

A MODEL FOR THE EFFECT OF TURBULENCE ON THE MOLECULAR VISCOSITY OF GENERALIZED NEWTONIAN FLUIDS

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Abstract. A model is derived to account for the effect of turbulence on the molecular viscosity of a purely viscometric fluid obeying a power-law equation. The Generalized Newtonian model is modified to mimic extensional effects and this introduces a second kinematic invariant into the viscosity equation. The dependence of viscosity on the strain rate invariant is also non-linear and consequently affected by turbulence. The molecular viscosity model derived initially is modified to account for this extra dependence.

Keywords. Drag reduction, turbulence model, elasticity, extensional viscosity

1. Introduction

Drag reduction in turbulent pipe flow of non-Newtonian fluids has been known since at least 1948 when it was first reported by Toms (1948). Ever since, many researchers have dedicated their time at understanding the behaviour of viscoelastic fluids under turbulent flow conditions. Such efforts culminated in the mid nineteen seventies in a fair amount of phenomenological understanding as is well documented in the reviews of Hoyt (1972) and Virk (1975). For the next twenty years research efforts were aimed at a more in-depth physical understanding of the details of such wall-dominated flows, made possible by developments in optical diagnostic techniques. Examples of detailed investigations with non-intrusive optical techniques are the earlier works of Achia and Thompson (1974) and Reischman and Tiederman (1975), later those of Luchik and Tiederman (1987,1988) and Pinho and Whitelaw (1990) in the eighties and more recently Pereira and Pinho (1994) and Escudier et al (1999), amongst others.

In contrast to experimental work, progress in turbulence models for viscoelastic flow predictions has been rather slow. After an initial effort by various research groups in the late nineteen seventies (Durst and Rastogi, 1977; Hassid and Poreh, 1975, 1978; Poreh and Hassid, 1977) very few developments have taken place henceforth. In some of these initial investigations, the $k - \epsilon$ turbulence model for Newtonian fluids was used with very specific modifications in wall functions (standard model) or damping functions (low Reynolds number approach). In Hassid and Poreh (1975) a one-equation model was suggested, but in general these were rapidly being discarded for better models due to difficulties in defining the appropriate length scale. In Hassid and Poreh (1978) and Poreh and Hassid (1977) the same modified version of the low Reynolds number $k - \epsilon$ model of Jones and Launder (1973) was used. In essence, these modifications were unable to deal with drag reduction with generality, but showed that adequate modifications of the law of the wall or of damping functions would lead to correct predictions. This deficiency stems from the lack of connection between turbulence model and fluid rheology. A solution to this shortcoming was attempted by Mizushima et al (1973): the Van Driest damping factor in a zero-equation turbulence model was modified to account for the viscoelasticity of the fluid by incorporating Rouse's relaxation time (Bird et al,1987) to be determined from rheological experiments. This approach enabled predictions of several flow conditions with the same fluid, a situation that not even the later models of Hassid and Poreh (1975,1977, 1978) and Durst and Rastogi (1977) could perform.

More recent excursions into the subject were also of limited application: Politis (1989) and Cruz et al (2000) independently deduced the relevant $k - \epsilon$ equations for turbulent flows of inelastic power-law fluids but the intense drag reduction of elastic origin could not be predicted. However, some of the ideas of the former work are pursued in the present work.

Recently, self-consistent DNS investigations of turbulent channel flow with viscoelastic constitutive equations derived from kinetic (FENE-P model) and network (Giesekus model) theories have been carried out by Sureshkumar et al (1997) and Dimitropoulos et al (1998, 2001). These works were able to predict elastic drag reduction and have shown qualitative agreement with experimental findings. In fact, the simulations with the FENE-P and Giesekus models showed similar amounts of drag reduction when their parameters were chosen to match the plateau extensional viscosity (Dimitropoulos et al, 2001). This latter work has also provided budgets of turbulence kinetic energy, Reynolds stresses and vorticity and consequently it is a major reference for the development of single-point turbulence closures. Drag reduction was found to be directly related to the extensibility of the polymer chains and a pre-requisite for drag reduction is a sufficiently enhanced extensional viscosity in agreement with the findings from various other sources. The recent experiments of Ptasiniski et al (2001) also provided budgets of mean energy and of turbulent kinetic energy, the results of which confirm some findings by DNS.

There is clearly the need for further progress in turbulence modeling for viscoelastic fluids, in particular taking into account the extensional behaviour of the drag reducing fluids. This paper constitutes the first step of a work aimed at developing and testing turbulence models for drag reducing fluids obeying a specific family of constitutive equations: here, a Generalised Newtonian fluid model, that was modified to mimic some effects of the extensional viscosity, was

chosen as constitutive equation. Instead of adopting immediately a complex differential viscoelastic constitutive equation, for which modeling would be more difficult, a Generalized Newtonian fluid modified to mimic extensional viscosity enhancement effects will lead to a turbulence model with some similarities to a Newtonian turbulence model. Turbulence models for Newtonian fluids do not suffer from the difficulty of choosing a constitutive equation and yet they often rely on damping functions and other approaches to compensate for inadequate or incomplete physics modeling. There is no reason to assume that it will not be so for viscoelastic fluids. If a more complex, but faithful, viscoelastic constitutive equation is adopted, it is likely that there will be many ad-hoc assumptions and simplifications than if a simpler Newtonian-like rheological expression is used. This will provide us with the experience for more elaborate turbulent closures in the future.

The paper is organised as follows: the constitutive model adopted and its modifications are explained and the corresponding time-averaged conservation equations of mass, momentum and turbulent kinetic energy are presented. Briefly, results of an order of magnitude analysis are stated in order to explain the relevance of the new terms involving the time-average molecular viscosity. Then, using high Reynolds number turbulence arguments a closed form expression for this time-average viscosity is derived.

2. Adopted constitutive equation

The most important rheological property of polymer solutions that must be taken into account is the viscometric viscosity. It can be constant, as with Newtonian fluids, but most often exhibits some degree of shear-thinning. A Generalised Newtonian fluid model (GNF) is adequate to predict this fluid property accurately.

For a long time there was controversy as to what rheological properties caused drag reduction and this was discussed in Oliveira and Pinho (1998). There, the relevance of extensional viscosity was made clear and the recent DNS works of Dimitropoulos et al (1998, 2001) and De Angelis et al (2000) have extensively confirmed it.

Experimentally, Escudier et al (1999) were probably the first to measure drag reduction and detailed velocity profiles with polymer solutions for which they provided extensional and viscometric viscosity data. The former was measured with an opposed-jet nozzle rheometer. Their measurements with different types of polymers showed shear-thinning of the viscometric viscosity and confirmed the strain-thickening of the Trouton ratio, relevant rheological features for drag reduction.

The constitutive equation adopted is simple: the GNF model modified to mimic extensional viscosity strain-thickening. This was the model used by Oliveira and Pinho (1998), here with small modifications. The elongational viscosity is introduced into the GNF constitutive equation by making it a function of the strain rate $\dot{\epsilon}$ as explained there in more detail. The GNF fluid, with dependence on the shear rate $\dot{\gamma}$ and strain rate $\dot{\epsilon}$, is written as

$$\boldsymbol{\sigma} = 2\mu\mathbf{S} \quad (1)$$

where μ is the viscosity function and \mathbf{S} is the rate of deformation tensor defined by

$$\mathbf{S} \equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (2)$$

The viscosity function $\mu = \mu(\dot{\gamma}, \dot{\epsilon})$ depends on $\dot{\gamma}$ and $\dot{\epsilon}$ which are the following invariants of \mathbf{S}

$$\dot{\gamma} = \sqrt{-4II_S} = \sqrt{2S_{ij}S_{ij}} \quad \text{and} \quad \dot{\epsilon} = \frac{6\det\mathbf{S}}{tr\mathbf{S}^2} = \frac{2tr\mathbf{S}^3}{tr\mathbf{S}^2} = \frac{2(S_{ik}S_{kj})S_{ij}}{S_{ij}S_{ij}} \quad (3)$$

An algebraic form for the viscosity function μ can be a Bird-Carreau type of equation but, for simplicity in the derivation of the turbulence model, a power-law based equation is preferred. The viscosity equation adopted is

$$\mu = K_v \left[\dot{\gamma}^2 \right]^{\frac{n-1}{2}} K_e \left[\dot{\epsilon}^2 \right]^{\frac{p-1}{2}} \quad (4)$$

where some constraints to the various parameters may have to be imposed (Cruz et al, 2003). This viscosity model is the product of a shear rate dependent term by a strain rate dependent term. Both terms do not have to be dimensionally identical but their product must be a viscosity and Eq. (4) must also obey some limiting conditions and agree with rheological measurements. The meaning of the various model parameters are specified in Cruz et al (2003) and at this stage it is only important to consider K_v , K_e , n and p as known fluid properties.

For this GNF constitutive equation it is now necessary to deduce the corresponding conservation equations for turbulent flow, bearing in mind that there are fluctuations in the viscosity because of its non-linear dependence on the flow kinematics. Although a specific viscosity expression is presented in Eq. (4), Section 3 is totally independent of the

adopted equation for μ . A specific viscosity model is only required from Section 4 onwards, but even then part of this work remains general and independent of specific forms adopted for μ .

3. Conservation equations

Modern developments of one-point closures for turbulence are usually based on first or second order models. Since this work is part of the development of a first-order coupled turbulence-rheology closure, conservation equations for mass, momentum, turbulent kinetic energy, and its rate of dissipation need to be derived for these fluids of variable viscosity, where the first major conceptual difference relative to a Newtonian fluid is the existence of fluctuations of viscosity. The equation for the turbulent kinetic energy results from contraction of indices of the transport equation for the Reynolds stress tensor components.

The derivation of all transport equations is tedious and has similarities to that for Newtonian fluids. It was carried out elsewhere (Pinho, 2002) and here only the final forms of the various equations, except the equation for ε , are presented. Throughout the paper the Reynolds decomposition is used and the average of the fluctuating quantities, including the viscosity, is zero. Capital letters or an overbar designate average values, small letters or a prime designate fluctuating quantities, the exception being the average pressure represented by p .

The Reynolds-averaged momentum equation for a GNF fluid is

$$\rho \frac{\partial U_i}{\partial t} + \rho U_k \frac{\partial U_i}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \frac{\partial (2\bar{\mu} S_{ik} + 2\overline{\mu' s_{ik}} - \rho \overline{u_i u_k})}{\partial x_k} \quad (5)$$

Relative to the momentum equation for a Newtonian fluid there is a new diffusive term ($2\overline{\mu' s_{ik}}$) and the classical term ($2\bar{\mu} S_{ik}$) is modified. Both need to be evaluated later for closure. As will be seen in Section 4, $\bar{\mu}$ also depends on s_{ij} and S_{ij} although at high Reynolds numbers the dependence on s_{ij} is more important. Its determination constitutes the main objective of this work.

In the recent experimental and DNS investigations of turbulent duct flows of viscoelastic fluids using the FENE-P model reviewed in Section 1, the total stress is written as the sum of the solvent, the polymer and the Reynolds shear stress tensors: $\tau_{ij} = \tau_{ij,s} + \tau_{ij,p} - \rho \overline{u_i u_j}$. It is important to understand that $\tau_{ij,p}$ accounts not only for an elastic contribution but also for an average viscous contribution of the polymer. In the present formulation, however, the separation of solvent and polymer effects is not carried out because both $2\bar{\mu} S_{ij}$ and $2\overline{\mu' s_{ij}}$ include purely viscous and extensional contributions of the polymer solution (notice that $\overline{\mu' s_{ij}} \neq 0$ for an inelastic shear-thinning fluid). Thus, $\tau_{ij,s}$ is totally included in $2\bar{\mu} S_{ij}$ but so is also part of $\tau_{ij,p}$. The new diffusive term was seen to be negligible for fluids with $n=1$ (Oliveira and Pinho, 1998), but otherwise, although small, it will need to be properly quantified. At this stage of development the term $2\overline{\mu' s_{ij}}$ is dropped but constitutes an important goal for future developments (some preliminary work on modeling $2\overline{\mu' s_{ij}}$ has shown the term to be small, thus justifying its neglect as a first approximation). Consequently, in the momentum equation the term $2\bar{\mu} S_{ij}$ bears a significant amount of the effects of drag reduction, hence the importance of an adequate determination of the time-average molecular viscosity, the objective of this work.

The Reynolds stress transport equation is given by

$$\begin{aligned} \rho \frac{D\overline{u_i u_j}}{Dt} + \rho \overline{u_j u_k} \frac{\partial U_i}{\partial x_k} + \rho \overline{u_i u_k} \frac{\partial U_j}{\partial x_k} = & -\rho \frac{\partial}{\partial x_k} \overline{u_i u_j u_k} - \left(\frac{\partial}{\partial x_i} \overline{p' u_j} + \frac{\partial}{\partial x_j} \overline{p' u_i} \right) + p' \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) + \bar{\mu} \frac{\partial^2 \overline{u_i u_j}}{\partial x_k \partial x_k} - \\ & 2\bar{\mu} \frac{\partial \overline{u_i}}{\partial x_k} \frac{\partial \overline{u_j}}{\partial x_k} + \frac{\partial \bar{\mu}}{\partial x_k} \frac{\partial \overline{u_i u_j}}{\partial x_k} + \frac{\partial \bar{\mu}}{\partial x_k} \left(\frac{\partial \overline{u_k u_j}}{\partial x_i} + \frac{\partial \overline{u_k u_i}}{\partial x_j} - 2\overline{u_k s_{ij}} \right) + \overline{\mu'} \frac{\partial^2 \overline{u_i u_j}}{\partial x_k \partial x_k} - 2\overline{\mu'} \frac{\partial \overline{u_i}}{\partial x_k} \frac{\partial \overline{u_j}}{\partial x_k} + \frac{\partial \overline{\mu'}}{\partial x_k} \frac{\partial \overline{u_i u_j}}{\partial x_k} + \\ & \frac{\partial \overline{\mu'}}{\partial x_k} \left(\overline{u_j} \frac{\partial \overline{u_k}}{\partial x_i} + \overline{u_i} \frac{\partial \overline{u_k}}{\partial x_j} \right) + \overline{\mu' u_j} \frac{\partial^2 U_i}{\partial x_k \partial x_k} + \overline{\mu' u_i} \frac{\partial^2 U_j}{\partial x_k \partial x_k} + \overline{u_j} \frac{\partial \overline{\mu'}}{\partial x_k} \left(\frac{\partial U_i}{\partial x_k} + \frac{\partial U_k}{\partial x_i} \right) + \overline{u_i} \frac{\partial \overline{\mu'}}{\partial x_k} \left(\frac{\partial U_j}{\partial x_k} + \frac{\partial U_k}{\partial x_j} \right) \end{aligned} \quad (6)$$

and contraction of indices gives the transport equation for the turbulent kinetic energy ($k \equiv \overline{u_i^2}/2$)

$$\rho \frac{Dk}{Dt} = -\frac{\partial \overline{u_j p}}{\partial x_j} - \frac{\partial}{\partial x_j} \left[\frac{1}{2} \rho \overline{u_i u_j u_i} - 2\bar{\mu} \overline{u_i s_{ij}} - 2\overline{\mu' u_i} S_{ij} - 2\overline{\mu' u_i s_{ij}} \right] - 2\bar{\mu} \overline{s_{ij}^2} - 2\overline{\mu' s_{ij}^2} - 2\overline{\mu' s_{ij}} S_{ij} - \rho \overline{u_i u_j} S_{ij} \quad (7)$$

Equations (6) and (7) include various non-Newtonian terms, but even some of the Newtonian terms are presented in an unusual way. The reader can recover the classical equation found in many papers and textbooks bearing in mind properties due to the symmetry of the rate of deformation tensor S_{ij} .

3.1. The rate of dissipation of turbulent kinetic energy

For Newtonian fluids the average rate of dissipation (ϵ) of turbulent kinetic energy (k), per unit of mass, is defined as

$$\rho\epsilon_n \equiv 2\mu \overline{s_{ij}^2} \quad (8)$$

where the subscript n is used here to distinguish it from the rate of dissipation of k for GNF fluids. Eq. (8) is the time-average of the instantaneous rate of dissipation defined by

$$\rho\hat{\epsilon}_n \equiv 2\hat{\mu}s_{ij}^2 \quad (9)$$

By analogy, for GNF fluids one can define an instantaneous rate of dissipation by using the instantaneous viscosity:

$$\rho\hat{\epsilon} \equiv 2\hat{\mu}s_{ij}^2 \quad (10)$$

Time-averaging Eq. (10) provides the average rate of dissipation for the GNF fluid

$$\rho\epsilon = 2\overline{\hat{\mu}s_{ij}^2} = 2\overline{(\bar{\mu} + \mu')s_{ij}^2} = 2\bar{\mu}\overline{s_{ij}^2} + 2\overline{\mu' s_{ij}^2} \quad (11)$$

These definitions of instantaneous and average rates of dissipation are identical to those used by Politis (1989) in his derivation of a $k - \epsilon$ turbulence model for purely viscous shear-thinning fluids.

The definition of the average rate of dissipation of turbulent kinetic energy also deserves a comment in light of the literature working with the FENE-P or similar models. Several works reviewed in Section 1 reported the existence of a deficit in Reynolds shear stress and the consequent existence of a polymer stress, here called τ_{ij}^P . The fluctuations of τ_{ij}^P contribute to increase or decrease turbulence via the term $-\overline{\tau_{ij}^P s_{ij}}$ which can take positive or negative values, respectively (Warholic et al, 1999). In a viscoelastic formulation the transport equations of k and $\overline{u_i u_j}$ also contain a term for the interaction between the elastic stress and the rate of strain as shown in the DNS investigation for a FENE-P fluid of Dimitropoulos et al (2001) (their ϵ_{ij}^v term). Their results do confirm that ϵ_{ij}^v (in combination to the interaction between τ_{ij}^P and u_i) acts as a turbulence production term near the wall and as a dissipation term elsewhere.

By defining the total stress tensor as $\tau_{ij} = 2\bar{\mu}S_{ij} - 2\overline{\mu' s_{ij}} - \rho\overline{u_i u_j}$, and the rate of dissipation as in Eq. (11), the two dissipative terms in Eq. (7) already include such interaction between fluctuating stress and fluctuating shear rate. ϵ is defined as a single quantity but still it is the sum of a positive definite term ($2\bar{\mu}\overline{s_{ij}^2}$) with a term that can be either a source or a sink of dissipation ($2\overline{\mu' s_{ij}^2}$). So, in a limiting situation of a 1-D shear flow, where the Reynolds shear stresses are found to be negligible (due to the presence of additives according to experimental results of Warholic et al, 1999), under the classical equilibrium condition the definition of Eq. (11) results in negligible production of turbulence and negligible dissipation. However, for the flow to be turbulent there must be some other mechanism of turbulence production, as well as of turbulence dissipation; what is happening in such situation may well be that $-2\overline{\mu' s_{ij}^2} = 2\bar{\mu}\overline{s_{ij}^2}$. However, note that recently Ptasiniski et al (2001) showed that in pipe flow the Reynolds stress stays definitely non-zero even at maximum drag reduction.

4. The time-average molecular viscosity

In turbulent flow, the molecular viscosity of a variable-viscosity fluid depends on the fluctuating rates of shear and strain, a major difference relative to a Newtonian fluid. Hence, the contribution of this work is the relationship between the time-average molecular viscosity and turbulent quantities to ensure closure of the set of equations. In order to arrive at such relationship it is now necessary to adopt the specific form for the viscosity function of Eq. (4).

In terms of instantaneous values the viscosity is given by

$$\hat{\mu} = K_v \left[\hat{\gamma}^2 \right]^{\frac{n-1}{2}} K_e \left[\hat{\epsilon}^2 \right]^{\frac{p-1}{2}} \quad (12)$$

with $\hat{\gamma}$ and $\hat{\epsilon}$ following from Eq. (3).

The maximum value of $\hat{\epsilon}$ was estimated by Oliveira and Pinho (1998) to be

$$\hat{\epsilon}_{\max} = \sqrt{\frac{2}{3}} \hat{S}_{ij} \hat{S}_{ij} \quad (13)$$

For a high Reynolds number turbulent flow Tennekes and Lumley (1972) have shown that $\hat{S}_{ij} \hat{S}_{ij} \cong s_{ij} s_{ij}$, thus

$$\hat{\epsilon}_{\max} = \sqrt{\frac{2}{3}} s_{ij} s_{ij} \quad (14)$$

Typical values of $\hat{\epsilon}$ being smaller, let us assume that in general

$$\hat{\epsilon} = \frac{\sqrt{s_{ij} s_{ij}}}{A_\epsilon} \quad (15)$$

where the value of A_ϵ is to be found from experimental data but must be higher than $\sqrt{3/2}$.

Back-substituting these definitions into the viscosity model and simplifying gives

$$\hat{\mu} = \frac{K_v K_e}{A_\epsilon^{p-1}} 2^{\frac{n-1}{2}} \left[s_{ij}^2 \right]^{\frac{n+p-2}{2}} \quad (16)$$

This expression can now be used to calculate the instantaneous rate of dissipation in Eq. (10)

$$\rho \hat{\epsilon} = 2^{\frac{n+1}{2}} \frac{K_v K_e}{A_\epsilon^{p-1}} \left[s_{ij}^2 \right]^{\frac{n+p}{2}} \quad (17)$$

Equations (16) and (17) are combined to eliminate s_{ij}^2 and yielding a relationship between the instantaneous viscosity and rate of dissipation

$$\hat{\mu} = \left[\frac{K_v K_e}{A_\epsilon^{p-1}} \right]^{\frac{2}{n+p}} 2^{\frac{1-p}{n+p}} (\rho \hat{\epsilon})^{\frac{n+p-2}{n+p}} \quad (18)$$

Introducing parameters

$$m \equiv \frac{n+p-2}{n+p} \quad \text{and} \quad B \equiv \left[\frac{K_v K_e}{A_\epsilon^{p-1}} \right]^{1-m} 2^{\frac{(n-1)-m(n+1)}{2}} \rho^m \quad (19-a,b)$$

for compactness, Eq. (18) assumes the simple form

$$\hat{\mu} = B \hat{\epsilon}^m \quad (20)$$

The average viscosity and the average rate of dissipation are determined using their probability distribution functions. By definition,

$$\bar{\mu} = \int_0^\infty B \hat{\epsilon}^m P(\hat{\epsilon}) d\hat{\epsilon} \quad (21)$$

Since the instantaneous viscosity is always a positive quantity, the instantaneous rate of dissipation ($\hat{\varepsilon}$) is positive-definite and is associated with small scale motion, assumed here to be locally isotropic at high Reynolds number flows. Thus, as explained by Monin and Yaglom (1975), $\hat{\varepsilon}$ follows a log-normal distribution

$$P(\hat{\varepsilon}) = \frac{1}{\hat{\varepsilon} \sigma \sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\ln \hat{\varepsilon} - M}{\sigma}\right)^2\right\} \quad (22)$$

with M and σ standing for the mean and standard deviation of $\hat{\varepsilon}$. Still, following Monin and Yaglom (1975) (pages 614-615),

$$\int_0^{\infty} \frac{\hat{\varepsilon}^m}{\hat{\varepsilon} \sigma \sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\ln \hat{\varepsilon} - M}{\sigma}\right)^2\right\} d\hat{\varepsilon} = \exp\left(mM + \frac{m^2 \sigma^2}{2}\right) \quad (23)$$

so, the average viscosity comes out as

$$\bar{\mu} = B \exp\left(mM + \frac{m^2 \sigma^2}{2}\right) \quad (24)$$

The average rate of dissipation is also obtained from the probability distribution function of $\hat{\varepsilon}$

$$\varepsilon = \int_0^{\infty} \hat{\varepsilon} P(\hat{\varepsilon}) d\hat{\varepsilon} = \int_0^{\infty} \hat{\varepsilon} \frac{1}{\hat{\varepsilon} \sigma \sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\ln \hat{\varepsilon} - M}{\sigma}\right)^2\right\} d\hat{\varepsilon} = \exp\left(M + \frac{\sigma^2}{2}\right) \quad (25)$$

Now, $\bar{\mu}$ and ε can be related to each other by solving Eq. (24) to get e^M and substituting it back into Eq. (25). The result is

$$\bar{\mu} = B \varepsilon^m e^{\frac{m\sigma^2(m-1)}{2}} \quad (26)$$

with m and B given by Eqs. (19-a) and b).

In Eq. (26), σ^2 is the variance of the distribution of $\ln \hat{\varepsilon}$ which is given by

$$\sigma^2 = A_1 + A_2 \ln\left(\frac{L}{\eta}\right) \quad (27)$$

with L representing an external turbulence length scale, such as a large eddy scale, and η an appropriately defined internal length scale such as the Kolmogorov microscale (Monin and Yaglom, 1975). A_1 is a parameter depending on the turbulence macrostructure and A_2 is a universal constant. The extensive discussion of these quantities, in pp. 612 to 626 of Monin and Yaglom (1975), suggests that $A_2 \approx 0.3$ to 0.5 , although the most appropriate value seems to be between 0.4 and 0.5 , thus 0.45 is considered. A_1 depends on the form of the space regions and there is very little information concerning it (page 634 in Monin and Yaglom, 1975), so it is assumed to be zero. Finally, σ^2 is given by

$$\sigma^2 = A_2 \ln\left(\frac{L}{\eta}\right) \quad (28)$$

The Kolmogorov scale is

$$\eta \equiv \left[\frac{\bar{v}^3}{\varepsilon}\right]^{1/4} = \left[\frac{\bar{\mu}^3}{\rho^3 \varepsilon}\right]^{1/4} \quad (29)$$

where the average molecular viscosity was used. The large length scale for the energy containing eddies is given by the inviscid estimate proportional to $k^{1.5}/\varepsilon$. A commonly used equation for L derived from inviscid theory arguments is adopted here as given by Younis (1996)

$$L \approx \frac{2C_{\mu}^{0.75}k^{1.5}}{\varepsilon} \quad (30)$$

where C_{μ} is a universal constant that usually assumes the numerical value of 0.09 for Newtonian fluids.

Finally, combining Eqs. (26) to (30) provides the final form of an explicit expression for $\bar{\mu}$

$$\bar{\mu} = \left(C_{\mu} \rho \right)^{\frac{3m(m-1)A_2}{8+3m(m-1)A_2}} 2^{\frac{4m(m-1)A_2}{8+3m(m-1)A_2}} k^{\frac{6m(m-1)A_2}{8+3m(m-1)A_2}} \varepsilon^{\frac{[8-3(m-1)A_2]m}{8+3m(m-1)A_2}} B^{\frac{8}{8+3m(m-1)A_2}} \quad (31)$$

with m and B defined above. In the limiting case of a Newtonian fluid ($n = p = 1$) a constant average molecular viscosity of $\bar{\mu} = K_v K_e$ is recovered.

With this relationship the full closure of the set of transport equations can be ensured provided the new non-Newtonian terms of the conservation equations are adequately modeled. This is beyond the scope of this work, but a quick overview is necessary to reassure the reader. The various transport equations are modified in this way:

- i) having neglected term $2\bar{\mu}'s_{ij}$ in the momentum equation, the other modified term is now known;
- ii) in the perspective of a $k - \varepsilon$ model, the k equation is simplified as follows: term $2\bar{\mu}'s_{ij}S_{ij}$ is neglected for obvious reasons, all turbulent diffusion terms are modeled as usual for Newtonian fluids and the term $2\bar{\mu}'u_i S_{ij}$ is also neglected as explained in Oliveira and Pinho (1998);
- iii) the dissipation equation is even more complex than for Newtonian fluids, but is modeled following the same philosophy, i.e., using physical and dimensional arguments as well as arguments based on single relevant large turbulent scales. Still, there are two new terms in the dissipation equation that require modeling. For fully-developed duct flow one of the terms vanishes due to symmetry and the other term requires modeling, as shown in detail in Pinho (2002). Still, its relevance is small and it can also be dropped as a first approximation.

5. Conclusions

A coupled turbulence- rheological model was developed based on the Generalised Newtonian fluid model modified to account for extensional viscosity effects. The conservation equations of momentum, Reynolds stress and turbulence kinetic energy were derived and some new terms appeared as a consequence of the adopted constitutive equation. Of those, the most important depend on the time-average molecular viscosity which is affected by the non-linear dependence on the fluctuating kinematics. In this work, an expression is successfully derived for this average molecular viscosity (Eq. 31) that depends only on rheological parameters and accounts for the effects of turbulence via ε and k .

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7. References

- Achia, B.U. and Thompson, D.W., 1974, "Structure of the turbulent boundary layer in drag reducing pipe flow" J Fluid Mech, vol. 81, N 3, pp. 439-464.
- Bird, R.B., Curtiss, C.F., Armstrong, R.C. and Hassager, O., 1987, "Dynamics of Polymeric Liquids- Vol. 2: Kinetic Theory." 2nd edition, John Wiley & Sons.
- Cruz, D.O.A., Maneschy, C.E., Macedo, E.N. and Quaresma, J.N.N., 2000, "A turbulence model for computing the flow of power law fluids within circular tubes" Hybrid Methods in Eng., vol. 2, pp. 1-13.
- Cruz, D.O.A., Maneschy, C.E. and Pinho, F.T., 2003, "A low Reynolds $k - \varepsilon$ model for viscoelastic fluids" Paper 285, 17th International Congress of Mechanical Engineering, COBEM 2003, São Paulo, Brasil.
- De Angelis, E., Casciola, C.M. and Piva, R., 2002, "DNS of wall turbulence: dilute polymers and self-sustaining mechanisms" Computers & Fluids, vol. 31, pp. 495-507.
- Dimitropoulos, C.D., Sureshkumar, R. and Beris, A.N., 1998, "Direct numerical simulation of viscoelastic turbulent channel flow exhibiting drag reduction: effect of the variation of rheological parameters" J Non-Newt Fluid Mech, vol. 79, pp. 433-468.
- Dimitropoulos, C.D., Sureshkumar, R., Beris, A.N. and Handler, R.A., 2001, "Budgets of Reynolds stress, kinetic energy and streamwise enstrophy in viscoelastic turbulent channel flow" Phys of Fluids, vol. 13, N 4, pp. 1016-1027.
- Durst, F. and Rastogi, A.K., 1977, "Calculations of turbulent boundary layer flows with drag reducing polymer additives" Phys of Fluids, vol. 20, pp. 1975-1985.

- Escudier, M.P., Presti, F. and Smith, S., 1999, "Drag reduction in the turbulent pipe flow of polymers" *J Non-Newt Fluid Mech*, vol. 81, pp. 197-213.
- Hassid, S. and Poreh, M., 1975, "A turbulent energy model for flows with drag reduction" *Trans ASME J. Fluids Eng.*, vol. 97, pp. 234-241.
- Hassid, S. and Poreh, M., 1978, "A turbulent energy dissipation model for flows with drag reduction" *J Fluids Eng*, vol. 100, pp. 107-112.
- Hoyt, J.W., 1972, "The effect of additives on fluid friction" *J Basic Eng.*, vol. 94, pp. 258-285.
- Jones, W.P. and Launder, B.E., 1973, "The calculation of a low-Reynolds-number phenomena with a two-equation model of turbulence" *Int. J. Heat and Mass Transf.*, vol. 16, pp. 1119-1130.
- Luchik, T.S. and Tiederman, W.G., 1987, "Time scale and structure of ejections and bursts in turbulent channel flows" *J Fluid Mech*, vol. 174, pp. 529-552
- Luchik, T.S. and Tiederman, W.G., 1988, "Turbulent structure in low-concentration drag-reducing channel flows" *J Fluid Mech*, vol. 190, pp. 241-263.
- Mizushima, T., Usui, H. and Yoshida, T., 1973, "Turbulent pipe flow of dilute polymer solutions" *J. Chem. Eng. Japan*, vol. 7, pp. 162-167.
- Monin, A.S. and Yaglom, A.M., 1975, "Statistical Fluid Mechanics"- vol. 2 MIT Press, Cambridge, US
- Oliveira, P.J. and Pinho, F.T., 1998, "A qualitative assessment of the role of a third invariant of the rate of deformation tensor dependent viscosity on turbulent flow" *J Non-Newt Fluid Mech*, vol. 78, pp. 1-25.
- Pereira, A.S. and Pinho, F.T., 1994, "Turbulent pipe flow characteristics of low molecular weight polymer solutions" *J Non-Newt Fluid Mech*, vol. 55, pp. 321-344.
- Pinho, F.T. and Whitelaw, J.W., 1990, "Flow of non-Newtonian fluids in a pipe" *J Non-Newt Fluid Mech*, vol. 34, pp. 129-144.
- Pinho, F. T. 2002. "A GNF framework for turbulent flow models of drag reducing fluids and proposal for a $k - \epsilon$ type closure. Paper submitted to *Journal of Non-Newt. Fluid Mechanics*.
- Politis, S., 1989, "Turbulence modelling on inelastic power-law fluids" Internal report of project Brite Project RIIB.0085.UK (H), Department of Mechanical Engineering, Imperial College of Science, Technology and Medicine, London, UK
- Poreh, M. and Hassid, S., 1977, "Mean velocity and turbulent energy closures for flows with drag reduction" *Phys Fluids*, vol. 20, N 10, pp. S193-S196.
- Ptasinski, P.K., Nieuwstadt, F.T.M., Van Den Brule, B.H.A.A. and Hulsen, M.A., 2001, "Experiments in turbulent pipe flow with polymer additives at maximum drag reduction" *Flow, Turbulence and Combustion*, vol. 66, pp. 159-182.
- Reischman, M.M. and Tiederman, W.G., 1975, "Laser-Doppler anemometer measurements in drag-reducing channel flows" *J Fluid Mech*, vol. 70, N 2, pp. 369-392.
- Sureshkumar, R, Beris, A.N. and Handler, R.A., 1997, "Direct numerical simulation of the turbulent channel flow of a polymer solution" *Phys Fluids*, vol. 9, N 3, pp. 743-755.
- Tennekes, H. and Lumley, J.L., 1972, "A first course in turbulence". The MIT Press, Cambridge, US.
- Toms, B. A., 1948, "Some observations on the flow of linear polymer solutions through straight tubes at large Reynolds numbers" *Proc. 1st International Congress on Rheology*, vol. 2, pp. 135-141, North Holland, Amsterdam
- Warholic, M.D., Massah, H. and Hanratty, T.J., 1999, "Influence of drag reducing polymers on turbulence: effects of reynolds number, concentration and mixing" *Exp. in Fluids*, vol. 27, pp. 461-472.
- Virk, P.S., 1975, "Drag reduction fundamentals" *AIChEJ*, vol. 21, n 4, pp. 625-656.
- Younis, B.A., 1996, "Progress in turbulence modelling for open-channel flows." Chapter 9 in *Floodplain processes* edited by MG Anderson, DE Wallace and PD Bates, John Wiley & Sons, pp. 299-332.

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