}\frac{\partial u}{\partial y}+\bar{e}_{36}^{(k)}\frac{\partial v}{\partial x}+\bar{e}_{32}^{(k)}\frac{\partial v}{\partial y}+\frac{h_k\bar{e}_{15}^{(k)}}{2}\frac{\partial^2 w}{\partial x^2}+\frac{h_k\left(\bar{e}_{14}^{(k)}+\bar{e}_{25}^{(k)}\right)}{2}\frac{\partial^2 w}{\partial x\partial y}$$
$$+\frac{h_k\bar{e}_{24}^{(k)}}{2}\frac{\partial^2 w}{\partial y^2}+\frac{h_k\bar{e}_{15}^{(k)}+(z_k+z_{k+1})\bar{e}_{31}^{(k)}}{2}\frac{\partial\beta_x}{\partial x}+\frac{h_k\bar{e}_{25}^{(k)}+(z_k+z_{k+1})\bar{e}_{36}^{(k)}}{2}\frac{\partial\beta_x}{\partial y}$$
$$+\frac{h_k\bar{e}_{14}^{(k)}+(z_k+z_{k+1})\bar{e}_{36}^{(k)}}{2}\frac{\partial\beta_y}{\partial x}+\frac{h_k\bar{e}_{24}^{(k)}+(z_k+z_{k+1})\bar{e}_{32}^{(k)}}{2}\frac{\partial\beta_y}{\partial y}$$
$$+\frac{\bar{\xi}_{33}^{(k)}}{h_k}\phi_k-\frac{h_k\bar{\xi}_{11}^{(k)}}{3}\frac{\partial^2\phi_k}{\partial x^2}-\frac{2h_k\bar{\xi}_{12}^{(k)}}{3}\frac{\partial^2\phi_k}{\partial x\partial y}-\frac{h_k\bar{\xi}_{22}^{(k)}}{3}\frac{\partial^2\phi_k}{\partial y^2}=-\bar{q}_{ek}. \tag{25}$$

The last of the equations above must be written for each sensor piezoelectric layer.

4. Results

In this section, specific Navier type solutions are developed for simply supported piezoelectric laminated plates according to the equations derived in Section 3. Three examples are considered in this section. In all of them, the material properties are listed in Tab. (1) and Tab. (2):

Table 1 – Piezoelectric material properties.

	PZT	PVDF		PZT	PVDF
Elastic properties (GPa)			Piezoelectric coefficients (C/m^2)		
C_{11}	148	3.61	e_{15}	9.2	-15.93 x 10^{-3}
C_{22}	148	3.13	e_{24}	9.2	-12.65 x 10^{-3}
C_{33}	131	1.63	e_{31}	-2.1	32.075 x 10^{-3}
C_{12}	76.2	1.61	e_{32}	-2.1	-4.07 x 10^{-3}
C_{13}	74.2	1.42	e_{33}	9.5	-21.19 x 10^{-3}
C_{23}	74.2	1.31	Electric permittivity (air: ε_o = 8.85 x 10^{-12} F/m)		
C_{44}	25.4	0.55	$\varepsilon_{11}/\varepsilon_o$	460	6.1
C_{55}	25.4	0.59	$\varepsilon_{22}/\varepsilon_o$	460	7.5
C_{66}	35.9	0.69	$\varepsilon_{33}/\varepsilon_o$	235	6.7

Table 2 – Aluminum material properties

	Aluminum
Elastic properties (GPa)	
E	70
G	26

The Navier solutions are achieved by assuming

$$u(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}U_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\qquad v(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}V_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$w(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \qquad \beta_x(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}X_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\beta_y(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Y_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \qquad \phi_k(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Phi_{mn}^{(k)}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{26}$$

where a and b are the dimensions of the rectangular plate with thickness h.

The first example considers a simple supported cross-ply plate configured as actuator. In the second example, an angle-ply laminate composed of two piezoelectric layers is configured as actuator. The third one considers the same plate of the first example now configured as sensor. Figure (1) shows the laminate geometry used in the examples.

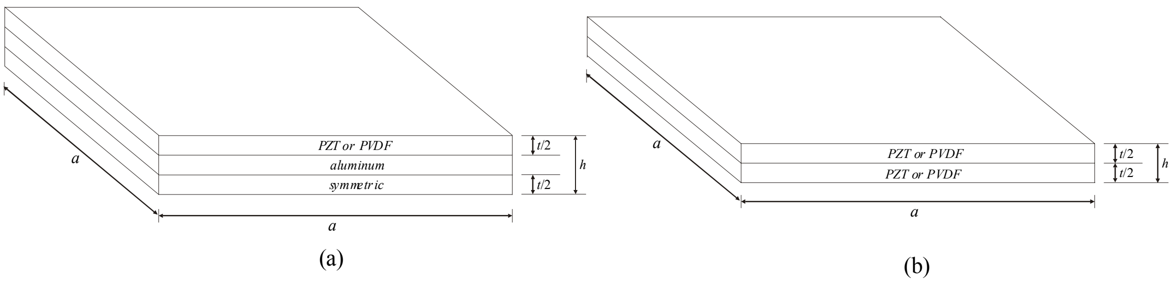

Figure 1.(a): three-layer symmetric cross-ply laminate. (b): two-layer antisymmetric angle-ply laminate.

4.1. Cross-ply laminate configured as actuator

A squared simple supported laminate is considered in three different *a/h* ratios. The laminate is configured as a symmetric three-layer plate consisting of piezoelectric materials in the top and bottom laminae and aluminum material in the middle lamina. A unit potential difference is applied to the top and bottom layers. Figure (2) provides the deflection at the center of the plate for various piezoelectric thickness ratio *t/h*. Results are achieved for PZT and PVDF layers and they perfectly match the values presented by Mitchell and Reddy (1995) obtained from a third-order shear approximation.

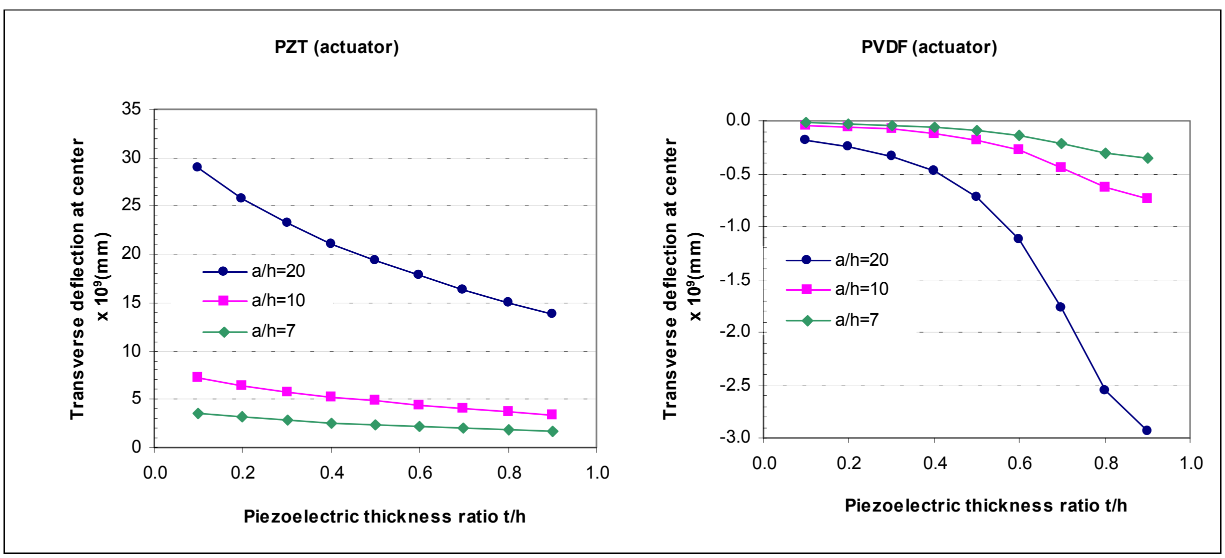

Figure 2.Transverse deflection *versus* piezoelectric thickness ratio for prescribed unit potential difference (PZT and PVDF).

4.2. Angle-ply laminate configured as actuator

Exact solutions are possible to be obtained for a squared angle-ply laminate configured as an actuator. The following example considers a two-layer laminate submitted to a unit voltage applied to the top and bottom layers. Considering the plate simple supported, the deflection at the center of the plate is presented in Fig. (3) for PZT and PVDF layers. Due to the transverse isotropy, a PZT angle-ply laminate will exhibit the same behaviour for any layer orientation.

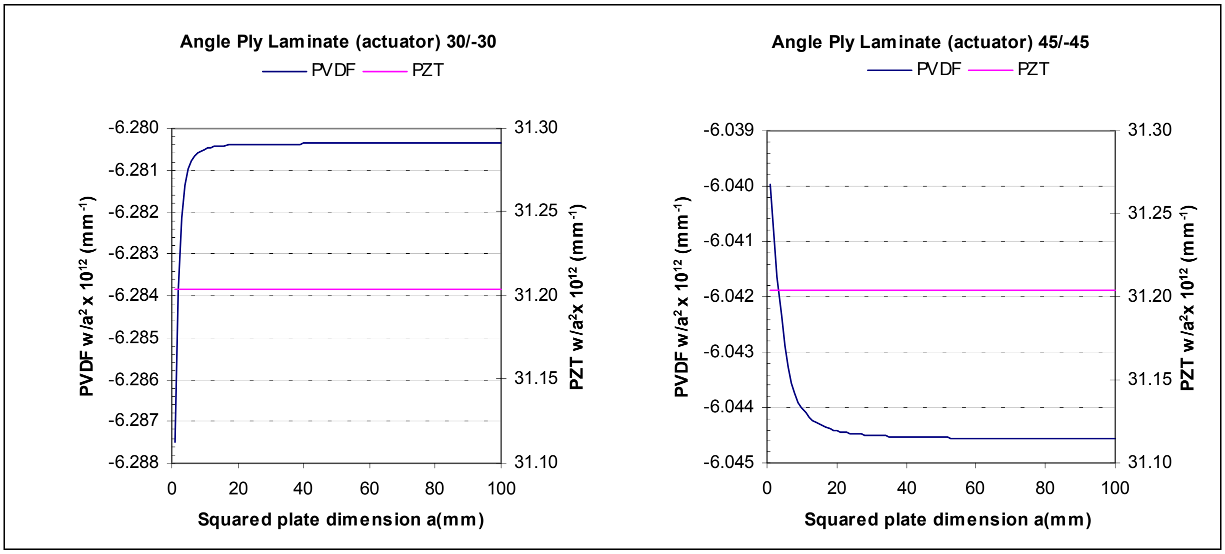

Figure 3. Transverse deflection of angle-ply laminates for prescribed unit potential difference (PZT and PVDF).

4.3. Cross-ply laminate configured as sensor

Simple supported laminates, with the same arrangement described in the first example, are submitted to a distributed unit load applied to the top surface. The deflection at the center of the plate and the voltage at the center of bottom layer for various piezoelectric thickness ratios are plotted in Fig. (4) and Fig. (5) for PZT and PVDF layers, respectively.

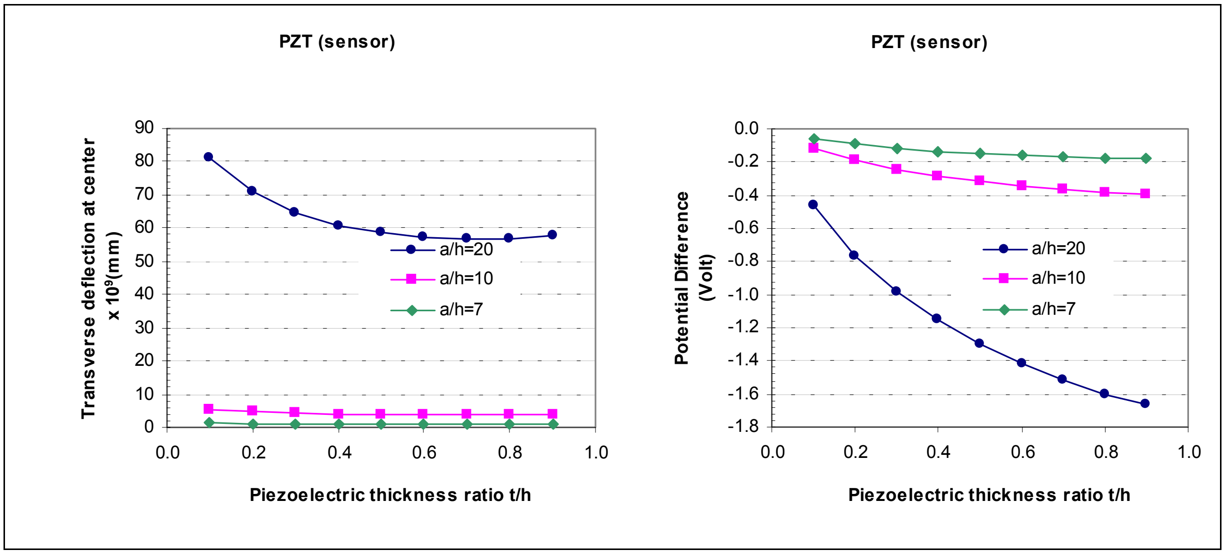

Figure 4. Transverse deflection and potential *versus* piezoelectric thickness ratio for a distributed unit load (PZT).

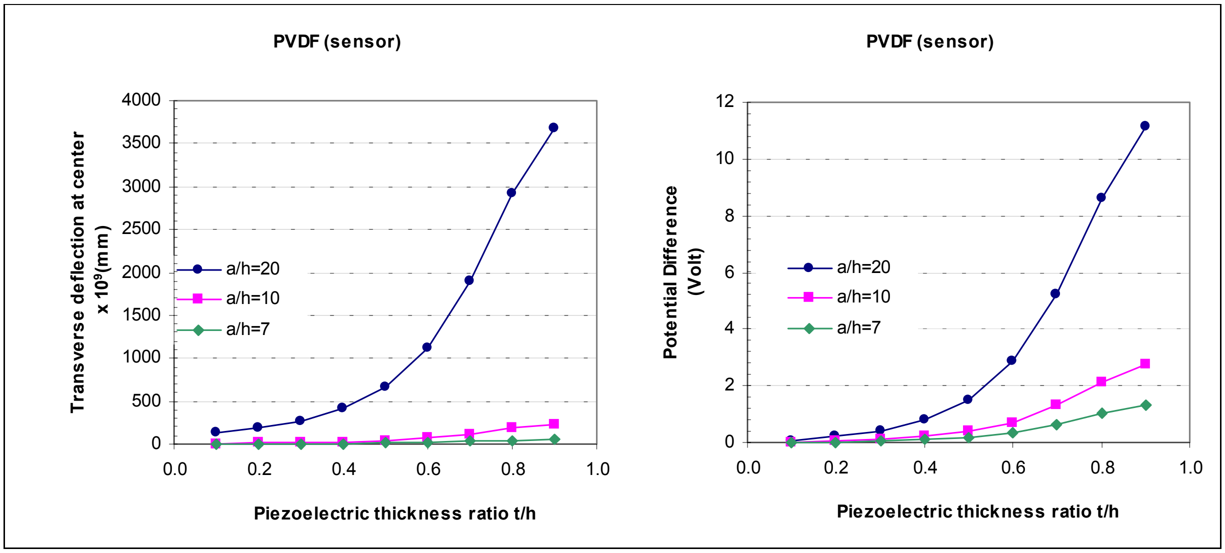

Figure 5. Transverse deflection and potential *versus* piezoelectric thickness ratio for a distributed unit load (PVDF).

5. Closure

All the Navier solutions were obtained quite straight forward with the proposed theory. It is observed in the first example, dealing with a cross-ply laminate, that the presented result matches perfectly with that one derived from a theory based on a third-order shear approximation (Mitchell and Reddy, 1995). Exact solutions were also obtained for the same cross-ply laminate configured as sensor and for angle-ply laminate composed of two active piezoelectric layers operating as actuators.

6. References

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