NUMERICAL MODELLING OF 2D ELASTODYNAMIC PROBLEMS USING BOUNDARY ELEMENTS AND THE OPERATIONAL QUADRATURE METHOD

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Abstract. *The Boundary Element Method (BEM) has been widely used in the solution of 2D elastodynamic problems in homogeneous and linear elastic domains. The analysis can be carried out using either the fundamental solution in the time domain or transformed-domain formulation wich solve the problem in a transformed frequency or Laplace domain. The latter starts from the solution in the transformed-domain and gets the time domain results applying numerical inverse transform algorithms. An elegant way of solving problems of this type is using the Operational Quadrature Method (OQM), where the convolution integral, present in time-domain BEM formulations, is substituted by a quadrature formula, whose weights are computed using the Laplace transform of the fundamental solution and a multistep method. The final solution to the problem however is obtained in the time domain. Examples of application are presented here to illustrate the validity of this OQM-BEM formulation.*

Keywords. *Boundary Element Method, Elastodynamic 2D, Operational Quadrature Method.*

1. Introduction

The subject of structural dynamics is a field where new alternatives are presented continuously. The boundary element method (BEM), (Brebbia, Telles and Wrobel, 1984) is in constant evolution and has been successfully applied to solve several problems in science and engineering. The technique transforms the differential equations that govern the problem in boundary integral equations upon the introduction of fundamental solutions, corresponding to discrete singularities, satisfying the original differential operator.

In the study of 2D elastodynamic problems, two formulations that are of great use: the first is a time-domain boundary element method (TD-BEM) (Mansur, 1983) that presents good accuracy and good representation of the causality condition, the second is a frequency or Laplace domain boundary element method that uses the fundamental solution in the Fourrier or Laplace domains (Dominguez, 1993)

The present work discusses the time-domain formulation, but the analysis is carried out with the Laplace fundamental solution. This approximation is based on the operational quadrature method (OQM), (Lubich, 1988), in which the convolution integral is substituted by a quadrature formula, whose weights are computed using the Laplace transform of the fundamental solution and a linear multistep method. The final solution of the problem is obtained in the time domain.

The TD-BEM and the OQM have been recently applied to the analysis of scalar wave propagation problems (Abreu et all, 2002) and in the simulation of cracks or discontinuities, in such problems, using a numerical Green's function (NGF) (Telles et all, 2002; Vera-Tudela et all, 2002) procedure. In the latter, a new formulation involving the Time-domain boundary element method, the numerical Green's function and the operational quadrature procedure are used to solve linear scalar wave propagation problems in presence of cracks. The present paper extends the formulation to linear elastic general wave propagation problems. Examples are presented in the end of the text with the aim of validating the formulation.

2. The Operational Quadrature Method

The Operational Quadrature Method was developed by Lubich, 1988 and the basic steps are outlined below, an extension to elastodynamic 2D will be discussed in the next section. The convolution integral is:

$$y(t) = \int_0^{t^+} u^*(t-\tau)\, f(\tau)\, d\tau = u^*(t) * f(t) \tag{1}$$

where $u^*(t)$ is the fundamental solution in the time-domain Boundary Element Method. Equation (1), in discretized form, reads:

$$y(n\,\Delta t) = \sum_{k=0}^{n} \omega_{n-k}(\Delta t)\, f(k\,\Delta t) \qquad\qquad n = 0,1,\ldots,N \tag{2}$$

In Equation (2), N is the total number of time steps and ω_n represents the quadrature weights (or integration weights) that constitute the coefficients of the power series that approximate the Laplace transform $\hat{u}^*(s)$ of the fundamental solution $u^*(t)$, that is:

$$\hat{u}^*(s) = \hat{u}^*\left(\frac{\gamma(z)}{\Delta t}\right) = \sum_{n=0}^{\infty} \omega_n(\Delta t)\, z^n \tag{3}$$

where $s = \gamma(z)/\Delta t$ and *z* is a complex variable. It is also required that $\left|\hat{u}^*(s)\right| \to 0$ when $\left|s\right| \to \infty$ for $\Re(s) \geq q$ (*q* is a real positive number).

The coefficients of the series in Eq. (3) are furnished by the Cauchy integral formula given below:

$$\omega_n(\Delta t) = \frac{1}{2\pi i} \int_{|z|=\rho} \hat{u}^*\left(\frac{\gamma(z)}{\Delta t}\right) z^{-n-1} dz = \frac{\rho^{-n}}{L} \sum_{l=0}^{L-1} \hat{u}^*\left(\frac{\gamma\left(\rho\, e^{il2\pi/L}\right)}{\Delta t}\right) e^{-inl2\pi/L} \tag{4}$$

where ρ is the radius of a circle in the domain of analyticity of the function $\hat{u}^*(s)$.
If a polar coordinate system is adopted, the integral presented in Eq. (4) is approximated by means a trapezoidal rule with *L* equal steps $(2\pi/L)$.

The function $\gamma(z)$, previously utilized in Eqs. (3) and (4), is the quotient of the polynomials generated by a linear multistep method. If this method is employed for approximating a certain function, say $x(t)$, which, by its turn, is the solution of the first order differential equation $dx(t)/dt = sx(t) + f(t)$ (with $x(0) = 0$), one has:

$$x(t) \approx \sum_{j=0}^{k} \alpha_j\, x_{n-j} = \Delta t \sum_{j=0}^{k} \beta_j (s\, x_{n-j} + f((n-j)\,\Delta t)) \tag{5}$$

and then:

$$\gamma(z) = \frac{\alpha_0 + \ldots + \alpha_k\, z^k}{\beta_0 + \ldots + \beta_k\, z^k} \tag{6}$$

The function $\gamma(z)$ clearly characterizes the multistep method and must be $A(\alpha)-$stable with a positive angle α, stable in a neighborhood of infinity, strongly zero-stable, and consistent of order *p*. If an error δ is assumed in the computation of $\hat{u}^*(s)$ in Eq. (4) , the choice $L = N$ and $\rho^N = \sqrt{\delta}$ leads to an error in ω_n of order $O\left(\sqrt{\delta}\right)$, Lubich, 1994.

3. The Boundary Element Method

This section shows the formulation of the time-domain Boundary Element Method (Mansur, 1983). It is based on Navier's Equation as outlined below:

$$G u_{j,kk} + (\lambda + G) u_{k,kj} = \rho\, \ddot{u}_j \tag{7}$$

where G and λ are the Lamé's constants. Equation (7) is referred to as the displacement equation of motion and constitute a linear system of hyperbolic differential equations for the dependent variable u_i.

The Boundary Element Method consists in the transformation of the partial differential equation (Eq. (7)) describing the behavior of the unknown inside and on the boundary of the domain into an integral equation relating only boundary values, and then finding the numerical solution to this equation. For this purpose, a Green's Function (fundamental solution) for infinite domain together with a weighted residual statement is employed. Thus, the boundary integral equation can be written in the form

$$4\pi C_{ij}(\xi)u_j(\xi,t)=\int_0^{t^+}\int_{\Gamma}u_{ij}^*(x,t;\xi,\tau)\,p_j(x,\tau)\;d\,\Gamma(x)\,d\,\tau-\int_0^{t^+}\int_{\Gamma}p_{ij}^*(x,t;\xi,\tau)u_j(x,\tau)\;d\,\Gamma(x)\,d\,\tau \tag{8}$$

Where u_{ij}^* and p_{ij}^* are the fundamental solution components for displacements and tractions and the coefficient C_{ij} is similar to the static case.

If the procedure described in Eq. (2) for the Operational Quadrature Method is applied to each integral of Eq. (8), the result is as follows:

$$\int_0^{t^+}\int_{\Gamma}u_{ij}^*(x,t;\xi,\tau)\,p_j(x,\tau)\;d\,\Gamma(x)\,d\,\tau=\sum_{k=0}^{n}{}^{n-k}g_{ij}^e(x,\xi,\Delta t)\;{}^{k}p_j^e(x)\qquad n=0,1,\ldots,N \tag{9}$$

and

$$\int_0^{t^+}\int_{\Gamma}p_{ij}^*(x,t;\xi,\tau)u_j(x,\tau)\;d\,\Gamma(x)\,d\,\tau=\sum_{k=0}^{n}{}^{n-k}h_{ij}^e(x,\xi,\Delta t)\;{}^{k}u_j^e(x)\qquad n=0,1,\ldots,N \tag{10}$$

The procedure of Eq. (4) applied to Eqs. (9) and (10) yields the weights ***g*** and ***h*** for the BEM formulation as given below:

$${}^{n}g_{ij}^e(x,\xi,\Delta t)=\frac{\rho^{-n}}{L}\sum_{l=0}^{L-1}\int_{\Gamma^e}\hat{u}_{ij}^*\left(x,\xi,\frac{\gamma\left(\rho\,e^{il2\pi/L}\right)}{\Delta t}\right)\Phi^e(x)\;d\,\Gamma(x)\;e^{-inl2\pi/L} \tag{11}$$

and

$${}^{n}h_{ij}^e(x,\xi,\Delta t)=\frac{\rho^{-n}}{L}\sum_{l=0}^{L-1}\int_{\Gamma^e}\hat{p}_{ij}^*\left(x,\xi,\frac{\gamma\left(\rho\,e^{il2\pi/L}\right)}{\Delta t}\right)\Phi^e(x)\;d\,\Gamma(x)\;e^{-inl2\pi/L} \tag{12}$$

where $\Phi^e(x)$ represent the quadratic interpolation functions utilized in each boundary element Γ^e.

The fundamental solution for elastodynamic 2D problems in the Laplace domain (Barra, 1996) is given by:

$$\hat{u}_{ij}^*(x,\xi,s)=\frac{1}{\rho c_s^2}\left[\varphi(r)\delta_{ij}-\chi(r)r,_i r,_j\right] \tag{13}$$

and the fundamental traction:

$$\begin{aligned}\hat{p}_{ij}^*(x,\xi,s)=\Bigg\{&\left[\frac{d\varphi(r)}{dr}-\frac{\chi(r)}{r}\right]\left(\delta_{ij}\frac{\partial r}{\partial n}+r,_j n_i\right)-2\frac{\chi(r)}{r}\left(n_j r,_i-2r,_i r,_j\frac{\partial r}{\partial n}\right)\\&-2\frac{d\chi(r)}{dr}r,_i r,_j\frac{\partial r}{\partial n}+\left(\frac{c_p^2}{c_s^2}-2\right)\left(\frac{d\varphi(r)}{dr}-\frac{d\chi(r)}{dr}-\frac{\chi(r)}{r}\right)r,_i n,_j\Bigg\}\end{aligned} \tag{14}$$

where $\chi(r)$ and $\varphi(r)$ are defined as:

$$\chi(r) = \kappa_2\left(\frac{sr}{c_s}\right) - \frac{c_s^2}{c_p^2}\kappa_2\left(\frac{sr}{c_p}\right) \tag{15}$$

and

$$\varphi(r) = \kappa_0\left(\frac{sr}{c_s}\right) + \left(\frac{sr}{c_s}\right)^{-1}\left[\kappa_1\left(\frac{sr}{c_s}\right) - \frac{c_s}{c_p}\kappa_1\left(\frac{sr}{c_p}\right)\right] \tag{16}$$

In which *r* is the distance between ξ and *x*; c_p and c_s are, respectively, the velocities of the pressure and shear waves of the medium; κ_j the modified Bessel functions of second type and order *j* and δ_{ij} is the Kroenecker delta symbol.

Eq. (8) can now be written in discretized form:

$$4\pi\, C_{ij}(\xi)\, u_j(\xi, t_n) = \sum_{e=1}^{E}\sum_{k=0}^{n} {}^{n-k}g_{ij}^e(x,\xi,\Delta t)\; {}^{k}p_j^e(x) - \sum_{e=1}^{E}\sum_{k=0}^{n} {}^{n-k}h_{ij}^e(x,\xi,\Delta t)\; {}^{k}u_j^e(x) \tag{17}$$

Equation (17) is applied to all the nodes of the boundary elements and the following system of equations is obtained:

$$\boldsymbol{C}\,\boldsymbol{u}^{\boldsymbol{n}} = \sum_{\boldsymbol{k}=0}^{\boldsymbol{n}} \boldsymbol{G}^{\boldsymbol{n-k}}\,\boldsymbol{p}^{\boldsymbol{k}} - \sum_{\boldsymbol{k}=0}^{\boldsymbol{n}} \boldsymbol{H}^{\boldsymbol{n-k}}\,\boldsymbol{u}^{\boldsymbol{k}} \tag{18}$$

Here, $\boldsymbol{C}$ is a diagonal matrix that contains the coefficients $C_{ij}(\xi)$; *n* and *k* correspond to the discrete times $t_n = n\,\Delta t$ and $t_k = k\,\Delta t$, respectively.

The matrix equations are solved in stepwise manner, producing the unknown displacements and tractions at the end of each time step. For the first step, Eq. (18) can be written in the form:

$$(\boldsymbol{C} + \boldsymbol{H}^0)\,\boldsymbol{u}^1 = \boldsymbol{G}^0\,\boldsymbol{p}^1 + (\boldsymbol{G}^1\,\boldsymbol{p}^0 - \boldsymbol{H}^1\,\boldsymbol{u}^0) \tag{19}$$

The columns of matrices $\boldsymbol{H}$ and $\boldsymbol{G}$ are reordered according to the boundary conditions, giving:

$$\boldsymbol{A}^0\,\boldsymbol{y}^1 = \boldsymbol{f}^1 + \boldsymbol{f}^0 \tag{20}$$

where $\boldsymbol{f}^1$ is formed by the boundary contribution at $t = \Delta t$ and:

$$\boldsymbol{f}^0 = \boldsymbol{G}^1\,\boldsymbol{p}^0 - \boldsymbol{H}^1\,\boldsymbol{u}^0 \tag{21}$$

For $t_n = n\,\Delta t$, one has the following general expression:

$$\boldsymbol{A}^0\,\boldsymbol{y}^{\boldsymbol{n}} = \boldsymbol{f}^{\boldsymbol{n}} + \sum_{\boldsymbol{k}=0}^{\boldsymbol{n}-1} \boldsymbol{f}^{\boldsymbol{k}} \tag{22}$$

and

$$\boldsymbol{f}^{\boldsymbol{k}} = \boldsymbol{G}^{\boldsymbol{n-k}}\,\boldsymbol{p}^{\boldsymbol{k}} - \boldsymbol{H}^{\boldsymbol{n-k}}\,\boldsymbol{u}^{\boldsymbol{k}} \tag{23}$$

The solution process becomes slower at later times because vector $\boldsymbol{f}^{\boldsymbol{k}}$ depends on all the matrices from the previous steps.

4. Examples

As an assessment of accuracy of the proposed procedure, a simple example of longitudinal wave propagation has been worked out. It is a two dimensional rod clamped at one end and free on the other, subjected to a suddenly applied axial tension of a constant value in time as shown in Fig. (1).

The material constants are: Young modulus $E= 2,4$, Poisson modulus $\nu=0,2$, density $\rho=1$. The OQM-BEM model has been discretized with 36 quadratic boundary elements. Double nodes were utilized at the corners.

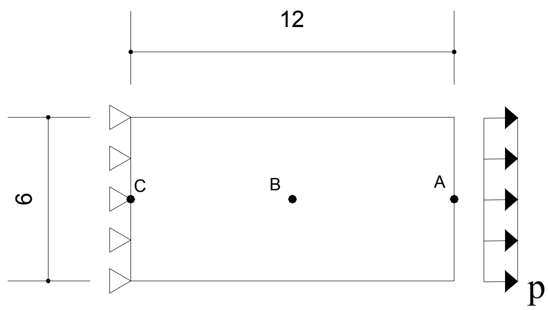

Figure 1. Rectangular bar subjected to Heaviside loading: geometry definitions and boundary conditions.

The results are compared with a Dual Reciprocity Boundary Element formulation (DR-BEM) (Vera-Tudela, 1999) discretized with 12 quadratic boundary elements and 1 internal point. Analytical values are also included for a complete comparison. Analytical, OQM-BEM and DR-BEM results, for the axial displacement at boundary node A, are depicted in Fig. (2).

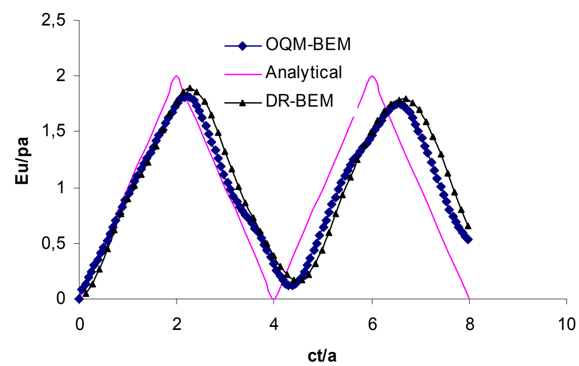

Figure 2. Normalized longitudinal displacement at point A.

The traction results, at boundary node C, are presented in Fig (3) and compared with the analytical values.

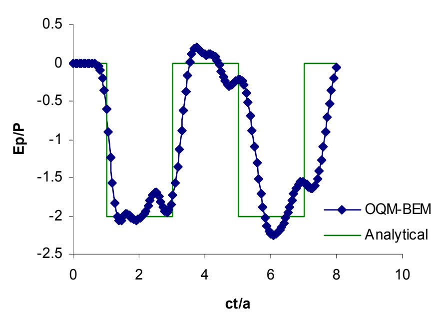

Figure 3. Normalized tractions at point C.

In the same way, analytical, OQM-BEM and DR-BEM results for axial displacement at internal point B, in the middle of the rod are depicted in Fig (4).

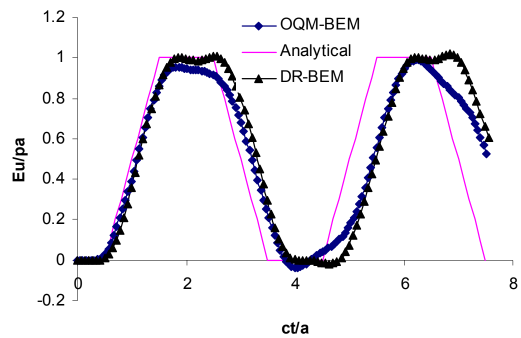

Figure 4. Results for internal point B (6,3)

5. Conclusions

A procedure for elastodynamic analysis using the Boundary Element Method with the Operational Quadrature Method has been proposed. It has been shown that the formulation produces accurate results for transient problems. It is worth mentioning, however, that the OQM implementation discussed here was found to be quite demanding in terms of computer resources, requiring considerably more CPU run time than the equivalent standard Laplace inverse transform of Barra and Telles, 1999.

6. Acknowledgement

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