SINGULAR SOLUTION OF AN INTEGRO-DIFFERENTIAL EQUATION IN ELASTODYNAMICS

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Abstract. *The propagation of steady-state time-harmonic waves in an unbounded elastic solid containing cylindrical cavities that are randomly distributed in a slab region is investigated. The solid is homogeneous and isotropic on either side of the slab. A general equation for the coherent motion in the solid is derived in terms of the average exciting displacement near a fixed cavity. As in Foldy's approach, it is assumed here that the average exciting displacement near a fixed cavity is equal to the average total displacement. The equation for the coherent motion reduces to an integro-differential equation, which is solved in closed form by using a Fourier transform technique. The closed form solution greatly reduces the risks of difficult and expensive computations of the numerical solution of the integro-differential equation and allows the evaluation of the amplitudes and phases of the reflected and transmitted waves outside the slab. Inside the slab, it is shown that the second derivative of the displacement has square-root singularities near the boundary of the slab region. This work is of interest in ultrasonic evaluation and seismic exploration of geomaterials.*

Keywords*. Elastodynamics, multiple scattering, porosity, singularity, Fourier transform.*

1. Introduction

Seismic and ultrasonic waves are used in non-destructive evaluation of material properties. For instance, when a plane wave is incident on a random distribution of solid or liquid inclusions contained in a slab of an elastic material, multiple scattering occurs inside the slab and reflected and transmitted waves propagate outside the slab. By measuring the reflected wave, we can infer the material properties of the inclusions.

In geophysical exploration, seismic waves are widely employed to detect the presence of hydrocarbons and other valuable resources buried below the ground. The objective is to reconstruct an image of the underground. This is accomplished by placing arrays of vibrators at regularly spaced intervals on surface areas of a square mile or more. By exciting the vibrators, one can send waves of low frequency into the ground and obtain an image of the earth structure at depths of a few hundred meters. Exploration and drilling costs, however, are too high. We investigate some aspects of wave propagation in earth-like structures in order to develop ways of making predictions about reflection, scattering, and transmission of waves at low cost.

In a recent paper, Aguiar and Angel (2000) investigated the reflection and transmission of antiplane waves from a random distribution of empty cylindrical cavities contained in a slab region. They use a probabilistic approach and derive an integro-differential equation that governs the average displacement inside and outside the slab region that contains the scatterers.

To solve this integro-differential equation, they have expressed the unknown function in the form of a Fourier series expansion and they have determined numerically the Fourier coefficients. In this paper, we will investigate the scattering of waves inside the slab region containing the cavities. We shall see that the solution of the integro-differential equation becomes unbounded inside this region.

From a physical and engineering viewpoint, it is expected that the propagation of waves through solids containing multiple scatterers produce an attenuated (coherent) wave and that the coherent wave lose energy at a constant rate with the distance traveled. These expectations are readily verified in the study of a similar problem investigated by Angel and Koba (1998), in which flat cracks (slits) are considered instead of cavities. Inside the slab region of Aguiar and Angel (2000), however, we shall see that the wave motion is more complex.

The presentation of the paper is as follows. In Section 2, we review the results of Aguiar and Angel (2000) that are needed in the present investigation. These results are obtained in the context of the multiple scattering model considered by Foldy. Then, in Section 3, we focus attention on the average displacement derivatives. We find that the first derivative is continuous everywhere, except on the boundary of the slab region, where it has a finite jump. The second derivative is evaluated next. After some lengthy analytical calculations, one finds that the second derivative has square-root singularities inside the slab region near the boundaries of the slab. These singularities occur at a distance of one cavity radius from the boundary of the slab region.

The results of Section 3 are illustrated in Section 4 by appropriate figures for both u' and u''/u, where a prime superscript denotes the derivative and u is the average displacement. These figures show that the ratio u''/u is nearly independent of y_1 in the range $|y_1| < w - 2a$, where $2w$ is the thickness of the slab region and a is the cavity radius.

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In the case of cracks, as considered by Angel and Koba (1998), the ratio u''/u is constant in the entire slab region $|y_1|<w$. In the case of cavities, the boundary regions $w-2a<|y_1|<w$ are transition regions where the classical engineering attenuation concept does not apply. Finally, in Section 4, we make some concluding remarks.

2. Preliminaries

We present here a summary of the results obtained in Aguiar and Angel (2000) concerning the average (coherent) displacement in the elastic solid of Fig. (1). We assume that the cavities are randomly distributed, have radius a, and extend to infinity in the $\pm y_3$ directions. The centers are contained in the infinite slab V_∞ of width $2w$ defined by

$$V_\infty = \{(y_1, y_2) \in \Re^2 : |y_1| < w\}. \tag{1}$$

The cavities are illuminated by a time-harmonic antiplane wave that propagates along the y_1-direction and is described, to within a factor $\exp(-i\omega t)$, by the displacement

$$u^I(y_1) = u_0 \exp(i\kappa y_1), \tag{2}$$

where u_0 is an amplitude factor in the y_3-direction and $\kappa = \omega/c$ is the wavenumber. The speed c of transverse waves in a solid without cavities is given by $c = \sqrt{\mu/\rho}$, where μ is the shear modulus and ρ is the mass density.

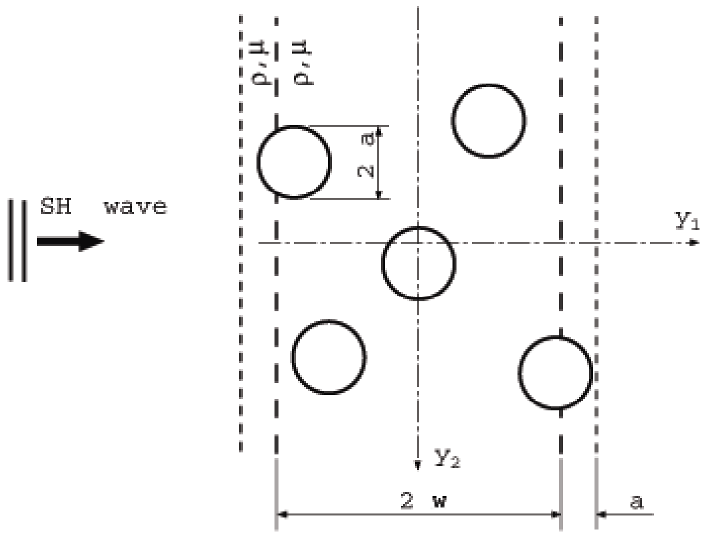

Figure 1. Cavities of diameter $2a$ centered in a slab region of width $2w$.

Averages are now taken over all possible configurations of cavities, assuming that there is on average a constant number n of cavities per unit area in V_∞, the cavity centers are exchangeable, and the cavities do not overlap.

Let u^S be the average scattered displacement generated in the solid by the incident wave (2). Then, the average total displacement u^T is the sum of the incident wave u^I and the average scattered displacement u^S.

The symmetry of the cavity distribution relative to the y_1-axis in Fig. (1) and the normal incidence of the wave (2) imply that both u^T and u^S are functions of y_1 only. Then

$$u^T(y_1) = u^I(y_1) + u^S(y_1). \tag{3}$$

Further, u^S is the sum of the displacements scattered by each cavity in V_∞. There is an infinite number of these scattered displacements. Since the slab is infinite in the y_2-direction, each average scattered motion has a translational invariance with respect to that direction, and it is sufficient to choose an observation point on the y_1-axis, where $y_2 = 0$. Thus, we write

$$u^S(y_1) = \lim_{N\to\infty} \sum_{i=1}^{N} \langle u^S \rangle_N \; (y_1, 0; z^i), \tag{4}$$

where z_i denotes the position vector of the *i*th cavity center and $\langle u^S \rangle_N$ is the average scattered displacement corresponding to the *i*th cavity.

We now recall that the cavities are exchangeable, which implies that the average $\langle u^S \rangle_N$ is independent of the position vector z_i. Thus, one has

$$\langle u^S \rangle_N \; (y_1, 0; z^i) \equiv \langle u^S \rangle_N \; (y_1, 0), \qquad \text{for all i.} \tag{5}$$

Next, we express the average $\langle u^S \rangle_N$, where all cavities are allowed to occupy random positions, in terms of a partial average $\langle u^S \rangle_{N-1}$, where one cavity is fixed at a position z and all the others occupy random positions. The equation that relates $\langle u^S \rangle_N$ to $\langle u^S \rangle_{N-1}$ is

$$\langle u^S \rangle_N \; (y_1, 0) \equiv \frac{n}{N} \int_{V_N} \langle u^S \rangle_{N-1} \; (y_1, 0; z) dz, \tag{6}$$

where V_N is a rectangle contained in the slab V_∞ of (1), and V_N approaches V_∞ as N approaches infinity. Combining (4), (5), and (6), and assuming that $\langle u^S \rangle_{N-1}$ has a limit as N tends to infinity for fixed values of both the coordinate y_1 and the position z of the cavity center, one finds from (4) that

$$u^S(y_1) \equiv n \int_{V_\infty} \langle u^S \rangle_\infty \; (y_1, 0; z) dz, \tag{7}$$

where $\langle u^S \rangle_\infty$ denotes the limit of $\langle u^S \rangle_{N-1}$.

The average displacement $\langle u^S \rangle_\infty$ corresponding to a cavity fixed at z, when all the other cavities are allowed to occupy random positions, is obtained as the solution of a one-cavity problem. The equations of this problem are the Helmholtz equation, the stress-free condition on the cavity boundary, which involves a single shear stress component for antiplane motions, and the radiation condition, which states that the scattered motion must propagate in the outward direction from the cavity center. The one-cavity problem is a well-known problem. In the present case, we obtain $\langle u^S \rangle_\infty$ as a series representation in terms of the average exciting displacement $\langle u^E \rangle_\infty$ on the boundary of the fixed cavity.

The average exciting displacement $\langle u^E \rangle_\infty$ on the cavity at z is the sum of the incident wave and of the average scattered displacements generated by all the other cavities. It takes into account the reflections of all orders between the cavities, and it cannot be evaluated by simple analytical methods.

For this reason, we assume, as in Foldy (1945) and Waterman and Truell (1961), that the average exciting displacement is equal to the average total displacement near the boundary of a fixed cavity. Thus, using the notation of (3), we write

$$\langle u^E \rangle_\infty \; (y_1, y_2; z) \equiv u^T(y_1). \tag{8}$$

Equation (8) can be rewritten in terms of local polar coordinates (r, θ) centered at $z = (z_1, z_2)$ such that

$$y_1 = z_1 + r\cos\theta, \qquad y_2 = z_2 + r\sin\theta. \tag{9}$$

From the invariance of the problem in the y_2-direction, it follows that the average exciting displacement $\langle \hat{u}^E \rangle_\infty$, evaluated in terms of the local coordinates (r, θ), takes the same value at (r, θ) when the cavity is centered at z and when it is centered at $z + qe_2$, where q is an arbitrary real number and e_2 is a unit vector in the y_2-direction. Thus, using (9), we write

$$< \hat{u}^E >_\infty (r,\theta; z_1 e_1) = < \hat{u}^E >_\infty (r,\theta; z) = < u^E >_\infty (y_1, y_2; z). \tag{10}$$

It follows now from (9) and (10) that the Foldy condition (8) can be written in the equivalent form

$$< \hat{u}^E >_\infty (r,\theta; z_1 e_1) = < u^T >_\infty (z_1 + r\cos\theta). \tag{11}$$

Now we return to the expression (7) for the average scattered displacement u^S. We observe that the integral sign in the right-hand side of (7) represents a double integral over the slab V_∞. The z_1 variable is integrated between -w and w, and the z_2 variable is integrated between $-\infty$ and $+\infty$. It is shown in Aguiar and Angel (2000) that the z_2 integration can be performed in closed form, which simplifies the analysis of the problem considerably.

Using (9) and (11), we can now write the average scattered displacement of (7) in terms only of the average total displacement u^T. Thus, substituting (7) into (3), one finds an integro-differential equation for u^T. To record this equation in a simple form, we introduce the dimensionless quantities

$$y = y_1 / a, \qquad z = z_1 / a, \qquad h = w / a, \qquad k = \kappa a, \qquad \varepsilon = n a^2, \tag{12}$$

$$u(y) = u^T (a y_1) / u_0 . \tag{13}$$

Then, the dimensionless average total displacement u satisfies the integro-differential equation

$$u(y) = \exp(iky) - \varepsilon \int_{-h}^{h} \left[\int_0^\pi u'(\eta)|_{\eta = z + \cos\zeta} \, S(\zeta, y - z) d\zeta \right] \exp(ik|y - z|) dz, \tag{14}$$

where the prime superscript denotes the derivative. The function S in (14) is defined by

$$S(\zeta, y) = \cos\zeta [Q_2(\zeta) - (\operatorname{sgn} y) Q_1(\zeta)], \tag{15}$$

$$Q_1(\zeta) = \frac{4i}{\pi k^2} \sum_{m=0}^{\infty} \frac{(-1)^m \cos((2m+1)\zeta)}{h_{2m+1}}, \tag{16}$$

$$Q_2(\zeta) = \frac{2}{\pi k^2} \sum_{m=0}^{\infty} \frac{\varepsilon_{2m} (-1)^m \cos(2m\zeta)}{h_{2m}}, \tag{17}$$

$$h_m = \left(H_m^{(1)} \right)' (\alpha)|_{\alpha = k}, \qquad \varepsilon_0 = 1, \qquad \varepsilon_m = 2, \qquad (m \geq 1), \tag{18}$$

and $H_m^{(1)}$ is the Hankel function of the first kind and order m. The series $Q_1(\zeta)$ and $Q_2(\zeta)$ in (16) and (17), respectively, converge uniformly in ζ for each k. Equation (14) is valid inside and outside the slab. Observe that the integral over z in (14) involves all the values of u' at $z + \cos\zeta$ when ζ is integrated over the interval $(0, \pi)$.

3. Evaluation of the displacement derivatives

From (15), one can see that the sign function causes the function $S(\zeta, y)$ to have a finite discontinuity at y=0. It follows that equation (14) can be written as

$$u(y) = \exp(iky) - \varepsilon \exp(iky) F^+(y) - \varepsilon \exp(-iky) F^-(y), \qquad |y| < h, \tag{19}$$

where $F^+(y)$ and $F^-(y)$ are defined by

$$F^+(y) = \int_{-h}^{y} s^+(z) \exp(-ikz) dz, \tag{20}$$

$$F^{-}(y) = \int_{y}^{h} s^{-}(z)\exp(ikz)dz. \tag{21}$$

In (20) and (21), the functions s^{+} and s^{-} are defined by

$$s^{\pm}(y) = \int_{0}^{\pi} u'(\eta)|_{\eta=y+\cos\zeta}\ S(\zeta,\pm 1)d\zeta. \tag{22}$$

Next, we differentiate (19) formally twice to obtain the derivatives u' and u'' of the average total displacement in the interval $|y|< h$. One has

$$u'(y) = ik\exp(iky) - \varepsilon\left[s^{+}(y) - s^{-}(y)\right] - ik\varepsilon\exp(iky)F^{+}(y) + ik\varepsilon\exp(-iky)F^{-}(y), \qquad |y|< h. \tag{23}$$

The second derivative u'' is obtained by differentiating (23) with respect to y. The result is

$$u''(y) = -k^{2}u(y) - \varepsilon\frac{d}{dy}\left[s^{+}(y) - s^{-}(y)\right] - ik\varepsilon\left[s^{+}(y) + s^{-}(y)\right], \qquad |y|< h. \tag{24}$$

Outside the slab region, where $|y|> h$, it follows from (19), (20), and (21) that

$$u(y) \equiv u_{T}(y) = T\exp(iky), \qquad y > h, \tag{25}$$

$$u(y) \equiv u_{R}(y) = R\exp(-iky) + \exp(iky), \qquad y < -h, \tag{26}$$

where we have defined the functions u_T and u_R , and where the transmission coefficient T and the reflection coefficient R are defined by

$$T = 1 - \varepsilon F^{+}(h), \qquad R = -\varepsilon F^{-}(-h). \tag{27}$$

The first and second derivatives of (25) and (26), which involve only exponential functions, can be evaluated without difficulty.

Now, we make the change of integration variables $\alpha = y + \cos\zeta$ in (22), where α is the new variable, and we obtain

$$s^{\pm}(y) = \int_{y-1}^{y+1} u'(\alpha)p^{\pm}(\alpha - y)d\alpha = \int_{-1}^{1} u'(y+\alpha)p^{\pm}(\alpha)d\alpha, \tag{28}$$

where the functions $p^{\pm}$ are defined by

$$p^{\pm}(y) = S(\arccos y, \pm 1)/\sqrt{1-y^{2}}. \tag{29}$$

We see from (28) and (29) that the second integrand in (28) has square-root singularities at the ends $\alpha = \pm 1$ of the interval of integration.

It follows from (28), (29), (15)-(17), and from the symmetry properties of Q_1 and Q_2, that the sum and difference of the quantities s^{+} and s^{-} are given by

$$s^{+}(y) - s^{-}(y) = \int_{0}^{1}\left[u'(y+\alpha) + u'(y-\alpha)\right]q_{1}(\alpha)d\alpha, \tag{30}$$

$$s^+(y)+s^-(y)=\int_0^1 [u'(y+\alpha)-u'(y-\alpha)]q_2(\alpha)d\alpha. \tag{31}$$

In (30) and (31), one has

$$q_1(\alpha)=-2\alpha Q_1(\arccos\alpha)/\sqrt{1-\alpha^2}, \qquad q_2(\alpha)=2\alpha Q_2(\arccos\alpha)/\sqrt{1-\alpha^2}. \tag{32}$$

To obtain the second derivative u'' of (24), we have to differentiate the expression (30) with respect to y. A direct differentiation of (30) with respect to y would yield the second derivative u'' under the integral sign. To avoid this situation, we first integrate the expression (30) by parts. At the same time, we keep in mind that the function q_1 contains a square-root singularity which, upon differentiation, will generate a non-integrable singularity of the type $(1-y^2)^{-3/2}$. In view of these remarks, we write q_1 in the equivalent form

$$q_1(\alpha)=\bar{q}_1(\alpha)-2Q_1(0)/\sqrt{1-\alpha^2}, \tag{33}$$

where

$$\bar{q}_1(\alpha)=-2[\alpha Q_1(\arccos\alpha)-Q_1(0)]/\sqrt{1-\alpha^2}. \tag{34}$$

Then we find that, in the limit as α approaches one, $\bar{q}_1$ and its derivative $\bar{q}_1'$ are such that

$$\bar{q}_1(\alpha)\to 0, \qquad \bar{q}_1'(\alpha)\approx 1/\sqrt{1-\alpha^2}, \qquad \text{as } \alpha\to 1. \tag{35}$$

Equation (33) allows us to decompose the integral of (30) into two integrals. The second integral contains the function $\bar{q}_1$ and, in view of (35), can be integrated by parts without difficulty. Thus, one has

$$s^+(y)-s^-(y)=-2Q_1(0)G(y)-\int_0^1 [u(y+\alpha)-u(y-\alpha)]\bar{q}_1'(\alpha)d\alpha, \tag{36}$$

where the function G(y) is given by

$$G(y)=\int_0^1 \left[\frac{u'(y+\alpha)+u'(y-\alpha)}{\sqrt{1-\alpha^2}}\right]d\alpha. \tag{37}$$

As shown in the next section, u' is discontinuous at $\pm h$. Thus, we define the jumps $\Delta(-h)$ and $\Delta(h)$ such that

$$\Delta(-h)=u'(-h)-u_R'(-h), \qquad \Delta(h)=u_T'(h)-u'(h). \tag{38}$$

The notations $u'(-h)$ and $u'(h)$ in (38) indicate that the derivatives of the displacement u are evaluated at $\pm h$ from inside the interval $(-h,h)$. Outside the interval, we use the notations u_T and u_R, which are defined in (25) and (26), respectively.

The arguments of the u and u' functions in (36) and (37) are either $y+\alpha$ or $y-\alpha$, and α varies in the interval (0,1). Thus, depending on the value of y, the arguments may be less than $-h$ or greater than h. We distinguish three cases.

Case 1: $-h<y<-h+1$. In this case, $y-\alpha$ is inside $(-h,h)$ if $0<\alpha<h+y$, $y+\alpha$ is inside $(-h,h)$ if $0<\alpha<1$, and

$$u(y-\alpha)=u_R(y-\alpha), \qquad \text{if } h+y<\alpha<1.$$

Case 2: $-h+1 < y < h-1$. Here, both $y-\alpha$ and $y+\alpha$ are inside $(-h,h)$ if $0 < \alpha < 1$.

Case 3: $h-1 < y < h$. In this case, $y+\alpha$ is inside $(-h,h)$ if $0 < \alpha < h-y$, $y-\alpha$ is inside $(-h,h)$ if $0 < \alpha < 1$, and

$$u(y+\alpha) = u_T(y+\alpha), \qquad \text{if } h-y < \alpha < 1.$$

The next step, before we take the derivative of (36), is to rearrange the function G of (37) in each of the three cases mentioned above. We again perform an integration by parts with respect to α so that G is expressed only in terms of the function u, instead of the derivative u'. This requires some caution since differentiation of the $1/\sqrt{1-\alpha^2}$ singularity yields a $\left(1-\alpha^2\right)^{-3/2}$ non-integrable singularity.

Using (24) and (30)-(38), we find that the second derivative u'' can be written in the form

$$u''(y) = -k^2 u(y) + 2\varepsilon Q_1(0) g_i(y) - 2\varepsilon Q_1(0) I_{1i}(y) - 4\varepsilon k^2 Q_1(0) I_{2i}(y) + 2\varepsilon I_3(y) - 2i\varepsilon k I_4(y), \tag{39}$$

where the subscript *i* takes the values 1, 2, 3, which correspond, respectively, to the three cases discussed above. In equation (39), one has

$$g_1(y) = \frac{\pi}{2} u'(y+1) - g^-(y) u'(-h) + \frac{\Delta(-h)}{\sqrt{1-(h+y)^2}}, \tag{40}$$

$$g_2(y) = \frac{\pi}{2} u'(y+1) - \frac{\pi}{2} u'(y-1), \tag{41}$$

$$g_3(y) = -\frac{\pi}{2} u'(y-1) + g^+(y) u'(h) + \frac{\Delta(h)}{\sqrt{1-(h-y)^2}}, \tag{42}$$

where

$$g^-(y) = \arcsin(h+y) + \sqrt{\frac{1-h-y}{1+h+y}}, \qquad g^+(y) = \arcsin(h-y) + \sqrt{\frac{1-h+y}{1+h-y}}, \tag{43}$$

and the jumps $\Delta(-h)$ and $\Delta(h)$ in, respectively, (40) and (42) are defined in (38). The functions I_{1i}, I_{2i}, I_3, and I_4, i = 1, 2, 3, in (39) are given by lengthy expressions not shown here. They are integrals whose integrands are expressed in terms of u', u_T, and u_R. The integrals I_{2i}, I_3, and I_4 are continuous functions of y and their integrands are bounded everywhere in their intervals of integration. The integrals I_{1i}, on the other hand, have logarithmic singularities, but these are weaker than the square-root singularities discussed below.

The functions g_1 and g_3 in expressions (40) and (42) have square-root singularities at $y = -h+1$ and $y = h-1$, respectively. Thus, we conclude from (39) that $u''(y)$ becomes infinite as y approaches $-h+1$ from the left and as y approaches $h-1$ from the right. This result implies that $u'(y)$ has vertical tangents at $y = -h+1$ and at $y = h-1$. We recall here, as will be discussed in the next section, that $u'(y)$ is continuous everywhere except at $y = \pm h$, where it has finite jumps.

Based on the exposition above and by examining the expression (39) for the second derivative u'', and realizing that the expressions of u'' corresponding to the three cases are interconnected through translations of the type $y \to y-1$, or, of the type $y \to y+1$ in the arguments of the functions of (39), we arrive at the following conclusions.

1. The second derivative $u''(y)$ has square-root singularities as $y \to (-h+1)^-$ and as $y \to (h-1)^+$;
2. The second derivative $u''(y)$ has logarithmic singularities as $y \to (-h+2)^-$ and as $y \to (h-2)^+$;
3. The second derivative $u''(y)$ is bounded as $y \to -h^+$ and as $y \to h^-$.

4. Numerical Results

We recall from Aguiar and Angel (2000) that the integro-differential equation (14) can be rearranged by changing the order of integrations. The result, which is given by equation (3.42) in Aguiar and Angel (2000), has the form

$$u(y) = \exp(iky) - \varepsilon \int_{-h-1}^{h+1} u'(\eta) G(y,\eta) d\eta, \qquad y \in \Re, \tag{44}$$

where the kernel $G(y,\eta)$ is defined by

$$G(y,\eta) = \int_{\beta_1(\eta)}^{\beta_2(\eta)} S(\zeta, y-\eta+\cos\zeta)\exp(ik\,|\,y-\eta+\cos\zeta\,|)d\zeta. \tag{45}$$

The limits of integration in (45) are given by

$$\beta_1(\eta) = \arccos(\eta + h), \qquad \beta_2(\eta) = \pi, \qquad \text{if } -h-1 < \eta < -h+1;$$

$$\beta_1(\eta) = 0, \qquad \beta_2(\eta) = \pi, \qquad \text{if } -h+1 < \eta < h-1; \tag{46}$$

$$\beta_1(\eta) = 0, \qquad \beta_2(\eta) = \arccos(\eta - h), \qquad \text{if } h-1 < \eta < h+1.$$

Differentiating (44) with respect to y, one has

$$u'(y) + \varepsilon \int_{-h-1}^{h+1} u'(\eta) \frac{\partial G(y,\eta)}{\partial y} d\eta = ik\exp(iky), \qquad y \in \Re. \tag{47}$$

Equation (47), which is an integral equation for u', is solved numerically in Aguiar and Angel (2000) by using a Fourier series expansion of the form

$$v(y) = \sum_{m=-\infty}^{\infty} \vartheta_m \exp(-i\xi m y), \qquad \xi = \pi/(h+1). \tag{48}$$

In (48), v(y) is defined for all y in $\Re$ and is the periodic extension with period $2(h+1)$ of $u'(y)$, where y is now in the interval $(-h-1, h+1)$.

This method yields a linear system of equations for the coefficients ϑ_m. Convergent results are obtained in Aguiar and Angel (2000) by truncating the system of equations after a sufficiently large number of terms are chosen. In fact, it can be shown that the coefficients ϑ_m are of the order O(1/m) as m tends to infinity, from which we conclude that the series (48) is weakly convergent.

We have verified numerically that $u'(y)$, obtained from (47) by substituting v into this integral equation and then integrating term by term, and v(y), obtained from (48), take identical values in the interval $|y| < h+1$. The difference between $u'(y)$ and v(y) is that the series for $u'(y)$ converges faster than that for v.

Thus, using the series (48), we can evaluate the derivative $u'(y)$ for all y. In the interval $|y| < h$, $u'(y)$ is evaluated from (47). Outside the interval $|y| < h$, $u'(y)$ is evaluated from either (25) or (26). The reflection and transmission coefficients R and T, respectively, are obtained in terms of the coefficients ϑ_m of the series (48), as is explained in detail in Aguiar and Angel (2000).

With the derivative $u'(y)$ obtained in the manner described above, we evaluate the jumps $\Delta(-h)$ and $\Delta(h)$ of (38). In Fig. (2), the moduli of the jumps are shown versus the frequency for $h = 3$ and for three values of the cavity density ε. The solid lines correspond to $\Delta(h)$ and the dashed lines to $\Delta(-h)$. We see from Fig. (2) that the derivative $u'(y)$ is discontinuous at $y = \pm h$ for all non-zero frequencies, and the magnitude of the jump increases as the cavity density ε increases.

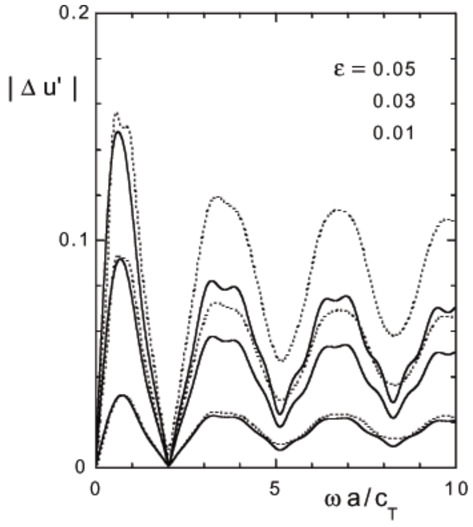

Figure 2. Jump of the first derivative versus the frequency at $y = h$ (solid lines) and $y = -h$ (dashed lines) for $h = 3$.

Next, we examine the second derivative $u''(y)$ in the interval $|y| < h$. One can evaluate $u''(y)$ by either substituting the series (48) into (44) and differentiating twice with respect to y or by using the equation (39) together with a Gaussian quadrature scheme, which is used to evaluate the finite integrals in (39). Both calculations yield the same result, except that the latter does not display high-frequency oscillations observed in the former when we truncate the series (48) after a sufficiently large number of terms.

We have evaluated $u''(y)$ numerically by using (39) for several values of both h and ε and we have found that $u''(y)$ tends to infinity at $y = \pm(h-1)$. Also, it appears that $u''(y)$ has a discontinuity at $y = \pm(h-2)$, which is consistent with the singularities discussed at the end of Section 3.

Next, we define the functions $\Omega(y)$ and K(y) such that

$$k^2\Omega(y) = -K^2(y) = \frac{u''(y)}{u(y)}. \tag{49}$$

Figure (3) shows both the real and imaginary parts of Ω versus the distance y for both $k = 1$ (solid lines) and $k = 5$ (dashed lines), for $h = 3$, and for three values of ε. We see from this figure that Ω becomes infinite on the left of $y = -2$ and on the right of $y = 2$; especially for $k = 1$. We also see an apparent discontinuity on the left of y = -1 and on the right of y = 1. These observations are consistent with the square-root and logarithmic singularities discussed at the end of Section 3. It also appears from Figure (3) that Ω is bounded at $y = \pm 3$, as expected. Further, Ω is nearly constant in the range |y| < 1. For other values of h and k, not shown here, we have observed that Ω has the same general characteristics as in the case analyzed here, ($h = 3$ and $k = 1$, or, 5), corresponding to Figure (3). When ε vanishes, one infers from (39) and (49) that $\Omega(y) = -1$, which means that $K^2(y) = k^2$ for all values of y in the interval $(-h, h)$.

5. Conclusion

We have examined in detail an integro-differential equation that describes the antiplane (SH) wave displacement in an elastic solid that contains a random distribution of empty cylindrical cavities. This equation is solved numerically by using a Fourier series representation. It is shown that the first derivative of the displacement is discontinuous at the boundary of the region where the cavities are centered. Outside this region, the wave motion propagates with the wavenumber that corresponds to a homogeneous elastic solid in the absence of cavities.

To investigate the wave motion inside the region where the cavities are centered, we have evaluated the second derivative of the displacement. Our analysis shows that the second derivative has square-root and logarithmic singularities near the boundary of the region where the cavities are centered. At the same time, at a distance greater than a cavity diameter from the boundary of the slab region, one finds that the second derivative of the displacement is nearly proportional to the displacement itself.

The derivation of the integro-differential equation of this work rests on a simple physical model to describe the scattering between multiple scatterers. This model, which is based on a similar model proposed by Foldy (1945), assumes that the incident field near a fixed scatterer is equal to the field that would exist at that location if the scatterer were not present. This assumption is believed to be valid for low concentrations of scatterers. Experimental results to test the validity of Foldy's assumption for cavities do not appear to be available yet.

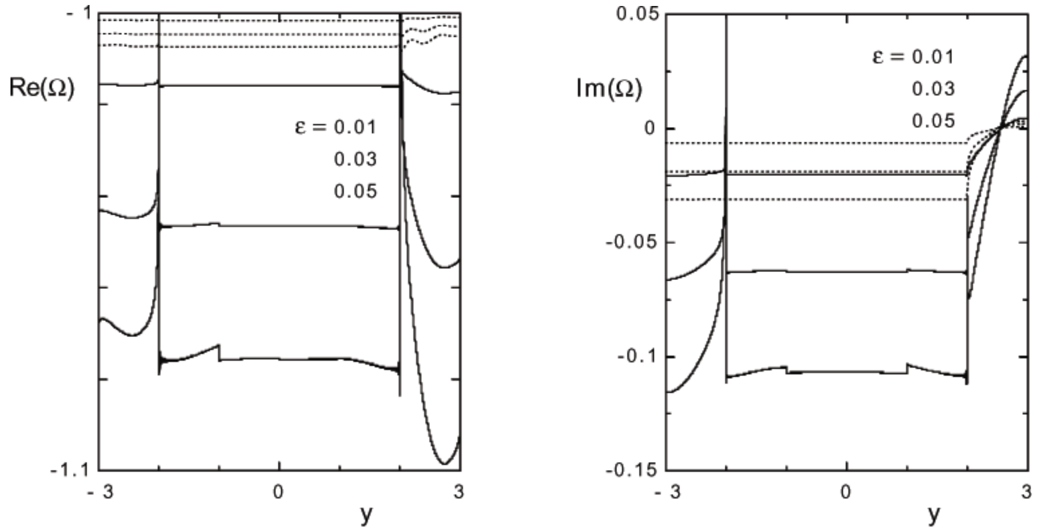

Figure 3. Real (left) and imaginary (right) parts of the ratio Ω of (49) versus the position y for $k = 1$ (solid lines), $k = 5$ (dashed lines), and $h = 3$.

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7. References

Aguiar, A. R. and Angel, Y. C., 2000, "Antiplane Coherent Scattering from a Region Containing a Random Distribution of Cavities", Proc. R. Soc. Lond. A, Vol. 456, pp. 2883-2909.

Angel, Y. C. and Koba, Y. K., 1998, "Complex-valued Wavenumber, Reflection and Transmission in an Elastic Solid Containing a Cracked Slab Region", Int. J. Solids Struct., Vol. 35, pp. 573-592.

Foldy, L. L., 1945, "The Multiple Scattering of Waves I. General Theory of Isotropic Scattering by Randomly Distributed Scatterers", Phys. Rev., Vol. 67, pp. 107-119.

Waterman, P. C. and Truell, R., 1961, "Multiple Scattering of Waves", Journal of Mathematical Physics, Vol. 2, pp. 512-537.

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