ON THE FLOW NEAR THE LEADING EDGE OF A FLAT PLATE

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Abstract. *In the present work, Kaplun limits are applied to the full Navier-Stokes equations to investigate the flow in the vicinity of the leading edge of a semi-infinity flat plate. The emphasis is on determining the flow asymptotic structure and on providing experimental evidence for it.*

Keywords. *Leading edge, boundary layer, turbulence, Kaplun limits.*

1. Introduction

We will enunciate the objective of this work right at its outset: to study the turbulent boundary layer asymptotic structure near the leading edge of a flat plate by applying Kaplun limits to the Navier-Stokes equations. More than that, we will gather experimental evidence to support our theoretical results.

Unlike the laminar flow case, whose leading edge solution has been studied since the forties, the turbulent problem poses some questions that still have to be understood and answered. Of course, many difficulties stem from the introduction of the time-averaged equations. To make these equations a determined system, closure conditions must be introduced to relate the Reynolds stresses to the mean flow velocities. The Reynolds stresses, the time averages of the fluctuating velocities, describe the effect of turbulent fluctuations on the mean flow; if they could be determined, the mean flow equations could be solved and the asymptotic structure unveiled. Many closure conditions have been proposed in literature but, unfortunately, none of them are generally valid.

In a previous work, Silva Freire and Loureiro(2002) showed how Kaplun limits could be applied directly to the Navier-Stokes equations so as to furnish a formal splitting of the solution domain where distinct asymptotic solutions could be obtained. Using the central hypothesis that the order of magnitude of the Reynolds stresses does not change throughout the boundary layer, Silva Freire and Loureiro splat the flow region into six distinct regions. The splitting resorted to four different scales, determining regions that covered the whole validity domain and overlapped each other.

Here, our major concern is to accumulate experimental evidence that the theory proposed by Silva Freire and Loureiro(2002) makes any sense. For that end a comprehensive experimental program has been developed. The tests were performed in a low turbulence wind tunnel and will be fully reported. Hot-wire anemometry was the experimental technique used in this work.

For the sake of completeness, part of the theory developed by Silva Freire and Loureiro(2002) will be repeated here.

2. Past results

The archetype of a singular perturbation problem is that of a viscous flow past a finite flat plate at high Reynolds number. It was this very problem that originated Prandtl's boundary layer theory. And yet, much still remains to be understood about it. Of course, this apparent lack of knowledge on the problem seems very odd on the eve of its one-century anniversary celebration.

In fact, this is just a simple remainder on how complex this problem is. Right before the beginning of the 20th century, the problem was regarded by many as unsolved. Only after Prandtl's proposition of the boundary layer equations in 1904 the problem was deemed treatable. D'Alembert's paradox had, at last, a rational explanation.

Despite Prandtl's essential contribution to the science of fluid mechanics, the progress in boundary layer theory was very slow. In reality, in proposing his boundary layer theory, Prandtl was well ahead of his time. For nearly 22 years the knowledge on the subject was restricted to his School in Goettingen.

Historically, the first problem to illustrate an application of the boundary layer equations was the flow along a infinite flat plate. In his doctor's thesis at Goettingen, Blasius realized that the velocity profiles at varying distances from the leading edge are similar to each other, so that under a suitable choice of scale factors the boundary layer equations could be reduced to a non-linear, third order, ordinary differential equation. Blasius found an analytical solution to his equation in 1908 in the form of a series expansion around the origin at the wall. Toepfer developed a first numerical solution to the equation in 1912. Finally, measurements to validate the theory were made by Burgers (1925), van der Hegge Zijner (1924) and Hansen (1930).

However, all the above authors were fully aware that the assumptions that led to the derivation of the boundary layer equations break down in the region of slow flow around the leading edge. The infinite velocity gradient demanded by the boundary conditions at the leading edge implies a singularity in the mathematical solution.

The nature of a viscous flow in the vicinity of the leading edge of a semi-infinite flat plate has been studied by Carrier and Lin (1948). Because no fundamental length can be attained to the problem, the appropriate Reynolds number is defined in terms of the leading edge distance. Thus, near the leading edge where Reynolds number is small, Stokes's equations are valid. By expanding a series for a small radius, Carrier and Lin proposed a solution to Stokes's equations that required the determination of three constants for its completion. Unfortunately, these constants depended on boundary conditions far outside the range of validity of solution so that they were undetermined in the context of the analysis. Thus, the approximation of Carrier and Lin was not an inner solution in a classical sense; it has no overlap with Blasius solution and that makes any matching process unfeasible. As it turned out, the problems were caused by the presence of logarithmic terms in the Stokes type solution.

Subsequently, Davis (1967) examined the problem of laminar incompressible flow past a semi-infinity flat plate by using the method of series truncation on the full Navier-Stokes equations. The first and second truncations were calculated at points on the plate away from the leading edge, whereas only the first truncation was calculated at the leading edge. Up to that time, and as we have just seen in the above paragraph, the approaches to the flat plate problem had consisted in finding local equations near the leading edge and far downstream from the leading edge. This resulted in a skin-friction coefficient that was singular at the leading edge. Davis's solution, however, also presented a singularity for the leading edge.

All results mentioned so far have dealt with laminar flow. For turbulent flow, a problem of considerable greater technological interest, almost nothing was found in literature. The majority of works on leading edge flow has been dedicated to the receptivity problem, with very little attention being paid to the mathematical aspects of the near edge solution. Of course, the problem of describing a turbulent flow is greatly complicated by the introduction of the averaged Navier-Stokes equations and the resulting Reynolds stress terms. In most cases, two kinds of approaches are used: in the first, asymptotic techniques are applied to the averaged equations without appealing to any closure model (Yajnik (1970), Mellor (1972)); in the second, eddy-viscosity (Bush and Fendell (1972)) or κ–ε (Deriat and Guiraud (1986)) models are used to find high order approximations.

For the zero-pressure-gradient boundary layer, the more recent theories divide the turbulent boundary layer into three regions (Long and Chen(1981), Sychev and Sychev(1987), Melnik(1989)). This structure considers an intermediate region in which a balance of inertia forces, and pressure and turbulent friction forces occurs.

3. Kaplun limits

The formulation to be presented here is only introductory to the ample set of results presented in Kaplun(1967) and in Lagerstrom and Casten(1972). For more details on the technique, the reader is referred to these two works. Complementary material is found in Meyer(1967), in Freund[16] and in Silva Freire and Hirata(1990).

Here we use the topology on the collection of order classes as introduced by Meyer(1967).

Let ε be a parameter on (0, 1] and x a variable in R^n with Euclidean norm $| x |$.

Let F be a function defined for ε and on some x-space domain with pointwise norm $| F |$. Our interest is to study the behaviour of F in the limit as ε tends to 0. In particular, we are interested in the cases where singularities arise. For example, passage of the limit may result in the loss of the highest order derivative term in a differential equation, and hence in the impossibility of satisfying all the boundary conditions. The idea of the Kaplun limit is to study the limit as as ε tends to 0 not for fixed x near a singularity point x_d, but for x tending to x_d in a definite relationship to ε specified by a stretching function $\eta(\varepsilon)$.

Taking $xd = 0$, we define

$$x(e) = \frac{x}{h(e)} \qquad\qquad G(x_h) = F(x;e) \tag{1}$$

with $h(e)$ a function defined in X (=space of all positive continuous functions on $(0, 1]$).

The Kaplun limit process is then defined as follows.

Definition 1 (Meyer(1967)). If the function

$$G(x_h;+0) = lim\ G(x_h;e), \qquad e \to 0, \tag{2}$$

exists uniformly on { x / | x_h | > 0 }; then we define $lim_h\ F(x;e) = G(x_h, +0)$.

If F is a function defined by a system of differential equations, then the above definition establishes to every order of η a correspondence original equation associated equation on that subset of X for which the associated equation exists. The passage of the η-limit process is a formal operation which results in a set of associated equations referred to by Kaplun(1967) as the "splitting" of the original differential equation; this operation establishes the basis for the definition of formal domain of validity.

Definition 2. The formal limit domain of an associated equation E is the set of orders e such that the η-limit process applied to the original equation yields E.

To evaluate how close two equations are, Kaplun needed to advance a measuring procedure. This was made through the definition of equivalent in the limit.

Definition 3. Two equations E_1 and E_2 are said to be equivalent in the limit for a given limit process, lim_h, and to a given order $d(e)$, if,

$$q = \frac{E_1(x_h;e) - E_2(x_h;e)}{d(e)} \to 0, \quad \text{as} \quad e \to 0. \tag{3}$$

Definition 4 (of formal domain of validity). The formal domain of validity to order $d(e)$ of an equation E of formal limit domain D is the set $D_e = D\ U\ D'_i$ s, where D_i's are the formal limit domains of all equations E_i such that E and E_i are equivalent to D_i to order $\delta(\varepsilon)$.

To relate the formal domain of validity of an equation to its actual domain of validity, Kaplun (1967) advanced two assertions, the Axiom of Existence and the Ansatz about domains of validity. These assertions are primitive and unverifiable assumptions of perturbation theory. They allow one to use definitions 1 to 4 to find approximate solutions to singular perturbation problems. Because the heuristic nature of the Axiom and of the Ansatz, comparison to experiments will always be important for validation purposes. The theory, however, as implemented through the above operations, is always very helpful in understanding the matching process and in constructing the appropriate asymptotic expansions.

Axiom (of existence) (Kaplun (1967)). If equations E_1 and E_2 are equivalent in the limit to order $d(e)$ for a certain region, then given a solution S_1 of E_1 which lies in the region of equivalence of E_1 and E_2, there exists a solution S_2 of E_2 such that as e tends to 0, | S_1 - S_2 |/δ tends to 0, in the region of equivalence of E_1 and E_2.

To this axiom, there corresponds an Ansatz to ensure that there exists a solution S_1 of E_1, which lies in the region of equivalence of E_1 and E_2.

Ansatz (about domains of validity) (Kaplun(1967)). An equation with a given formal domain of validity D has a solution whose actual domain of validity corresponds to D.

The word "corresponds to"' in the Ansatz was assumed by Kaplun to actually mean "is equal to". The above formulation ceases to be valid when small terms have large integrated effects. In the example to be studied here, however, the principle is expected to work. Switchback terms, which are deduced from inspection of formally higher order terms, can always be included in the original formulation if we backtrack to the lower order terms. Large integrated effects occur when singularities occur in the approximating functions; these are not expected to occur here

4. The Turbulent flow near the leading edge of a flat plate

The classical three-layered asymptotic structure must undergo modifications if the flow near to the leading edge of a flat plate is to be considered.

A major difficulty for a direct translation of the classical boundary layer model into a model that applies for the leading edge is the fact that the assumption.

$$\frac{\partial^2}{\partial y^2} >> \frac{\partial^2}{\partial x^2}$$

does not apply any more.

In addition, when the friction velocity, u_τ is used to develop the asymptotic structure of the boundary layer, a non-uniformity will occur near the leading edge point where $u_\tau = 0$. The result is that any theory advanced for the problem should explain in asymptotic terms how the far downstream two-deck structure reduces to an alternative structure near the leading edge.

To find the asymptotic structure of the boundary layer near a leading edge point we apply the following stretching transformation to the equations of the previous section

$$y_h = \frac{y}{h(\varepsilon)} \qquad\qquad x_D = \frac{x}{D(\varepsilon)} \tag{4}$$

with η and Δ defined on the Ξ.

The resulting flow structure is given by:

x-momentum equation.

$$ord\ \text{Ä} = ord\, 1:\ \hat{u}_1 \frac{\partial \hat{u}_1}{\partial x_{\text{Ä}}} + \hat{v}_1 \frac{\partial \hat{u}_1}{\partial y_{\text{ç}}} + \frac{\partial \hat{p}_1}{\partial x_{\text{Ä}}} = 0, \tag{5}$$

$$ord\ \mathbf{e}^2 < ord\ \text{Ä} < ord\, 1:\ \hat{u}_1 \frac{\partial \hat{u}_1}{\partial x_{\text{Ä}}} + \hat{v}_1 \frac{\partial \hat{u}_1}{\partial y_{\text{ç}}} + \frac{\partial \hat{p}_1}{\partial x_{\text{Ä}}} = 0, \tag{6}$$

$$ord\ \mathbf{e}^2 = ord\ \text{Ä}:\ \hat{u}_1 \frac{\partial \hat{u}_1}{\partial x_{\text{Ä}}} + \hat{v}_1 \frac{\partial \hat{u}_1}{\partial y_{\text{ç}}} + \frac{\partial \hat{p}_1}{\partial x_{\text{Ä}}} = -\frac{\partial \overline{\hat{u}'^2_1}}{\partial x_{\text{Ä}}} - \frac{\partial \overline{\hat{u}'_1 \hat{v}'_1}}{\partial y_h}, \tag{7}$$

$$ord\ 1/eR < ord\ \text{Ä} < ord\ \mathbf{e}^2 : 0 = -\frac{\partial \overline{\hat{u}'^2_1}}{\partial x_{\text{Ä}}} - \frac{\partial \overline{\hat{u}'_1 \hat{v}'_1}}{\partial y_h}, \tag{8}$$

$$ord\ 1/eR\, ord\ \text{Ä} : 0 = \frac{\partial^2 \hat{u}_1}{\partial x^2_{\text{Ä}}} + \frac{\partial^2 \hat{u}_1}{\partial y^2_h} - \frac{\partial \overline{\hat{u}'^2_1}}{\partial x_{\text{Ä}}} - \frac{\partial \overline{\hat{u}'_1 \hat{v}'_1}}{\partial y_h}, \tag{9}$$

$$ord\ \text{Ä} < ord\ 1/eR :\ 0 = \frac{\partial^2 \hat{u}_1}{\partial x^2_{\text{Ä}}} + \frac{\partial^2 \hat{u}_1}{\partial y^2_h}. \tag{10}$$

y-momentum equation

$$ord\ \text{Ä} = ord\, 1 : \hat{u}_1 \frac{\partial \hat{v}_1}{\partial x_{\text{Ä}}} + \hat{v}_1 \frac{\partial \hat{v}_1}{\partial y_{\text{ç}}} + \frac{\partial \hat{p}_1}{\partial y_{\text{ç}}} = 0, \tag{11}$$

$$ord\ \text{Ä} < ord\, 1 : \frac{\partial \hat{p}_1}{\partial y_h} = 0. \tag{12}$$

Note that in region $(D,\ h) = (e^2,\ e^2)$ the boundary layer formulation has to be modified so as to include an extra Reynolds stress term in the x-momentum equation. The y-momentum equation remains unchanged, with the pressure term dominating the leading order solution. Further close to the leading edge the complete viscous terms are recovered in the x-momentum equation; the pressure term still dominates the y-momentum equation.

The two principal equations in the x-direction are Eqs. 7 and 9. They cover the whole domain and overlap in $ord\ 1/eR < ord\ D < ord\ e^2$. In the y-direction the motion is dominated by the single principal equation, Eq. 11. The result is that near the leading edge the motion governing equations are not the boundary layer equation. Rather, alternative equations need to be considered which include extra Reynolds and viscous stress terms.

Thus, the region of approximate validity of these equations are as follows:

In addition, the y-momentum equation has to be considered in the analysis in the form of Eq. 11.

Some experimental evidence (Sreenivasan(1989)) has shown that

$$\begin{aligned} \mathbf{d} &\propto x^{4/5}, \\ \mathbf{e} &\propto x^{-1/10}, \\ y_p &\propto x^{1/2}. \end{aligned}$$

where y_p denotes the point of maximum turbulent stress.

The structure presented in Fig. 1 is consistent with these results.

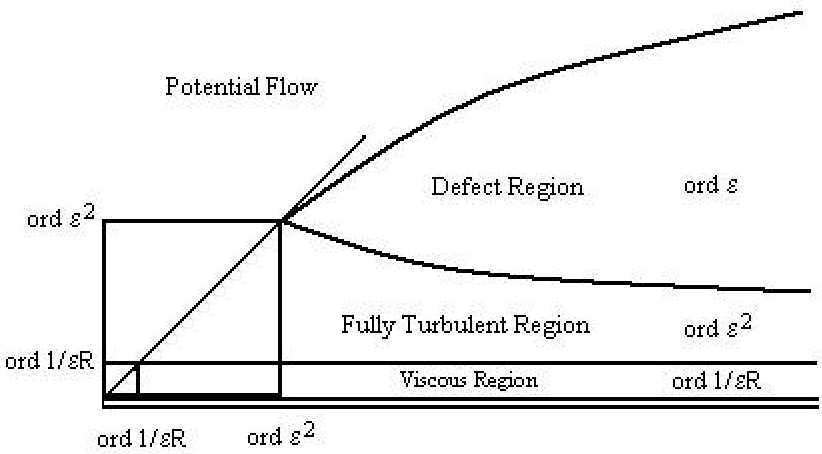

Figure 1. Asymptotic structure of the turbulent boundary layer near the leading edge of a flat plate.

5. Experimental evidence

The nature the flow in the leading of a flat plate will be experimentally studied in the low-turbulence wind tunnel of the Laboratory of Turbulence Mechanics of PEM/COPPE/UFRJ. This tunnel presents turbulence intensity levels of the order of 0.2% and can be set to run at velocities that can reach 13 m/s. The test section is 4 m long and the cross section area is 0.30 x 0.30 m. The tunnel has recently being adapted for the calibration of cold-wires with the inclusion of a new heating section. The heating section was built with four electrical resistances in series, each one of them consisting of strings distributed transversally to the flow. In the present experiments, however, the tunnel will run in isothermal conditions. The tunnel has honeycombs and screens to control the turbulence levels so as to guarantee a uniform flow. The computer-controlled traverse gear is two-dimensional and capable to position sensors with an accuracy of 0.1 mm

The experiments were conducted in a controlled environment, with the laboratory temperature set to 18.0 ^{o}C +/- 0.5 ^{o}C.

The general features of the wind tunnel are shown below.

- Circuit: open.
- Test section: 0.30 m high, 0.30 m wide and 2 m long.
- Wind speed: continuously variable from 0.5 to 16 m/s.
- Longitudinal pressure gradient: adjustable to zero by means of an adjustable ceiling.
- Turbulence intensity: below 0.2.
- Incoming flow temperature: variable from 20 to 35$^o C$.
- Number of resistences used to heat the incoming air: 4.
- Resistances capacity: 7 kW.

A general view of the low-turbulence wind tunnel is shown in Figure 2. Details of the flat plate assembly are shown in Figure 3.

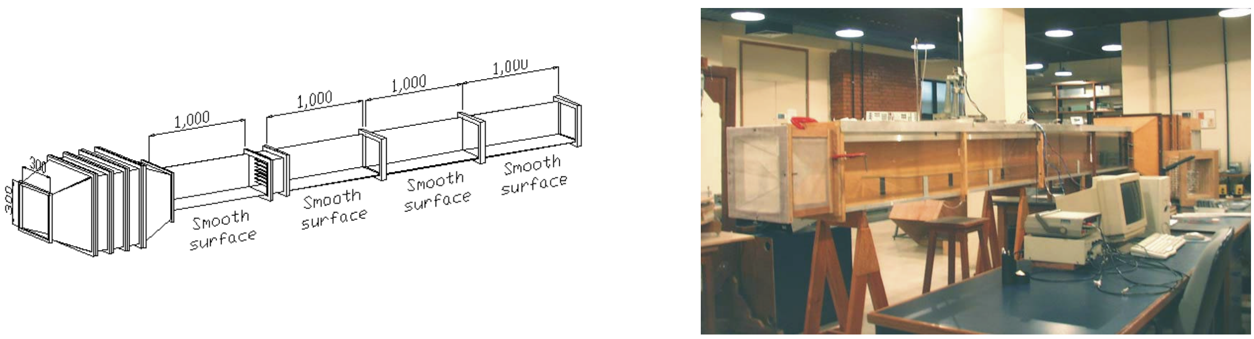

Figure 2. General view of wind tunnel. Dimensions in millimeters.

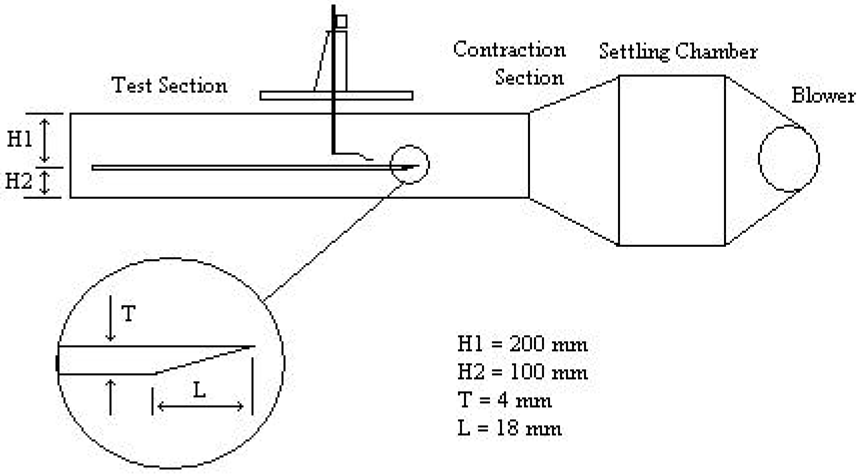

Figure 3. Assembly details of flat plate.

For the present measurements, a DANTEC 55M01 main unit together with a 55M20 constant current bridge was used. The boundary layer probe was of the type 55P76. A Pitot tube, an electronic manometer, and a computer controlled traverse gear were also used. In getting the data, 10.000 samples were considered. The reference mean temperature profiles were obtained through a chromel-constantan micro-thermocouple mounted on the same traverse gear system used for the hot-wire probe. An uncertainty analysis of the data was performed according to the procedure described in Kline (1985). Typically the uncertainty associated with the velocity and temperature measurements were: U = 0.0391 m/s precision, 0 bias (P=0.95); T = 0.2 ^{o}C precision, 0 bias (P=0.99).

To obtain accurate measurements, the mean and fluctuating components of the analogical signal given by the anemometer were treated separately. Two output channels of the anemometer were used. The mean velocity profiles were calculated directly from the untreated signal of channel one. The signal given by channel two was 1 Hz high-pass filtered leaving, therefore, only the fluctuating velocity. The latter signal was then amplified with a gain controlled between 1 and 500 and shifted by an offset so as to adjust the amplitude of the signal to the range of the A/N converter.

Velocity profiles in physical and logarithmic coordinates are shown in Figures 4 and 5.

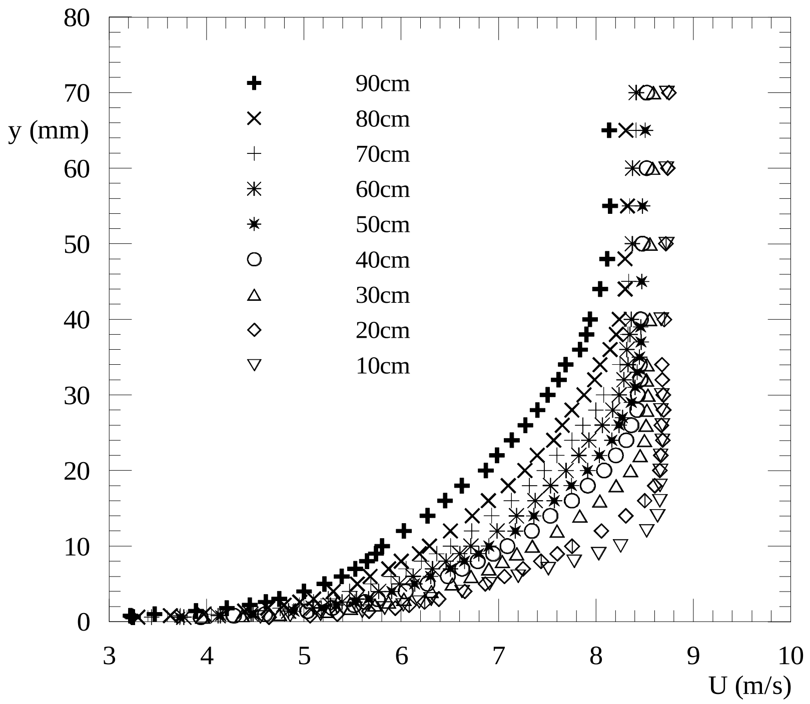

Figure 4. Velocity profiles in physical coordinates.

The pictures show that at distances over 10 cm the classical structure of the boundary layer has clearly been established. All three classical regions can easily be identified: the viscous region, the turbulent region and the defect region. For the nine velocity profiles shown in Fig. 4, the parameters shown in Table 1 have resulted.

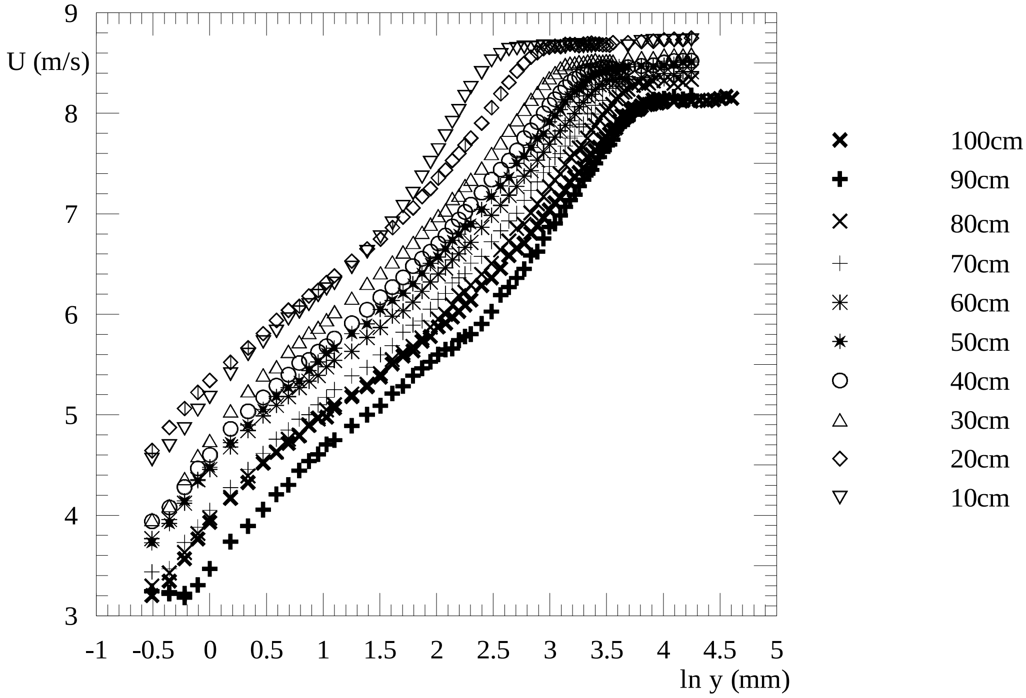

Figure 5. Velocity profiles in logarithmic coordinates.

Table 1. Velocity profile parameters.

Station (mm)	u_τ (m/s)	A	Π	δ (cm)
100	0.39	5.17	0.44	13
200	0.40	5.62	0.30	18
300	0.37	4.2	0.20	22
400	0.37	4.1	0.26	31
500	0.36	4.00	0.27	32
600	0.35	4.71	0.35	40
700	0.33	3.12	0.32	43
800	0.33	3.45	0.41	41
900	0.31	2.44	0.42	47

A closer view of the flow behaviour in the leading edge region is shown in Figure 6.

Despite all caution we have taken to try to prevent separation at the leading edge of the flat plate, the resulting flow configuration was clearly that of a separated flow. In literature, most researchers that investigate the flow in a leading edge use carefully tailored wing profiles that normally resort to the hodograph method to design the leading edge. These leading edges are, therefore, not blunt but aerodynamic.

Here, we have sharpened the tip of a 4 mm aluminum plate to simulate a semi-infinity flat plate. The tip of the plate was machined as shown in Figure 3. The flat plate when fitted into the wind tunnel was let with an angle of attack of -0.2^{o} to prevent separation.

Despite all that, a separation region extended over the first 60 mm of the plate. Even for the small dimensions defined here the plate is seen by the flow as a blunt body.

Despite our failure in obtaining an attached flow at the tip of the plate we have decided to show our results here to illustrate the degree of difficulty involved in the study of the flow in the leading edge region.

New experiments are currently under way to simulate the flow of an attached leading edge flow.

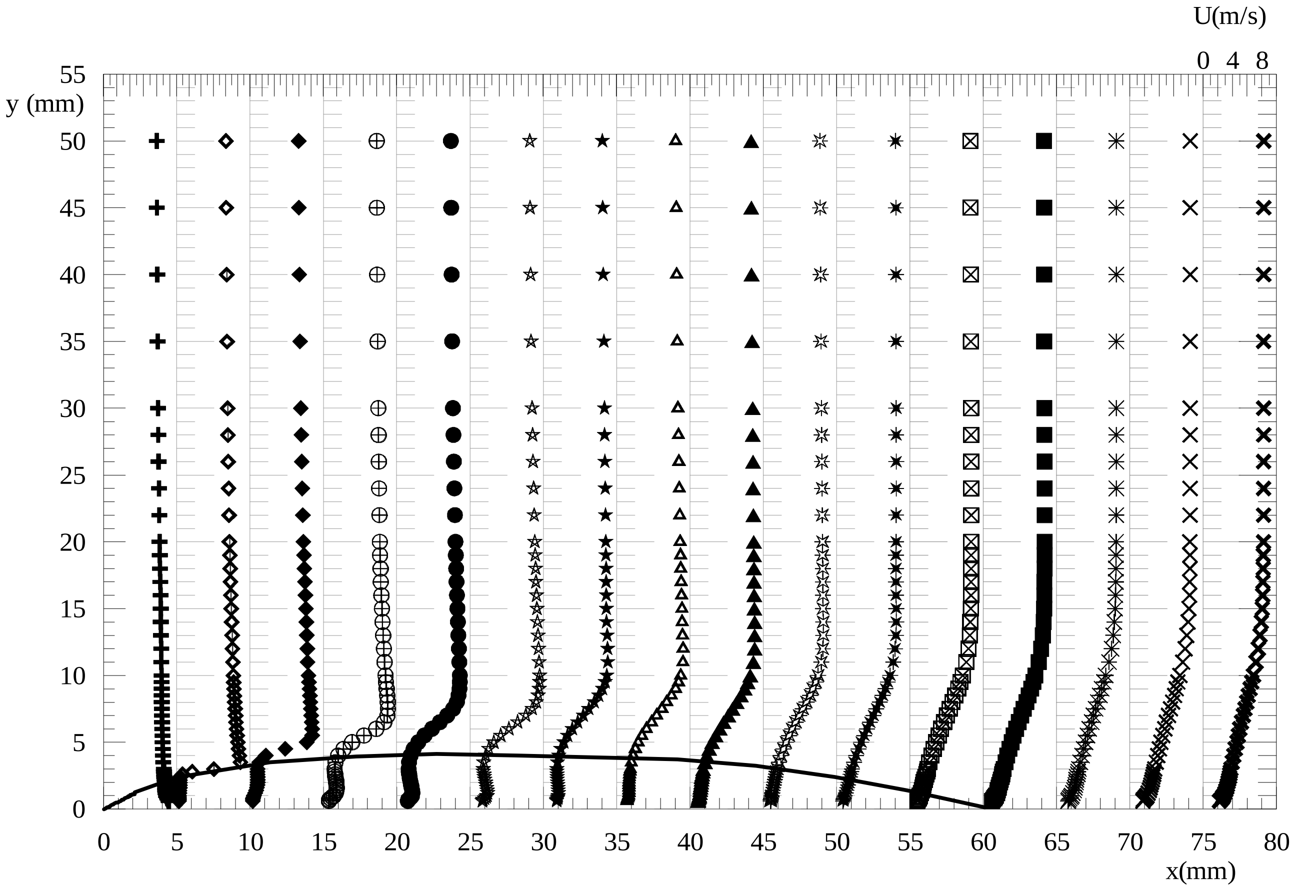

Figure 6. Velocity profiles in physical coordinates showing the separated flow region.

6. Final Remarks

In the present work, Kaplun limits were applied to the full Navier-Stokes equations to analyze the flow in the leading edge of a flat plate. The experiments have failed to produce an attached leading edge flow so that the results could be completely tested. However, the experiments have been extremely useful in provide information for future simulation of the phenomenon. In addition, the experiments have shown that at a relatively short distance from the leading edge a completely developed turbulent boundary layer flow arises. Figures 4 and 5 show that the law of the wall, law of the wake, regions are well defined, so that the transition region is restricted to the first 6 cm of the plate.

In the coming months, the experiments will be repeated under new conditions so that a better understanding of the leading edge flow behaviour can be achieved.

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