TOPOLOGY OPTIMIZATION OF CONTINUUM TWO-DIMENTIONAL STRUCTURES USING A GLOBAL STRESS OBJECTIVE FUNCTION

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Abstract. *This article presents an approach to find a solution to the problem of optimization of continuum structures considering the stresses. The problem is solved by a topology optimization methodology, formulated as finding the best material distribution into the design domain. The design is accomplished by distributing a fictitious density in the domain. An artificial power-law material parameterization relates the density to the elastic properties. The domain is discretized into simpler subdivisions, which are used to define both the finite element approximation of the structural response and the density approximation, taken constant in each element. Sequential linear programming is used to accomplish the minimization.An adjoint sensitivity analysis is performed for the von Mises failure criteria. A first order neighborhood filter is implemented to minimize the effects of checkerboard pattern areas and mesh dependency. Results are presented and compared to the existing literature.*

Keywords*. Topology optimization, Stress, Finite Element Method*

1. Introduction

More and more, the human being is increasingly aware of the necessity of saving natural resources. This fact is the main motivation for researching optimum designs.

Structural engineers also adopted this trend in the design of new structures or modification of existent ones. In this context, a structure can be considered as an amount of distributed material over a design domain, in order to support loads (static or dynamic), absorb and distribute energy and transmit it to the supports. One of the goals of optimum design is best distribute the available material into the design domain.

Initially, only relatively simple optimization problems could be addressed, due to the difficulty in solving equilibrium equation for more complicated structures. Maxwell (1872) derived analytical solutions for the minimum volume problem in uniaxial structures subjected to several types of loads. However, structural optimization became practical only when numerical methods began to be used, especially for solving the equilibrium equations. One very popular method is the Finite Element Method, where the continuum is approximated by an assembly of simpler geometric domains.

Mathematical programming is another important tool created to help the solution process in optimization problems. According to Rozvany et. al. (1995), before the arise of the mathematical programming, the updating of the design parameters was based on analytical methods (many of them heuristically decided), known as optimality criteria.

With the development of 2-D and 3-D finite elements, new contributions to the structural optimization field were developed. An important result was obtained by Cheng and Olhoff (1981). Studying the problem of optimum thickness distribution in plates under compliance and natural frequency constraints, they concluded that the geometrical irregularities obtained in the thickness distribution could be interpreted as ribs (stiffeners). They also concluded that the exact solution for plate optimization contains an infinite number of ribs, so the finer the finite elements mesh more ribs appear. This result showed the necessity of considering some kind of microstructure to find one valid macroscopic solid-void layout.

Addressing this problem, several authors worked with relaxed formulations, by relating the constitutive material properties with microstructural parameters. Works by Allaire and Kohn (1993), Niordson (1983) and Rossow and Taylor (1973) present different methods of material parameterization.
Rossow and Taylor (1973) for example, proposed a minimum compliance problem for membranes in plane stress behavior.

However, topology optimization only received more attention after the introduction of Homogenization Method by Bendsøe and Kikuchi (1988). This theory is considered a natural extension of previous works, like Reiss (1976) and Cheng (1981).

This material parameterization model considers the existence of periodic microstructures (Fig. (1)), from which composite material effective properties are computed. Mathematically, the different material scales are split using an asymptotic expansion (Sanchez-Hubert and Sanchez-Palencia, 1992).

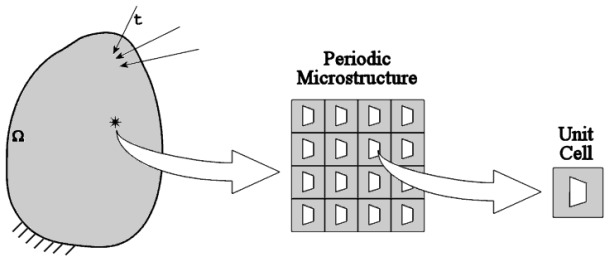

Figure 1. Representation of a composite material made of a periodic microstructure

Dealing either with isotropic or anisotropic material constitutive laws, this model considers the material stiffness as a function of microstructure and a density-like parameters, as follows:

$$E = E^0\left(\rho, \theta, \mu, \ldots\right) \tag{1}$$

Bendsøe (1989) proposed another type of material parameterization, nowadays named SIMP (Solid Isotropic Microstructure with Penalization). Differently from the homogenization, this approach considers the existence of only one design variable, a constant fictitious density (ρ) in each finite element. Therefore, the new stiffness parameterization is calculated as follows:

$$\begin{aligned} &E = \rho^{\eta} E^0 \\ &0 \le \rho \le 1 \end{aligned} \tag{2}$$

where η defines the amount of penalization. If η>1, material stiffness is penalized, avoiding the appearing of low stiffness elements (with intermediary densities). This parameterization is an extension of the work of Rossow and Taylor (1973).

Besides the common problems associated to the topology optimization solution (checkerboard patterns and mesh dependency, for example), stress constraints (as considered in this work) bring further ones. Firstly, stress is a local constraint, then, each infinitesimal point of the structure should have its stress level under control. Moreover, the singular optimum phenomenon can arise.

Stress singularity was firstly pointed by Sved and Ginos (1968). Performing an analytical study on the 3 bar truss problem with three load cases and stress constraints, they have found out that only removing one structural bar the global optimum could be reached. This apparently simple problem defined all mathematical programming algorithms, causing stress constraint violation, or even non-convergence. Moreover, algorithmic difficulties appear because in most codes finite elements can not be simply removed from the mesh. The physical cause is easily understandable: in bars, the cross sectional area of each bar (or density, in SIMP model, according to Duysinx and Bendsøe (1998)) is inversely proportional to the stress. Thus, when areas tend to vanish, stresses may increase unreasonably.

This problem was unsolvable for much time, when Cheng and Guo (1997) proposed a perturbation technique called epsilon-relaxation. Thus, reformulating the stress constrained problem, the design space is modified, including new sub domains to the design space. This is a manner to modify the dimension of this space, without add or remove bars. Then, the design space is successively diminished, by decreasing this perturbation value, so that the solution of these series of modified sub-problems converges to the correct solution of the original problem.

Duysinx and Bendsøe (1998) extended this technique for two-dimensional continuum problems. In this work, they also developed an analytical solution for the sensitivity analysis of Von Mises equivalent stress.

As one can see, to manage with several local constraints is an expensive task to be performed, even by fast computers. Recently, some works are addressing the stress optimization problem as a global constraint, where the peak level of the stresses is controlled.

In this same year, Duysinx and Sigmund (1998) proposed a global method to handle stress constraints in optimal material distribution. In this work, they defined two global stress constraints based on p-norm and p-mean of the ε-relaxed overall stress criteria in the finite elements mesh. For the problem of volume minimization, they proposed the following constraints:

a) p-norm function:

$$\left[\sum_{e=1}^{N}\left(\max\left\{0, \frac{\sigma_{VM}^{*}, e}{\rho_e^{n}\sigma_l} + \varepsilon - \frac{\varepsilon}{\rho_e}\right\}\right)^{p}\right]^{1/p} \le 1 \tag{3}$$

b) p-mean function:

$$\left[\frac{1}{N}\sum_{e=1}^{N}\left(\max\left\{0,\frac{\sigma_{VM}^{*},e}{\rho_e^n\sigma_l}+\varepsilon-\frac{\varepsilon}{\rho_e}\right\}\right)^p\right]^{1/p}\leq 1 \tag{4}$$

where N is the number of finite elements, σ_{VM}^{*},e is the von Mises equivalent stress, ε is the epsilon relaxation factor, ρ_e is the fictitious density and p is a given value.

The basic difference between both formulations is that for a given p, the maximum stress value is always bounded from above by the p-norm and from below by the p-mean. This proof can be found in this same article.

Pereira (2001), in his Ph.D. thesis used the Augmented Lagrangian Technique, incorporating a similar global constraint in the objective function. Thus, he performed a simple unconstrained optimization using mathematical programming.

The present paper proposes a different approach for the problem of stress optimization. Basically, we are minimizing the von Mises stress while controlling the amount of available material (a simple volume constraint). We propose a very simply p-mean objective function without ε-relaxation, because filtering the gradient of the move limits seems to reduce the effects of the singularity phenomenon. The source code was developed in MATLAB, including analysis and optimization sub-routines. The four-node bilinear isoparametric element is used to solve the equilibrium and Sequential Linear Programming (SLP) is the chosen first order decision algorithm.

2. Theoretical background

Topology optimization aims to find the best stiffness distribution within an admissible domain while satisfying the constraints. Figure (2) better illustrates this concept:

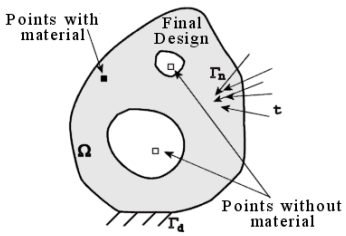

Figure 2. Representation of topology optimization problem, where the goal is to find the best material (or stiffness) distribution along the design domain

Mathematically, the material stiffness is formulated as follows:

$$E = 1_{\Omega^m} E^0$$

$$\text{where } 1_{\Omega^m} = \begin{cases} 1 \text{ if } x \in \Omega^m \\ 0 \text{ if } x \in \Omega \setminus \Omega^m \end{cases} \tag{5}$$

Solving this problem is a computationally intensive task, due to its combinatorial nature. To make the solution process easier, this problem is usually relaxed, by making use of microstructure parameters as design variables. Homogenization Method (as represented in Eq. (1)) or SIMP Method (Eq. (2)) are commonly used.

Due to its relative simplicity, SIMP method was used in this work. The optimization problem is solved by using sequences of isotropic materials. Doing this, the only design variable is the constant density in each finite element, designed as ρ. The next picture shows the relation between ρ and different penalization levels concerning to the material stiffness (in normal direction, for example):

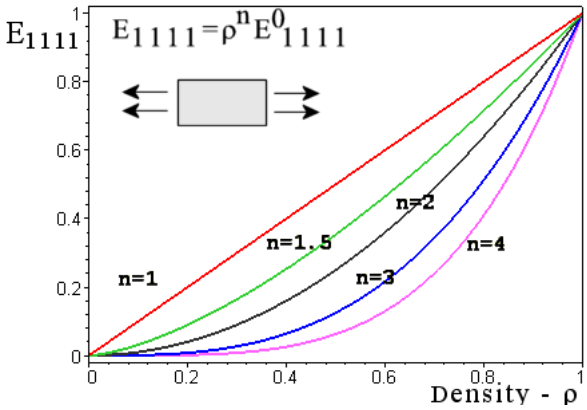

Figure 3. Relation between ρ and material stiffness

Intermediary densities represent an unknown isotropic microstructure with a known stiffness. In practical sense, these intermediary densities are not desirable, because their stiffness is too low and, at least nowadays, it is not commercially advantageous.

Penalizing intermediary densities make their stiffness low comparing to the stiffness of a solid-void structure. Thus, to respect equilibrium and constraint, the optimization decision algorithm makes the intermediary densities attain the upper and lower bounds.

Some numerical drawbacks are always present in optimization algorithms. Undoubtedly, the most common problem is the appearance of solid and void elements alternating themselves, known as checkerboard patterns. Figure (4) illustrates a typical example of this undesirable phenomenon.

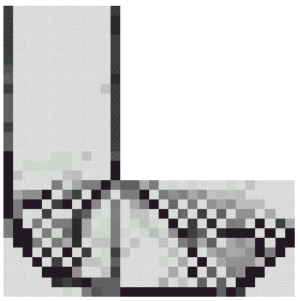

Figure 4. Example of checkerboard pattern

This phenomenon is a convergence problem caused by the incorrect evaluation of the strain energy by the finite element mesh. For example: the topology shown on Fig, (4) is really a local minimum solution of the finite element problem, but not the continuum problem.

In this work we have used the filtering strategy, by controlling the upper and lower moving limits gradients, according to the following equation set:

$$x_i = w_1 x_{i-1} + w_2 x_i + w_3 x_{i+1} \tag{6}$$

where w_j is the filter weight and x_k is the upper or lower density bound (calculated through the moving limits) in the direction X_1 or X_2, i; e., the mesh is firstly horizontally analyzed. When all x_i elements have their moving limits recomputed, we apply equation (6) to the vertical neighbors. A good setting for these weights is 0.01-0.98-0.01, i. e., to compute the density in the element i, 1% of the density from its neighbors is taken account, and "only" 98% from its density is considered.

There is another problem related only to stress constraints, called stress singularity. As described by Cheng and Jiang (1992), the cause of this problem is that the stress function is non-continuous when one element reaches the minimum value for cross section area (or density). When this density tends toward zero, stress value tends to infinitum. Thus, one algorithm based on Karush-Kuhn-Tucker (KKT) optimality conditions (such as SLP) can not reach the actual optimum, located in a subspace of smaller dimension than the dimension of the whole design domain.

To overcome this drawback, one usually uses the perturbation technique called ε-relaxation (Cheng and Guo, 1997), which basically increases the "size" of the original design domain. Thus, mathematical programming algorithms can reach the real optimum, by gradually decreasing the value of ε.

However, we have notice that for the analyzed examples in this work, ε-relaxation technique was not necessary. We have found desirable solutions for the proposed examples without using it.

3. Formulation and solution strategies

As earlier mentioned, this work tries to minimize the global von Mises stress objective function using volume constraint. The great advantage in using a global criterion for stress optimization problems is due to the fact that we can reduce the number of constraints, reducing computing time and allowing smoother meshes. Smooth finite element meshes can better represent the stress field, leading to better results in the optimization phase.

Thus, instead we have one constraint for each finite element, the peaks of stress levels are globally controlled, similarly to compliance problems, where the mechanical work is controlled in the whole mesh. This way, we have a weaker control of the local stress state.

Chen and Yang (1996) firstly proposed an equivalent integrated stress constraint, without taking the stress singularity in account.

In this work, we decided consider the integrated stresses as the objective functions, and not as constraint. Further, we have used a very simple global stress function, derivated from the original stress constrained problem ($\sigma_{VM}^{e} \leq \sigma_{l}$):

$$\begin{aligned} \min \quad & \left[\frac{1}{V_T} \sum_{e=1}^{N} v_e \left(\max \left\{ 0, \frac{\sigma_{VM}^{e}}{\sigma_l} - \frac{1}{\rho_e^{\varepsilon}} \right\} \right)^p \right]^{1/p} \\ s.t.: \quad & \sum_{e=1}^{N} \left[\rho_e^n + \alpha \rho_e \left(1 - \rho_e \right) \right] v_e \leq V_l \end{aligned} \tag{7}$$

where V_T is the total initial structural volume of the finite element mesh (considering all the densities equal to unity), v_e is the actual volume of element e, σ^{e}_{VM} is the von Mises stress already homogenized, σ_l is the stress limit, ρ_e is the density, ε is the ε-relaxation factor, V_l is the volume fraction limit, α and n are penalty factors and p is one given positive integer number.

When $\alpha = 0$ and $n = 1$, one has the structural volume as constraint. For any other values, we are penalizing the constraint, in such a way intermediary densities become very expensive.

Although von Mises stress is always positive, the difference $\frac{\sigma_{VM}^{e}}{\sigma_l} - \frac{1}{\rho_e^{\varepsilon}}$ can result in a negative value. This means that the element e is not activating the stress constraint. Since we are interested in control only the elements whose stress level is higher than the limit, we collect only the positive terms in the sum.

Although our formulation allows the use of ε-relaxation as an exponent applied in each density, preliminary experiments showed that this was not necessary. So, we set $\varepsilon = 0$, returning to the original (and local) formulation, when not considering the integral formulation:

$$\begin{aligned} \min \quad & \left[\frac{1}{V_T} \sum_{e=1}^{N} v_e \left(\max \left\{ 0, \frac{\sigma_{VM}^{e}}{\sigma_l} - 1 \right\} \right)^p \right]^{1/p} \\ s.t.: \quad & \sum_{e=1}^{N} \left[\rho_e^n + \alpha \rho_e \left(1 - \rho_e \right) \right] v_e \leq V_l \end{aligned} \tag{8}$$

Thus, by choosing one stress limit, our optimization process usually starts from one infeasible point (using a low volume fraction limit). So, gradually we increase this volume limit, up to the stresses be respected. Then, using a continuation method (Cardoso, 2000), we penalize the rigidity tensor, accordingly to Eq. (2). After the convergence, if it is still necessary, we penalize the volume constraint in order to "cleaning" as much as possible the intermediary densities from the final topology.

Since this is a very non-linear problem, with several local minima (Bendsøe and Sigmund, 2002), sometimes, even after penalizing some grey areas are still present. In this case, a post-processing stage should be performed, but this was out of the scope in this work.

3.1. Minimum von Mises stress with volume constraint

The problem we want to solve is shown in Eq. (8). Since we are using SLP, it is necessary to compute the derivatives with respect to the design variables. This task is known as sensitivity analysis, that informs how one given functional changes when the design variables changes.

3.1.1 Computation of the derivatives of the constraints and the objective function

The volume constraint equation is shown below:

$$V = \sum_{e=1}^{N} \left[\rho_e^n + \alpha \rho_e \left(1 - \rho_e\right) \right] v_e \tag{9}$$

Differentiating Eq. (9) with respect to ρ_e we obtain (repeted sub-indexes indicates implicit sum):

$$\frac{\partial V}{\partial \rho_e} = \left[n\left(\rho_e^{n-1}\right) + \alpha\left(1 - 2\rho_e\right) \right] v_e \tag{10}$$

The global stress function is shown in the following equation:

$$\sigma\left(\rho\right) = \left[\frac{1}{V_T} \sum_{e=1}^{N} v_e \left(\max\left\{ 0, \frac{\sigma_{VM}^e}{\sigma_l} - 1 \right\} \right)^p \right]^{1/p} \tag{11}$$

To compute its sensitivity, we make use of the adjoint sensitivity analysis, in which we have to add the equilibrium to the function one wants to calculate the derivative. Adding the equilibrium equation, we are not changing the problem, because we are actually adding zero to the functional. So:

$$\sigma\left(\rho\right) = \underbrace{\left[\frac{1}{V_T} \sum_{e=1}^{N} v_e \left(\max\left\{ 0, \frac{\sigma_{VM}^e}{\sigma_l} - 1 \right\} \right)^p \right]^{1/p}}_{f} + \lambda^T \left(Ku - f \right) \tag{12}$$

Thus, differentiating Eq. (12) with respect to the design variables, we obtain:

$$\frac{d\sigma}{d\rho_j} = \frac{\partial f}{\partial \sigma_k} \frac{\partial \sigma_k}{\partial u} \frac{\partial u}{\partial \rho_j} + \lambda^T \left(K \frac{\partial u}{\partial \rho_j} + \frac{\partial K}{\partial \rho_j} u - \cancelto{0}{\frac{\partial f}{\partial \rho_j}} \right) \tag{13}$$

Grouping the terms multiplying $\frac{\partial u}{\partial \rho_j}$:

$$\frac{d\sigma}{d\rho_j} = \left(\frac{\partial f}{\partial \sigma_k} \frac{\partial \sigma_k}{\partial u} + \lambda^T K \right) \frac{\partial u}{\partial \rho_j} + \lambda^T \frac{\partial K}{\partial \rho_j} u \tag{14}$$

Thus, it is not necessary directly to compute the derivative of the displacement vector with respect to the design variables, which is a very expensive calculation. Finding λ, the Lagrange multipliers, it can be calculated the desirable derivative. To compute λ, we set the terms between parenthesis to zero. Then:

$$K\lambda = -\frac{\partial f}{\partial \sigma_k} \frac{\partial \sigma_k}{\partial u} \tag{15}$$

Some authors call λ as pseudo-displacements and the right hand side of Eq. (15) as pseudo-load. However, we still have to compute the derivatives that compound the pseudo-load.

For the first derivative to be computed is the following:

$$f = \left[\sum_{e=1}^{N} \left(\frac{v_e}{V_T} \left(\frac{\sigma_{VM}^e}{\sigma_l} - 1 \right) \right)^p \right]^{1/p} \tag{16}$$

Here we suppressed the "max" function because only the elements with active constraints must have its derivative computed. Thus, computing the derivative of equation (16) with respect σ_k, after some steps we reach to the desirable analytical formula:

$$\frac{\partial f}{\partial \sigma_k} = \left[\sum_{e=1}^{N} \left(\frac{v_e}{V_T} \left(\frac{\sigma_{VM}^e}{\sigma_l} - 1 \right) \right)^p \right]^{\frac{1-p}{p}} \left[\frac{v_e}{V_T} \left(\frac{\sigma_{VM}^k}{\sigma_l} - 1 \right) \right]^{p-1} \frac{v_k}{V_T \sigma_l} \tag{17}$$

The next derivative is computed based on the paper of Duysinx and Bendsøe (1998). The following set of equations is used:

$$\sigma_k = \sqrt{u^T M u}$$

Where:

$$M = T^T V T \tag{18}$$

$$T = \rho^\eta E^0 B$$

$$V = \begin{bmatrix} 1 & -½ & 0 \\ -½ & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

where E^0 is the constitutive tensor of the base material and B is the matrix of derivatives of the shape functions with respect to the axes position, from the finite elements formulation. Again, after some steps, we obtain one expression for computing the derivative:

$$\frac{\partial \sigma_k}{\partial u} = \frac{1}{\sigma_k^{VM}} M^0 u \tag{19}$$

4. Results

The chosen design problem for exemplify this method consists in finding the optimal distribution of material in the "L-shaped" design domain shown in Fig. (5). The structure should withstand a vertical load **(P = 1N)** applied on the middle of the right hand edge without exceeding the stress limit **(σ_l = 60 N/m^2)** of the material. The Young modulus is **E = 100 N/m^2** and the Poisson ratio is **ν = 0.3.** The design domain is discretized by a regular mesh of 1024 quadrilateral bi-linear isoparametric elements.

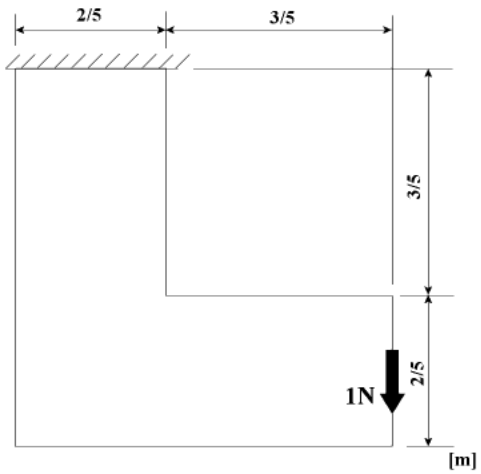

Figure 5. Geometry of L beam problem

For this problem, we started the optimization process with SIMP exponent equal to unit **(η = 1**, in Eq. (2)). After converge, we set **η = 3**, to obtain a cleaner design. After converging again, we penalize the volume constraint, using **p = 0.125** and **α = 0.7** in Eq. (9). The final topology can be seen in Fig. (6) and the stresses are on Fig. (7):

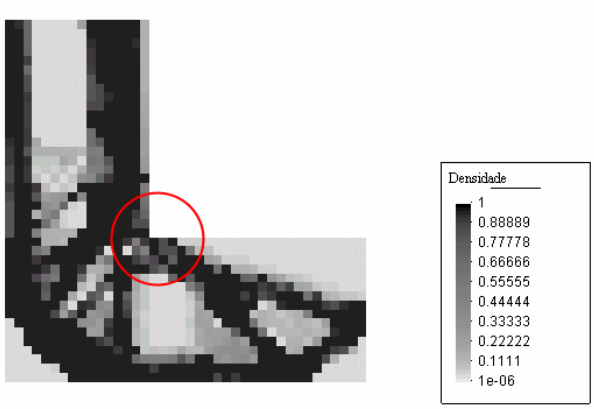

Figure 6. Topology for $\sigma_l = 60$ N/m^2

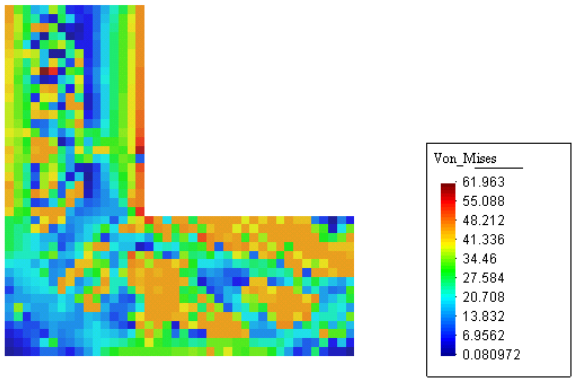

Figure 7. Stress field (σmax = 61.9 N/m^2)

The next picture shows one equivalent obtained by Sant'Anna (2002), using a local stress formulation:

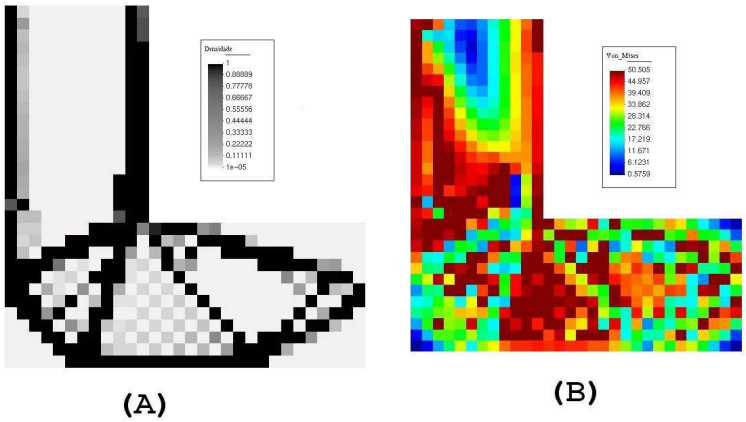

Figure 8. (A) Final topology. (B) Stress field ($\sigma_{max} = 50$ N/m^2)

The most important stress concentration is located the inner corner of the beam because of the stress singularity. Comparing the detached region on Fig. (7) with the same region of Fig. (8-A), we can see the algorithm attempting to minimize this effect, by creating a curved boundary.

To conclude, we also compared our result with one obtained by Duysinx and Sigmund (1998). Using Eq. (4) and a stress limit of 5 N/m^2, they obtained the following result:

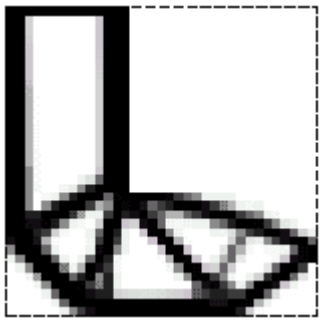

Figure 9. Duysinx and Sigmund (1998) result for a p-mean local stress constraint

5. Conclusions

Our main conclusion is that there is still much work to be done. We developed a code that satisfactorily reaches to a final topology respecting the stress levels. But sometimes, much adjusts in the theoretical parameters must be done (like penalizations).

Dealing with stress constraints is always a hard task. The local constrained problem is much more stable but, even with the inherent difficulties imposed by local stress function, this one leads to better results, as can be seen from the pictures above.

6. Acknowledgements

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