Abstract

Multiaxial fatigue damage occurs when the principal stress directions vary during the loading induced by several independent forces, such as out-of-phase bending and torsion. Uniaxial damage models cannot be reliably applied in this case. Besides the need for multiaxial damage models, another key issue to reliably model such problems is how to calculate the elastic-plastic stresses from the multiaxial strains. Hooke’s law cannot be used to correlate stresses and strains for short lives due to plasticity effects. Ramberg-Osgood cannot be used either to directly correlate principal stresses and strains under multiaxial loading, because this model has been developed for the uniaxial case. The purpose of this work is to critically review and compare the main fatigue crack initiation models under multiaxial loading. The studied models include stress-based ones such as Sines, Findley and Dang Van, and strain-based ones such as the $\gamma N$ curve, Brown-Miller, Fatemi-Socie and Smith-Watson-Topper models. Modified formulations of the strain-based models are presented to incorporate Findley’s idea of using critical planes that maximize damage. To incorporate plasticity effects, four models are studied and compared to correlate stresses and strains under proportional loading: the method of the highest $K_t$, the constant ratio model, Hoffmann-Seeger’s and Dowling’s models.

Keywords: multiaxial fatigue, crack initiation, life prediction models, stress-strain models.

1 Introduction

Real loads can induce combined bending, torsional, axial and shear stresses, which can generate bi- or tri-axial variable stress/strain histories at the critical point (in general a notch root), causing the so-called multiaxial fatigue problems. The load history is said to be proportional when it generates stresses with principal axes which maintain a fixed orientation, while non-proportional loading is associated with principal directions which change in time during the loading history.
For periodic loads with same frequency, one can also define the concept of in-phase and out-of-phase loading. In-phase loading always leads to proportional histories, however the opposite is not true: e.g., the stresses $\sigma_x = \sigma_I$ and $\sigma_y = \sigma_{II}$ induced on a plate by perpendicular (⊥) forces $F_x$ and $F_y$ are always proportional, because the principal axes maintain a fixed direction even if $F_x$ and $F_y$ are out-of-phase.

On the other hand, out-of-phase axial and torsional stresses always generate non-proportional (NP) loading [1]. The non-proportionality factor $F_{np}$ of the applied loads can be obtained from the shape of the ellipse that encloses the history of normal and shear strains induced by them, $\varepsilon$ and $\gamma$. Considering $a$ and $b$ ($b \leq a$) as the semi-axes of the ellipse which encloses the strain path in the Mises diagram $\varepsilon \times \gamma/\sqrt{3}$, then the non-proportionality factor $F_{np}$ is defined as $b/a$ ($0 \leq F_{np} \leq 1$), see Fig. 1. A further discussion on enclosing ellipses, and hyper-ellipsoids, can be found in [2].

All proportional loadings have shear strains $\gamma$ proportional to the normal strains $\varepsilon$, with $F_{np} = 0$ and a straight-line trajectory in the $\varepsilon \times \gamma/\sqrt{3}$ diagram. Any loading history with $F_{np} > 0$ is NP. Note e.g. that the loading $(\sigma_a \sin \omega t + \tau_a \cos \omega t)$ with $\tau_a = \sigma_a/\sqrt{3}(1 + \nu)$, caused by a traction and a properly scaled torsion $90^\circ$ out of phase, has $F_{np} = 1$, therefore the maximum possible non-proportionality.

![Diagram](image)

Figure 1: Diagram $\varepsilon \times \gamma/\sqrt{3}$, and associated non-proportionality factors (Socie and Marquis, 1999).

Predictions with NP histories can be very complex, because they involve at least three potential problems:

1. NP hardening: the cyclic hardening coefficient $H_c$ and the ratio $\Delta \sigma / \Delta \varepsilon$ of a few materials increase under NP loading, which significantly decreases the fatigue life of parts subject to a constant $\Delta \varepsilon$;
2. Damage calculation: the SN and $\varepsilon_N$ curves, measured under proportional loading, cannot be directly used when principal directions vary, because in this case the crack propagation plane in general does not match the one from the tests; and
3. Cycle counting: the traditional rain-flow counting techniques cannot be applied to variable amplitude NP loading, because the peaks and valleys of $\varepsilon$ in general do not match with the ones of $\gamma$, becoming impossible to decide $a priori$ which points should be accounted for.

The first two problems will be addressed in this work. A NP hardening model will be presented, to allow for the correct calculation of the equivalent stresses, and multiaxial models based on stress...
or strain measurements will be used to calculate the damage generated both by proportional and NP loadings.

A few classical models that correlate stresses or strains with multiaxial fatigue life are studied below. Stress-based models (which can be applied for long life predictions) proposed by Sines, Findley and Dang Van are presented, as well as strain-based models proposed by Brown-Miller, Fatemi-Socie and Smith-Watson-Topper (SWT), which must be used for short lives.

One problem with the application of the Fatemi-Socie or SWT models is the need to calculate the elastic-plastic stresses from the multiaxial strains, because Ramberg-Osgood is only valid for uniaxial stresses. Another challenge in multiaxial fatigue life calculations is the modeling of the notch effect. The elastic stress concentration factor $K_{\sigma}$ and strain concentration factor $K_{\varepsilon}$ are the same for uniaxial loading, but in general in the multiaxial case $K_{\sigma}$ is different from $K_{\varepsilon}$ even under elastic stresses.

Therefore, even in the elastic case, it is not trivial to study the notch effect under multiaxial loading. The problem is worse in the elastic-plastic case, where even uniaxial loadings can generate NP multiaxial stress and strain histories, due to the tri-axial stress state at the notch root and to the difference between the elastic and plastic Poisson coefficients. Typically, metallic alloys have $1/4 \leq \nu_{el} \leq 1/3$ and $\nu_{pl} = 0.5$. In the following sections the multiaxial stress-strain models are presented and compared, including notch effects.

2 Non-proportional loading

A few materials under NP cyclic loading can harden much more than it would be predicted from the traditional cyclic $\sigma \varepsilon$ curve. This phenomenon, called NP hardening, depends on the load history (through the NP factor $F_{np}$) and on the material (through a constant $\alpha_{np}$ of NP hardening, where $0 \leq \alpha_{np} \leq 1$). The NP hardening can be modeled in general using the same Ramberg-Osgood plastic exponent $h_{c}$ from the uniaxial cyclic $\sigma \varepsilon$ curve, and using a new coefficient $H_{cnp} = H_{c} \cdot (1 + \alpha_{np} \cdot F_{np})$, where $H_{c}$ is the uniaxial Ramberg-Osgood plastic coefficient, see Fig. 2. Note that the NP hardening can multiply the uniaxial strain hardening coefficient $H_{c}$ by a value as high as 2.

The largest NP hardening occurs when $F_{np} = 1$, e.g. under a properly scaled traction-torsion loading $90^\circ$ out of phase which generates a circle in the $\varepsilon \times \gamma/\sqrt{3}$ Mises diagram.

Typically, the NP hardening effect is high in austenitic stainless steels at room temperature ($\alpha_{np} \cong 1$ in the stainless steel 316), medium in carbon steels ($\alpha_{np} \cong 0.3$ in the 1045 steel) and low in aluminum alloys ($\alpha_{np} \cong 0$ for Al 7075). Note that proportional histories do not lead to NP hardening.

The NP hardening happens in materials with low fault stacking energy (which in austenitic stainless steels is only 23mJ/m$^2$) and well spaced dislocations, where the slip bands generated by proportional loading are always planar. In these materials, the NP loads activate crossed slip bands in several directions (due to the rotation of the maximum shear planes), therefore increasing the hardening effect ($\alpha_{np} \gg 0$) with respect to the proportional loadings. But in materials with high fault stacking energy (such as aluminum alloys, with a typical value of 250mJ/m$^2$) and with close dislocations, the crossed slip bands already happen naturally even under proportional loading, therefore the NP histories do not cause any significant difference in hardening ($\alpha_{np} \cong 0$).

But the Coffin-Manson or the Morrow crack initiation equations cannot account for the influence of
NP hardening. This implies that the use of traditional $\varepsilon N$ equations, which were developed to model uniaxial fatigue problems, can be non-conservative when the loading histories are NP.

However, it must be noted that the NP hardening reduces fatigue life only in strain-controlled problems (such as in $\varepsilon N$ specimen tests or very sharp notches, e.g.), because the stresses $\Delta \sigma$ caused by a given $\Delta \varepsilon$ are higher than in the proportional case. But in stress-controlled problems (the most common case in practice), the $\Delta \varepsilon$ generated by a given $\Delta \sigma$ is lower under NP loading, therefore the fatigue life is higher than in the proportional case (the uniaxial $\varepsilon N$ equations can lead to conservative predictions in this case). In the following sections, the multiaxial models to predict NP damage are studied.

3 Stress-based multiaxial fatigue damage models

It is well known that Tresca or Mises equivalent stresses must be used to predict crack initiation lives, which depend on the cyclic movement of dislocations. However, crack initiation can and should be divided into:

- formation of microcracks, which is almost insensitive to mean stresses and hydrostatic pressure in metals, because it only depends on dislocation movement; followed by
- propagation of the dominant microcrack, which also depends on the crack face opening and the friction between the faces, becoming increasingly sensitive to the applied mean stress $\sigma_m$ as the microcrack grows.

Microcracks are cracks with sizes up to the order of the metal grain sizes. Their modeling using classical fracture mechanics is questionable, as opposed to long cracks (typically larger than 1 or 2mm), which have crack propagation rates controlled by $\Delta K$. 
However, SN and εN tests bring test specimens (TS) to fracture, or to the generation of a small, finite crack, therefore they include both microcrack initiation and propagation phases. Thus, since the shear stress Δτ controls the initiation of a microcrack, while the normal stress σ⊥ perpendicular to its plane (or the hydrostatic stress σh, invariant defined as the mean of the normal stresses) controls its opening, both are important to predict the fatigue lives of SN and εN specimens.

In fact, a component under uniaxial traction σx = σ and another under torsion τxy = σ/2 work under the same Tresca equivalent stress, but the microcracks on the plane of τmax in the first component are subject to a normal stress σ⊥ perpendicular to that plane that tends to keep their mouth open, exposing the crack tips and decreasing the crack face friction, see Fig. 3. Therefore, the fatigue damage generated by Δσ can be greater than the one caused by the pure torsion Δτ = Δσ/2.

The Mises equivalent stress is able to, at least in part, consider such effect, because the component under torsion would have σMises = τxy√3 = 0.866 · σx < σ, however σMises is insensitive to the hydrostatic stress σh. The Mises shear strain τMises, which acts on the octahedric planes, does not consider as well the effects of σh, relating with σMises through:

\[
\sigma_{Mises} = \frac{3}{\sqrt{2}} \tau_{Mises} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2)}
\]

Figure 3: Mohr circles showing why under the same Δσ_{Tresca} equivalent loading the component under pure torsion can have a higher fatigue life than the one under pure traction.
Sines [3] has proposed a fatigue failure criterion under proportional multiaxial stresses, based on $\Delta \tau_{\text{Mises}}$ and on $\sigma_{hm} = (\sigma_{xm} + \sigma_{ym} + \sigma_{zm})/3$, the hydrostatic component of the mean stresses (insensitive to the shear stresses):

$$\frac{\Delta \tau_{\text{Mises}}}{2} + \alpha_S \cdot (3 \cdot \sigma_{hm}) = \beta_S \tag{2}$$

where $\alpha_S$ and $\beta_S$ are adjustable constants for each material, and

$$\Delta \tau_{\text{Mises}} = \frac{1}{3} \sqrt{(\Delta \sigma_1 - \Delta \sigma_2)^2 + (\Delta \sigma_1 - \Delta \sigma_3)^2 + (\Delta \sigma_2 - \Delta \sigma_3)^2} \tag{3}$$

In this way, according to the Sines criterion, a component will have infinite fatigue life under proportional loading if

$$\Delta \tau_{\text{Mises}}/2 + \alpha_S \cdot (3 \cdot \sigma_{hm}) < \beta_S \tag{4}$$

On the other hand, the Findley [4] criterion, which is also applicable to NP multiaxial loadings, assumes that the crack initiates at the critical plane of the critical point. This idea is interesting, because it is on this plane that the damage caused by the combination $\Delta \tau/2 + \alpha_F \cdot \sigma_\perp$ is maximum, where $\Delta \tau/2$ is the shear stress amplitude on that plane and $\sigma_\perp$ is the normal stress perpendicular to it. Thus, according to Findley the fatigue failure criterion at the critical plane of the critical point is

$$\left( \frac{\Delta r}{2} + \alpha_F \cdot \sigma_\perp \right)_{\max} = \beta_F \tag{5}$$

where $\alpha_F$ and $\beta_F$ are constants which must be fitted by measurements in at least two types of fatigue tests, e.g., under rotating bending and under pure torsion, or in push-pull tests under two different $R$ ratios.

The critical plane can vary at each $i$-th event of the NP loads, even when the critical point remains the same, but Findley predicts fatigue failure based on the plane where the sum of the damages associated with $[\Delta \tau_i(\theta)/2 + \alpha_F \cdot \sigma_{\perp,i}(\theta)]$ is maximum, where $\theta$ is the angle of such plane with respect to a reference direction.

Under pure torsion, Eq. (5) can be written as

$$\sqrt{1 + \alpha_F^2} \cdot \frac{\Delta \tau}{2} = \beta_F \tag{6}$$

And under cyclic uniaxial traction with alternate component $\sigma_a$ and maximum component $\sigma_{max}$, it can be shown that Findley’s criterion can be written as

$$0.5 \sigma_a \left[ \sqrt{1 + \left( \frac{2\alpha_F}{1-R} \right)^2} + \frac{2\alpha_F}{1-R} \right] = \beta_F \tag{7}$$

where $R = \sigma_{min}/\sigma_{max}$ is the stress ratio, which quantifies the mean stress effects.

Therefore, from Findley it is possible to estimate the fatigue limit $S_L(R)$ under any ratio $R$ from $\alpha_F$ and the fatigue limit $S_L$ (obtained under zero mean loads, i.e., with $R = -1$, see Fig. 4) through
From the principle that the damage associated with the initiation of fatigue microcracks cannot be detected from macroscopic measurements, Dang Van [5] proposed a model that considers the variable micro stresses that act inside a characteristic volume element (VE) of the material, where the macroscopic stresses and strains are supposedly constant. The VE is the unit used in structural analysis to represent the material properties, such as its Young modulus and its several strengths. Thus the VE must be small compared to the component’s dimensions, but large compared to the parameter that characterizes the intrinsic anisotropy of the material. For instance, a VE of only 1mm$^3$ is sufficient for most structural metal alloys, which have a grain size $g$ typically between 10 and 100µm (the grain itself, being a monocrystal, is intrinsically anisotropic).

Inside a VE, the local micro stresses $[\sigma_{ij}]_\mu = \sigma_\mu$ and strains $[\varepsilon_{ij}]_\mu = \varepsilon_\mu$ acting between grains, or between them and small imperfections such as inclusions, e.g., can significantly differ from the macro stresses $[\sigma_{ij}]_M = \sigma_M$ and strains $[\varepsilon_{ij}]_M = \varepsilon_M$, assumed constant in the macroscopic analysis normally used in mechanical design. Therefore, these micro quantities can significantly influence crack initiation. Note that if the term “microscopic” is reserved to the scale associated with interatomic stresses, domain of solid state physics, then it is recommended to use the term “mesoscopic” to describe the intra or intergranular stresses. Thus, the macroscopic stresses reflect the average of the mesoscopic stresses in a VE: $\sigma_M = \int \sigma_\mu dV/V$, where $V$ is the volume of the VE. Similarly, $\varepsilon_M = \int \varepsilon_\mu dV/V$.

In other words, the macroscopic stresses and strains are assumed constant at the characteristic volume element VE of the material, however the mesoscopic intergranular stresses can vary a lot, influencing crack initiation.
Since the microcracks initiate at persistent slip bands, Dang Van assumed that fatigue damage was caused by the mesoscopic shear strain history $\tau_\mu(t)$ and influenced by the mesoscopic hydrostatic stress history $\sigma_{\mu h}(t)$. The simplest failure criterion involving these components is the linear combination given by:

$$\tau_\mu(t) + \alpha_{DV} \cdot \sigma_{\mu h}(t) = \beta_{DV} \quad (9)$$

Note that the Sines, Findley and Dang Van criteria can be included in the general class of Mohr models against material failure, which use combinations of the shear stress $\tau$ that acts on a certain plane with the normal or hydrostatic stresses $\sigma$ on this plane:

$$\tau + \alpha \cdot \sigma = \beta \quad (10)$$

The Sines criterion uses the Mises or octahedral plane and the hydrostatic stresses, therefore $\tau \equiv \Delta \tau_{\text{Mises}}/2$, $\sigma \equiv 3 \cdot \sigma_{\text{hm}}$, $\alpha \equiv \alpha_S$, $\beta \equiv \beta_S$; Findley uses the shear stress on the critical plane and the normal stress perpendicular to it, thus $\tau \equiv \Delta \tau/2$, $\sigma \equiv \sigma_{\perp}$, $\alpha \equiv \alpha_F$, $\beta \equiv \beta_F$; and Dang Van can be obtained from $\tau \equiv \tau_\mu(t)$, $\sigma \equiv \sigma_{\mu h}(t)$, $\alpha \equiv \alpha_{DV}$, $\beta \equiv \beta_{DV}$. Other similar criteria can be found in [1] and [6].

Finally, it is important to remember that the SN and $\varepsilon N$ tests involve both microcrack initiation (sensitive to $\tau$) and propagation (more sensitive to $\sigma$) phases, and therefore fatigue damage can be more influenced by $\tau$ or $\sigma$, depending on the percentage of the life spent at each phase. Therefore, materials with large values of $\alpha$ are more sensitive to $\sigma$ (normal stresses are more important to them), probably spending more cycles to propagate than to initiate the microcrack.

4 Strain-based multiaxial fatigue damage models

The three multiaxial failure criteria presented above are based on macroscopic stresses that are supposedly elastic, therefore they are only applicable when $\sigma_{\text{Mises}}$ is much smaller than the cyclic yielding strength $S_{yc}$. Thus, as in the case of the SN method, they should only be used to predict long fatigue lives. Otherwise, it is imperative to use fatigue damage criteria based on applied strains instead of stresses [1], using the principles studied in the so-called $\varepsilon N$ method.

One of the simplest models is the one based on the $\gamma N$ curve, similar to Coffin-Manson’s equation, which uses the largest shear strain range $\Delta \gamma_{\text{max}}$ acting on the specimen ($\gamma_{ij}$ ( $2 \varepsilon_{ij}$, $i \neq j$)) to predict fatigue life

$$\frac{\Delta \gamma_{\text{max}}}{2} = \frac{\tau_c}{G}(2N)^{b_\gamma} + \gamma_c(2N)^{c_\gamma} \quad (11)$$

where $\tau_c$, $b_\gamma$, $\gamma_c$ and $c_\gamma$ are parameters similar to the ones used in Coffin-Manson’s equation. In this way, since the shear modulus $G = E/ [2(1 + \nu)]$, $\nu$ being Poisson’s coefficient, if no experimental data is available, then the $\gamma N$ curve can be estimated assuming $\tau_c \cong \sigma_c/\sqrt{3}$, $b_\gamma \cong b$, $\gamma_c \cong \varepsilon_c \sqrt{3}$ and $c_\gamma \cong c$, resulting in
Evaluation of stress-strain models and fatigue life prediction methods under proportional loading

\[ \frac{\Delta \gamma_{\text{max}}}{2} \geq \frac{\sigma_c}{E} \frac{2(1 + \nu)}{\sqrt{3}} (2N)^b + \varepsilon_c \sqrt{3} (2N)^c \]  \hspace{1cm} (12)

The \( \gamma N \) curve is only recommended to model fatigue damage in materials that are more sensitive to shear strains (which have small \( \alpha \) in the Mohr models), and if the mean loads are zero. It would be expected that such materials would have a shorter torsional fatigue life than similar materials more sensitive to normal stresses.

The Brown-Miller [7] model can consider the mean stress effects, combining the maximum range of the shear strain \( \Delta \gamma_{\text{max}} \) to the range of normal strain \( \Delta \varepsilon_{\perp} \) (through the term \( \Delta \gamma_{\text{max}}/2 + \alpha_{BM} \cdot \Delta \varepsilon_{\perp} \)) and the mean normal stress \( \sigma_{\perp m} \) perpendicular to the plane of maximum shear strain, to obtain the fatigue life \( N \):

\[ \frac{\Delta \gamma_{\text{max}}}{2} + \alpha_{BM} \cdot \Delta \varepsilon_{\perp} = \beta_1 \frac{\sigma_c - 2\sigma_{\perp m}}{E} (2N)^b + \beta_2 \varepsilon_c (2N)^c \]  \hspace{1cm} (13)

where \( \alpha_{BM} \) is a fitting parameter (\( \alpha_{BM} \approx 0.3 \) for ductile metals in lives near the fatigue limit), \( \beta_1 = (1 + \nu + (1 - \nu) \cdot \alpha_{BM} \), and \( \beta_2 = 1.5 + 0.5 \cdot \alpha_{BM} \).

This equation was adapted from Morrow to fit uniaxial traction test data, where the mean stress \( \sigma_m \) is equal to \( 2\sigma_{\perp m} \) (because \( \sigma_{\perp m} \) acts perpendicularly to the plane of \( \gamma_{\text{max}} \), therefore it is worth half of \( \sigma_m \)).

The values of \( \beta_1 \) and \( \beta_2 \) are obtained assuming uniaxial traction, see Fig. 5:

\[ \frac{\Delta \gamma_{\text{max}}}{(1 + \nu) \Delta \varepsilon} = \frac{\Delta \gamma_{\text{max}}}{(1 - \nu) \Delta \varepsilon / 2} \Rightarrow \frac{\Delta \gamma_{\text{max}}}{2} + \alpha_{BM} \Delta \varepsilon_{\perp} = \frac{\Delta \varepsilon}{2} \left[ (1 + \nu) + \alpha_{BM} (1 - \nu) \right] \]  \hspace{1cm} (14)

Figure 5: Mohr circles for stresses and strains under uniaxial traction.
From Eq. (14), the coefficients $\beta_1 = (1 + \nu) + (1 - \nu) \cdot \alpha_{BM}$ and $\beta_2 = 1.5 + 0.5 \cdot \alpha_{BM}$ are obtained, because $\nu = 0.5$ for plastic strains, which preserve volume. The original Brown-Miller model assumes that the elastic strains have $\nu = 0.3$, therefore $\beta_1 \cong (1 + 0.3) + (1 - 0.3) \cdot \alpha_{BM} = 1.3 + 0.7 \cdot \alpha_{BM}$.

The Brown-Miller model is frequently used in multiaxial fatigue, even though it is not reasonable to assume that $\Delta \varepsilon_\perp$ can control the opening and closure of microcracks, because the range $\Delta \varepsilon$ does not include information about maximum stresses or strains. E.g., two microcracks with the same $\Delta \gamma_{\max}$ and $\Delta \varepsilon_\perp$ can have very different fatigue lives if one is opened (under traction) and the other closed (under compression) due to the mean load effect. The use of $\sigma_\perp m$ compensates in part for this model flaw, however the mean stress effect is only considered in the elastic part.

Fatemi and Socie [8] suggested replacing $\Delta \varepsilon_\perp$ by the maximum normal stress $\sigma_{\perp \max}$ perpendicular to the plane of maximum shear strain, applying it to the $\gamma N$ curve:

$$\frac{\Delta \gamma_{\max}}{2} \left(1 + \alpha_{FS} \frac{\sigma_{\perp \max}}{\sigma_c}ight) = \frac{\tau_c}{G}(2N)^{b_\gamma} + \gamma_c(2N)^{c_\gamma}$$

(15)

Note that the value of $\alpha_{BM}$ and $\alpha_{FS}$ indicates whether the material is more sensitive to $\tau$ ($\alpha_{BM}$ or $\alpha_{FS} \ll 1$) or to $\sigma$ ($\alpha_{BM}$ or $\alpha_{FS} \gg 1$).

If the propagation phase of the microcracks (more sensitive to $\sigma$) is dominant over initiation, the Smith-Watson-Topper (SWT) multiaxial model can be used [9]:

$$\frac{\Delta \varepsilon_1}{2} \cdot \sigma_{\perp \max} = \frac{\sigma_c^2}{E}(2N)^{2b} + \sigma_c \varepsilon_c(2N)^{b+c}$$

(16)

where $\Delta \varepsilon_1$ is the range of the maximum principal strain and $\sigma_{\perp \max}$ is the stress peak in the direction perpendicular to $\varepsilon_1$.

Figure 6 summarizes the parameters used in the above strain-based models. In addition, there are several other models based on the plastic energy dissipated by the hysteresis loops, and other combining energy with critical planes, see [1].

![Figure 6: Parameters which affect the strain-based multiaxial models.](image)

It is important to note that the plane of maximum shear strain amplitude $\Delta \gamma_{\max}/2$ (used in Brown-Miller’s and Fatemi-Socie’s models) is in general different from the planes that would maximize the respective damage parameters ($\Delta \gamma/2 + \alpha_{BM} (\Delta \varepsilon_\perp$ for Brown-Miller, and $\Delta \gamma((1 + \alpha_{FS})(\sigma_{\perp \max}/\sigma_c))/2$ for Smith-Watson-Topper).
for Fatemi-Socie. But if these are the parameters that cause damage, it is reasonable to argue that fatigue life should be calculated on the critical plane that maximizes them (in a similar way as done in Findley’s model), and not on the plane of $\Delta \gamma_{\text{max}}$. In this way, it is a good idea to modify the Brown-Miller and Fatemi-Socie models introducing a subtle but important change:

$$\frac{\Delta \gamma_{\text{max}}}{2} + \alpha_{BM} \cdot \Delta \varepsilon_\perp \Rightarrow \left( \frac{\Delta \gamma}{2} + \alpha_{BM} \cdot \Delta \varepsilon_\perp \right)_{\text{max}}$$

(17)

$$\frac{\Delta \gamma_{\text{max}}}{2} \left( 1 + \alpha_{FS} \frac{\sigma_{\text{max}}}{S_{yc}} \right) \Rightarrow \left( \frac{\Delta \gamma}{2} + \alpha_{FS} \frac{\Delta \gamma \cdot \sigma_{\perp}}{2 S_{yc}} \right)_{\text{max}}$$

(18)

The use of critical planes that maximize the damage parameters in each model has the advantage of predicting not only the fatigue life but also the dominant planes where the crack will initiate. However, these calculations are not simple and require the use of sophisticated numerical methods.

This idea can also be applied to the SWT model, calculating the critical plane where the product between the normal strain range $\Delta \varepsilon_\perp$ and the normal stress peak $\sigma_{\perp \text{max}}$ is maximized, adopting the modification

$$\frac{\Delta \varepsilon_1}{2} \cdot \sigma_{\perp \text{max}} \Rightarrow \left( \frac{\Delta \varepsilon_1}{2} \cdot \sigma_{\perp \text{max}} \right)_{\text{max}}$$

(19)

A great advantage of the Fatemi-Socie (or SWT) model is to be able to consider the effect of NP hardening from the peak of normal stress $\sigma_{\perp \text{max}}$ (or $\sigma_{\perp \text{max}}$). In stainless steels, e.g., a NP history leads to a much higher damage than a proportional one with the same $\Delta \gamma_{\text{max}}$ and $\Delta \varepsilon_\perp$, because the NP hardening increases the value of $\sigma_{\perp \text{max}}$ (or $\sigma_{\perp \text{max}}$). Note that Brown-Miller would wrongfully predict the same damage in both histories (because $\Delta \gamma_{\text{max}}$ and $\Delta \varepsilon_\perp$ would be the same), and only the Fatemi-Socie and SWT models would be able to correctly account for the greater damage of the NP loading (assuming that $H_{\text{cnp}}$ would be used to obtain $\sigma_{\perp \text{max}}$ and $\sigma_{\perp \text{max}}$).

5 Multiaxial stress-strain relations

Hooke’s law cannot be used to correlate stresses and strains for short multiaxial fatigue life predictions, due to plasticity effects. The hookean stresses and strains, $\tilde{\sigma}$ and $\tilde{\varepsilon}$, defined as the values of $\sigma$ and $\varepsilon$ obtained assuming that the material would be linear elastic (using Hooke’s law and, at the notches, considering elastic $K_\sigma$ and $K_\varepsilon$), can only be applied for long life predictions.

In addition, Ramberg-Osgood cannot be used either to directly correlate principal stresses and strains $\sigma_i$ and $\varepsilon_i$ ($i = 1, 2, 3$) of a multiaxial history, because this model has been developed for the uniaxial case.

However, if the elastic nominal stress range $\Delta \sigma_n$ is caused by in-phase loading, then it is trivial to calculate the elastic-plastic stresses and strains at the notch root using the “highest $K_i$ method”. In this approximate method, the equivalent nominal stress range $\Delta \sigma_n$ calculated from Tresca or Mises is used to obtain $\Delta \sigma$ and $\Delta \varepsilon$ at the notch root using Ramberg-Osgood and (for safety, because the method is conservative) the highest $K_i$ in Neuber’s rule. Remember that the multiaxial loadings can result, at
the same notch root, in different values of $K_t$ for traction, bending, torsion and shear loadings, but only the maximum one is used. To generate more accurate predictions for notches under combined stresses, it is recommended to use multiaxial $\sigma$-$\varepsilon$ relations.

Several models have been proposed to correlate $\sigma_i$ and $\varepsilon_i$ in proportional histories, e.g.: the constant ratio model [1], Hoffmann-Seeger’s model ([10], and Dowling’s model [11]. To present these three models, it is necessary to define a few variables involved in their formulation:

- $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, $\tilde{\sigma}_3$, $\tilde{\varepsilon}_1$, $\tilde{\varepsilon}_2$, $\tilde{\varepsilon}_3$: hookean principal stresses and strains at the notch root (elastically calculated using Hooke’s law and elastic $K_\sigma$ and $K_\varepsilon$);
- $\tilde{\sigma}_{\text{Mises}}$, $\tilde{\varepsilon}_{\text{Mises}}$: hookean Mises stress and strain (at the notch root), calculated using the above variables;
- $\sigma_1$, $\sigma_2$, $\sigma_3$, $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$: elastic-plastic principal stresses and strains (notch root);
- $\lambda_2$, $\lambda_3$: ratios between pairs of principal stresses, where $\lambda_2 = \sigma_2/\sigma_1$ and $\lambda_3 = \sigma_3/\sigma_1$, both between -1 and 1;
- $\phi_2$, $\phi_3$: ratios between pairs of principal strains, where $\phi_2 = \varepsilon_2/\varepsilon_1$, $\phi_3 = \varepsilon_3/\varepsilon_1$, both between -1 and 1; and
- $\lambda_{\text{Mises}}$, $\varphi_{\text{Mises}}$: Mises ratios $\lambda_{\text{Mises}} = \sigma_{\text{Mises}}/\sigma_1$ and $\varphi_{\text{Mises}} = \varepsilon_{\text{Mises}}/\varepsilon_1$.

From the above definitions, it is possible to obtain

$$\lambda_{\text{Mises}} = \frac{\sigma_{\text{Mises}}}{\sigma_1} = \frac{1}{\sqrt{2}} \sqrt{(1 - \lambda_2)^2 + (1 - \lambda_3)^2 + (\lambda_2 - \lambda_3)^2}$$

(20)

$$\varphi_{\text{Mises}} = \frac{\varepsilon_{\text{Mises}}}{\varepsilon_1} = \frac{1}{\sqrt{2(1 + \nu)}} \sqrt{(1 - \phi_2)^2 + (1 - \phi_3)^2 + (\phi_2 - \phi_3)^2}$$

(21)

The three models are described next.

### 5.1 Constant ratio model

The constant ratio model [1] assumes that, under a proportional history, the bi-axial ratios $\lambda_2$, $\lambda_3$, $\phi_2$ and $\phi_3$ remain constant even after yielding has occurred. Since the elastic Poisson coefficient $\nu_{el}$ is typically between 1/4 and 1/3 in most metal alloys, significantly different than the plastic $\nu_{pl} = 0.5$, these ratios are in fact not constant, but for small plastic strains this is a good approximation.

Thus, these ratios can be estimated from the elastic (hookean) stresses and strains, obtained from Hooke’s law using elastic $K_\sigma$ and $K_\varepsilon$:

$$\lambda_2 \approx \frac{\tilde{\sigma}_2}{\tilde{\sigma}_1}, \lambda_3 \approx \frac{\tilde{\sigma}_3}{\tilde{\sigma}_1}, \phi_2 \approx \frac{\tilde{\varepsilon}_2}{\tilde{\varepsilon}_1}, \phi_3 \approx \frac{\tilde{\varepsilon}_3}{\tilde{\varepsilon}_1}$$

(22)

Therefore, $\lambda_{\text{Mises}}$ is also a constant, leading to

$$\lambda_{\text{Mises}} \approx \frac{\tilde{\sigma}_{\text{Mises}}}{\tilde{\sigma}_1} \Rightarrow \tilde{\sigma}_{\text{Mises}} \approx \frac{\tilde{\sigma}_1}{\sqrt{2}} \sqrt{(1 - \lambda_2)^2 + (1 - \lambda_3)^2 + (\lambda_2 - \lambda_3)^2}$$

(23)

and, similarly, $\varphi_{\text{Mises}}$ can be calculated from $\phi_2$ and $\phi_3$. The cyclic $\sigma$-$\varepsilon$ relation is then defined using Mises and the Ramberg-Osgood uniaxial parameters.
Evaluation of stress-strain models and fatigue life prediction methods under proportional loading

\[
\varepsilon_{\text{Mises}} = \frac{\sigma_{\text{Mises}}}{E} + \left(\frac{\sigma_{\text{Mises}}}{H_c}\right)^{1/h_c}
\]

(24)

If no notches are present, then the above equation is used together with the estimates for \(\lambda_{\text{Mises}}, \varphi_{\text{Mises}}, \lambda_2, \lambda_3, \varphi_2\) and \(\varphi_3\) to obtain \(\sigma_i\) from \(\varepsilon_i\) (\(i = 1, 2, 3\)), or vice-versa. In notched components, \(\sigma_{\text{Mises}}\) (elastically calculated including the \(K_i\)s) is applied to a variation of the Neuber’s rule to calculate the Mises elastic-plastic stress \(\sigma_{\text{Mises}}\) and, finally, \(\varepsilon_{\text{Mises}}, \sigma_i\) and \(\varepsilon_i\) (\(i = 1, 2, 3\)):

\[
\frac{(\sigma_{\text{Mises}})^2}{E} = \sigma_{\text{Mises}} \cdot \varepsilon_{\text{Mises}} = \left(\frac{\sigma_{\text{Mises}}}{E}\right)^2 + \sigma_{\text{Mises}} \cdot \left(\frac{\sigma_{\text{Mises}}}{H_c}\right)^{1/h_c}
\]

(25)

After calculating \(\sigma_{\text{Mises}}\) and \(\varepsilon_{\text{Mises}}\), the constant ratio model obtains the principal stress and strain using:

\[
\begin{align*}
\sigma_1 &= \sigma_{\text{Mises}} \lambda_{\text{Mises}}, \\
\sigma_2 &= \lambda_2 \sigma_1, \\
\sigma_3 &= \lambda_3 \sigma_1 \\
\varepsilon_1 &= \varepsilon_{\text{Mises}} \phi_{\text{Mises}}, \\
\varepsilon_2 &= \phi_2 \varepsilon_1, \\
\varepsilon_3 &= \phi_3 \varepsilon_1
\end{align*}
\]

(26)

5.2 Hoffmann-Seeger’s model

Hoffmann-Seeger’s model [10] uses the same cyclic \(\sigma-\varepsilon\) relation and the same variation of Neuber’s rule presented above to calculate \(\sigma_{\text{Mises}}\) and \(\varepsilon_{\text{Mises}}\), but it assumes that:

- the critical point happens at the surface, with principal stresses \(\sigma_1\) and \(\sigma_2\);
- \(\sigma_3\) is defined normal to the surface, therefore \(\sigma_3 = 0\) (and then \(\lambda_3 = 0\)); and
- only the ratio \(\phi_2 = \varepsilon_2/\varepsilon_1\) is estimated using the linear elastic (hookean) values.

After calculating \(\sigma_{\text{Mises}}\) and \(\varepsilon_{\text{Mises}}\), \(\sigma_i\) and \(\varepsilon_i\) are estimated from:

\[
\begin{align*}
\sigma_1 &= \sigma_{\text{Mises}} \lambda_{\text{Mises}}, \\
\sigma_2 &= \lambda_2 \sigma_1, \quad \sigma_3 = 0 \\
\varepsilon_1 &= \frac{(1-\lambda_2\varphi_{\text{Mises}})}{\lambda_{\text{Mises}}}, \quad \varepsilon_2 = \phi_2 \varepsilon_1, \quad \varepsilon_3 = -\varphi_{\varepsilon_1} \frac{1+\lambda_2}{1-\lambda_3} \varepsilon_2 \\
\varphi &= \frac{1}{2} - \frac{(1-\nu_{\varepsilon_2}) \sigma_{\text{Mises}}}{E \cdot \varepsilon_{\text{Mises}}}, \quad \lambda_2 = \frac{\phi_2 + \varphi}{1 + \phi_2 \varphi} \lambda_{\text{Mises}} = \sqrt{1 - \lambda_2 + \lambda_2^2}
\end{align*}
\]

(27)

(28)

5.3 Dowling’s model

The model proposed in [11] also assumes that the principal stresses \(\sigma_1\) and \(\sigma_2\) act on the surface of the critical point (therefore \(\sigma_3\) is zero), and it considers \(\lambda_2\) and \(\varphi_2\) constant, estimating them from their hookean values

\[
\lambda_2 = \frac{\sigma_2}{\sigma_1} \approx \frac{\bar{\sigma}_2}{\bar{\sigma}_1} \approx \frac{\phi_2 + \varphi}{1 + \phi_2 \varphi}, \quad \phi_2 = \frac{\bar{\varepsilon}_2}{\bar{\varepsilon}_1} \approx \frac{\bar{\varepsilon}_2}{\bar{\varepsilon}_1} = \frac{\lambda_2 - \varphi}{1 - \lambda_2 \varphi}
\]

(29)

Exceptionally, \(\sigma_2\) is defined here as the lowest principal stress at the surface, even if \(\sigma_2\) is smaller than \(\sigma_3\) (i.e. the convention \(\sigma_3 \leq \sigma_2 \leq \sigma_1\) is violated if \(\lambda_2 < 0\)).
The greatest difference between the previous two models and Dowling’s is that the latter correlates \( \sigma_1 \) and \( \varepsilon_1 \) directly using effective Ramberg-Osgood parameters \( E^* \) and \( H_c^* \)

\[
E^* = \left( \frac{1 + \phi_2 \nu}{1 - \nu^2} \right) \cdot E, \quad H_c^* = H_c \cdot \left( \frac{2}{2 - \lambda_2} \right)^{h_c} (1 - \lambda_2 + \lambda_2^2)^{0.5(h_c - 1)}
\]

(30)

and the effective relation between \( \sigma_1 \) and \( \varepsilon_1 \) is [11]

\[
\varepsilon_1 = \frac{\sigma_1}{E^*} + \left( \frac{\sigma_1}{H_c^*} \right)^{1/h_c}
\]

(31)

Figure 7 presents the principal stress-strain relation for the 1020 steel, according to Dowling’s model.

Figure 7: Principal stress-strain relations under plane strain, plane stress and pure torsion, according to Dowling.

In notched components, another variation of Neuber’s rule must be used to calculate \( \sigma_1 \) (and then \( \varepsilon_1 \)) from \( \hat{\sigma}_{Mises} \):

\[
\left( \frac{\hat{\sigma}_{Mises}}{E} \right)^2 = \sigma_1 \cdot \varepsilon_1 = \frac{\sigma_1^2}{E^*} + \sigma_1 \cdot \left( \frac{\sigma_1}{H_c^*} \right)^{1/h_c}
\]

(32)

The other principal stresses and strains are obtained from \( \sigma_1 \) and \( \varepsilon_1 \):

\[
\sigma_2 = \lambda_2 \sigma_1, \quad \sigma_3 = 0
\]

\[
\varepsilon_2 = \phi_2 \varepsilon_1, \quad \varepsilon_3 = -\nu \varepsilon_1 \frac{1 + \lambda_2}{1 - \nu^2}, \quad \nu = \frac{1}{2} - \frac{1 - \nu}{E^*} \frac{\sigma_1}{\varepsilon_1}
\]

(33)

The largest shear strain \( \gamma_{\text{max}} \) can then be calculated from the maximum difference between the principal strains \( \varepsilon_i \) (i = 1, 2, 3), obtaining not only its magnitude but also the plane where this maximum occurs.

It is important to note that the three presented models (formulated using the cyclic \( \sigma-\varepsilon \) curve) can also be applied to the hysteresis loops equations, by replacing in each equation \( \varepsilon \) with \( \Delta \varepsilon/2 \) and also \( \sigma \) with \( \Delta \sigma/2 \). The presented models are compared next.
Evaluation of stress-strain models and fatigue life prediction methods under proportional loading

6 Comparison among the multiaxial models

The presented multiaxial models are compared considering a notched 1020 steel shaft with diameter $d$ equal to 60mm under alternate bending moment $M_a$ of 2kNm and torsion $T_a$ of 3kNm, in phase, with stress concentration factors in bending $K_{tM}$ equal to 3.4 and in torsion $K_{tT}$ equal to 2.4.

Assuming the alternate nominal stress $σ_{na}$ as elastic,

$$σ_{na} = \frac{\sqrt{(32M_a)^2 + 3(16T_a)^2}}{πd^3}$$  \hfill (34)

then $σ_{na} = 155\text{MPa}$. This stress is lower than the cyclic yielding strength $S_{yc} = 241\text{MPa}$, therefore the hypothesis of $σ_{na}$ elastic is valid.

Using the “highest $K_t$ method” through the highest $K_t = 3.4$, $σ_a$ and $ε_a$ are calculated using Mises and Neuber

$$\frac{(K_tσ_{na})^2}{2} = (3.4 \cdot 155)^2 = σ_aε_aE = σ_a^2 + 203000 \cdot σ_a\left(\frac{σ_a}{772}\right)^{1/0.18} \Rightarrow \begin{cases} σ_a = 279\text{MPa} \\ ε_a = 0.49% \end{cases}$$  \hfill (35)

and then the life $N$ estimated for the shaft is

$$\frac{Δε}{2} = ε_a = \frac{896}{203000}(2N)^{-0.12} + 0.41(2N)^{-0.51} \Rightarrow N = 5871\text{cycles}$$  \hfill (36)

To use the multiaxial stress-strain models, the hookean stresses at the notch root are calculated considering $K_{tM} = 3.4$ and $K_{tT} = 2.4$ as purely elastic:

$$\tilde{σ}_{aMises} = \sqrt{(K_{tM}σ_M)^2 + 3(K_{tT}τ_T)^2} = \sqrt{(3.4 \cdot 32 \cdot M_a)^2 + 3(2.4 \cdot 16 \cdot T_a)^2} \cdot \frac{1}{π(0.060)^3}$$  \hfill (37)

$$\tilde{σ}_{a1,2} = \frac{K_{tM}σ_M}{2} \pm \sqrt{\left(\frac{K_{tM}σ_M}{2}\right)^2 + (K_{tT}τ_T)^2} = 160 \pm 234\text{MPa}$$  \hfill (38)

Thus, the hookean stresses are $\tilde{σ}_{aMises} = 435\text{MPa}$, $\tilde{σ}_{a1} = 394\text{MPa}$, $\tilde{σ}_{a2} = -73\text{MPa}$ and $\tilde{σ}_{a3} = 0$, which can be correlated to the principal hookean strains from Hooke’s law (considering $ν = 0.3$):

$$ε_{a1} = \frac{[394 - 0.3(-73 + 94)]}{203000} = 0.205%$$

$$ε_{a2} = \frac{[-73 - 0.3(394 + 0)]}{203000} = -0.094%$$

$$ε_{a3} = \frac{[0 - 0.3(-73 + 394)]}{203000} = -0.047%$$

$$\tilde{ε}_{aMises} = \frac{1}{\sqrt{2(1 + ν)}} \sqrt{(ε_{a1} - ε_{a2})^2 + (ε_{a1} - ε_{a3})^2 + (ε_{a2} - ε_{a3})^2} = 0.214%$$  \hfill (39)

From the constant ratio and Hoffmann-Seeger models,
\[
\frac{a_{Mises}^2}{E} = 0.93 = \frac{\sigma_{aMises}^2}{E} + \sigma_{aMises} \cdot \left(\frac{\sigma_{aMises}}{772}\right)^{1/0.18} \Rightarrow \sigma_{aMises} = 250MPa
\]
(41)

\[
\varepsilon_{aMises} = \frac{\sigma_{aMises}}{203000} + \left(\frac{\sigma_{aMises}}{772}\right)^{1/0.18} \Rightarrow \varepsilon_{aMises} = 0.360% 
\]
(42)

Note, as expected, that \(\sigma_{aMises} < \sigma_{aMises}\) and \(\varepsilon_{aMises} > \varepsilon_{aMises}\).

From the constant ratio model, the hookean stresses and strains can be used to estimate \(\lambda_{Mises} = 1.105\), \(\lambda_2 = -0.185\), \(\lambda_3 = 0\), \(\varphi_{Mises} = 1.046\), \(\varphi_2 = -0.460\) and \(\varphi_3 = -0.231\), so the alternate principal stresses and strains are

\[
\sigma_{a1} = 259/1.1 = 235MPa, \quad \sigma_{a2} = \lambda_2\sigma_{a1} = -44MPa, \quad \sigma_{a3} = 0 \\
\varepsilon_{a1} = 0.359%/1.046 = 0.344\%, \quad \varepsilon_{a2} = \phi_2\varepsilon_{a1} = -0.158\%, \quad \varepsilon_{a3} = \phi_3\varepsilon_{a1} = -0.080\% 
\]
(43)

On the other hand, Hoffmann-Seeger’s model predicts

\[
\bar{\lambda}_2 = \frac{2\phi_2 + \bar{\rho}}{1 + 2\phi_2\bar{\rho}} = -0.46 + \bar{\rho} \\
\bar{\lambda}_{Mises} = \sqrt{1 - \lambda_2^2 + 0.0387} = 1.02 
\]
(44)

resulting in alternate principal stresses and strains

\[
\begin{align*}
\{ & \sigma_{a1} = 259/1.02 = 254MPa, \quad \sigma_{a2} = -0.0387 \cdot \sigma_{a1} = -10MPa, \quad \sigma_{a3} = 0 \\
& \varepsilon_{a1} = (1 - \bar{\lambda}_2\bar{\rho})0.360%/1.02 = 0.359\%, \quad \varepsilon_{a2} = \phi_2\varepsilon_{a1} = -0.165\% \\
& \varepsilon_{a3} = -\bar{\rho}\varepsilon_{a1}(1 + \bar{\lambda}_2)/(1 - \bar{\lambda}_2\bar{\rho}) = -0.146\% 
\end{align*}
\]
(45)

Dowling’s model uses the elastic ratios \(\lambda_2 = -0.185\) and \(\varphi_2 = -0.460\) to calculate the effective parameters of the hardening curve

\[
E^* = \left(1 + \frac{\phi_2\varphi_2}{1 - \varphi_2^2}\right) \cdot E = \left(1 - 0.46 \cdot 0.3\right) \cdot 203GPa = 192GPa
\]
(46)

\[
H_c^* = 772MPa \cdot \left(\frac{2}{2 - \lambda_2}\right)^{0.18} (1 - \lambda_2 + \lambda_2^2)^{0.5(0.18-1)} = 700MPa
\]
(47)

\[
\frac{(\bar{\sigma}_{aMises})^2}{E} = 0.93 = \sigma_{a1} \cdot 1.08 = \left(\frac{\sigma_{a1}}{E^*}\right) + \sigma_{a1} \cdot \left(\frac{\sigma_{a1}}{H_c^*}\right)^{1/0.18} \Rightarrow \left\{ \begin{array}{l}
\sigma_{a1} = 240MPa \\
\varepsilon_{a1} = 0.3888% 
\end{array} \right.
\]
(48)

\[
\left\{ \begin{array}{l}
\sigma_{a2} = \lambda_2\sigma_{a1} = -45MPa, \quad \sigma_{a3} = 0 \\
\varepsilon_{a2} = \phi_2\varepsilon_{a1} = -0.179\%, \quad \varepsilon_{a3} = -\bar{\rho}\varepsilon_{a1}(1 - \lambda_2\bar{\lambda}_2)/(1 - \lambda_2^2) = -0.127\% (\bar{\rho} = 0.436)
\end{array} \right.
\]
(49)

For all considered models, the maximum shear strain amplitude is calculated from \(\gamma_{a\max} = \varepsilon_{a1} - \varepsilon_{a2}\), assuming that the directions 1 and 2 are respectively the ones with maximum and minimum principal strains. The maximum normal strains and stresses in the plane of \(\gamma_{a\max}\) are
\[ \varepsilon_{a\perp} = (\varepsilon_{a1} + \varepsilon_{a2})/2 \quad \text{and} \quad \sigma_{a\perp} = (\sigma_{a1} + \sigma_{a2})/2 \] (50)

Since in this problem the mean stresses and strains are zero, the values used by the Brown-Miller, Fatemi-Socie and SWT strain-life models are respectively
\[ \Delta \varepsilon_{\perp} = 2\varepsilon_{a\perp}, \quad \sigma_{\perp\max} = \sigma_{a\perp} \quad \text{and} \quad \sigma_{\perp\max1} = \sigma_{a1}. \]

Table 1 summarizes the stresses and strains obtained from the hookean values (obtained assuming elastic stresses, which must not be used in life predictions in the presence of significant plasticity), from the “highest \( K_t \) method”, and from the three presented multiaxial stress-strain models: the constant ratio, Hoffmann-Seeger’s and Dowling’s.

<table>
<thead>
<tr>
<th></th>
<th>hookean values</th>
<th>highest ( K_t ) method</th>
<th>constant ratio</th>
<th>Hoffmann-Seeger</th>
<th>Dowling</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_{\alpha\text{Mises}} )</td>
<td>435</td>
<td>279</td>
<td>259</td>
<td>259</td>
<td>265</td>
</tr>
<tr>
<td>( \varepsilon_{\alpha\text{Mises}} )</td>
<td>0.214%</td>
<td>0.488%</td>
<td>0.360%</td>
<td>0.360%</td>
<td>0.418%</td>
</tr>
<tr>
<td>( \sigma_{\alpha1} )</td>
<td>394</td>
<td>253</td>
<td>235</td>
<td>254</td>
<td>240</td>
</tr>
<tr>
<td>( \sigma_{\alpha2} )</td>
<td>-73</td>
<td>-47</td>
<td>-44</td>
<td>-10</td>
<td>-45</td>
</tr>
<tr>
<td>( \sigma_{\alpha3} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \varepsilon_{\alpha1} )</td>
<td>0.205%</td>
<td>0.466%</td>
<td>0.344%</td>
<td>0.359%</td>
<td>0.388%</td>
</tr>
<tr>
<td>( \varepsilon_{\alpha2} )</td>
<td>-0.094%</td>
<td>-0.215%</td>
<td>-0.158%</td>
<td>-0.165%</td>
<td>-0.179%</td>
</tr>
<tr>
<td>( \varepsilon_{\alpha3} )</td>
<td>-0.047%</td>
<td>-0.108%</td>
<td>-0.080%</td>
<td>-0.146%</td>
<td>-0.127%</td>
</tr>
<tr>
<td>( \gamma_{\sigma\text{max}} )</td>
<td>0.299%</td>
<td>0.681%</td>
<td>0.502%</td>
<td>0.524%</td>
<td>0.567%</td>
</tr>
<tr>
<td>( \Delta \varepsilon_{\perp} )</td>
<td>0.111%</td>
<td>0.251%</td>
<td>0.186%</td>
<td>0.194%</td>
<td>0.209%</td>
</tr>
<tr>
<td>( \sigma_{\perp\max} )</td>
<td>160</td>
<td>103</td>
<td>95</td>
<td>122</td>
<td>98</td>
</tr>
</tbody>
</table>

Note from Table 1 that the “highest \( K_t \) method” is conservative, especially for the calculated strains, but not too much, therefore it could be used in practice. The three multiaxial models are in theory more accurate, predicting approximately the same values.

Now, using e.g. Dowling’s model, the fatigue life \( N \) can be obtained from the several damage models. Considering the \( \varepsilon N \) curve and using the Mises strain \( \varepsilon_{\alpha\text{Mises}} = 0.418\% \), then it is found that \( N = 8765 \) cycles.

If, instead of the \( \varepsilon N \) curve, the \( \gamma N \) curve is considered, estimating its coefficients from \( \tau_c \approx \sigma_c/\sqrt{3} \), \( b_c \approx b \), \( \gamma_c \approx \varepsilon_c\sqrt{3} \) and \( c_c \approx c \), and using \( \gamma_{\alpha\text{max}} = 0.567\% \), then it is found that \( N = 14693 \) cycles.

Considering the Brown-Miller’s model, with its constants estimated from \( \alpha_{BM} \approx 0.3 \), \( \beta_1 = 1.3 + 0.7 \cdot \alpha_{BM} = 1.51 \) and \( \beta_2 = 1.5 + 0.5 \cdot \alpha_{BM} = 1.65 \), with \( \Delta \varepsilon_{\perp} = 0.209\% \), then it is found that \( N = 10290 \) cycles.
Fatemi-Socie’s model, using $\alpha_{FS} \equiv S_{yc}/\sigma_c = 241\text{MPa}/896\text{MPa} \cong 0.27$ and the $\gamma N$ curve estimated as above, where $\sigma_{\perp max} = 98\text{MPa}$, results in $N = 11201$ cycles.

And finally, considering the SWT’s model, which is appropriate for materials more sensitive to normal stresses, with $\Delta \varepsilon_1/2 = \varepsilon_{a1} = 0.388\%$ and, since the mean loads are zero, $\sigma_{\perp1max} = \sigma_{a1} = 240\text{MPa}$, then $N = 13577$ cycles.

The above results, based on stresses and strains from Dowling’s model, are recalculated considering hookean values, the “highest $K_t$ method”, the constant ratio and Hoffmann-Seeger models, using the ViDa fatigue design software [12,13]. The results are shown in Table 2.

<table>
<thead>
<tr>
<th></th>
<th>Mises + $\varepsilon N$ curve</th>
<th>$\gamma N$ curve</th>
<th>Brown-Miller</th>
<th>Fatemi-Socie</th>
<th>SWT</th>
</tr>
</thead>
<tbody>
<tr>
<td>hookean values</td>
<td>59500</td>
<td>94300</td>
<td>63000</td>
<td>56200</td>
<td>18300</td>
</tr>
<tr>
<td>highest $K_t$ method</td>
<td>5900</td>
<td>9120</td>
<td>6440</td>
<td>6940</td>
<td>8470</td>
</tr>
<tr>
<td>constant ratio</td>
<td>13000</td>
<td>20300</td>
<td>14100</td>
<td>15500</td>
<td>18300</td>
</tr>
<tr>
<td>Hoffmann-Seeger</td>
<td>13000</td>
<td>18100</td>
<td>12600</td>
<td>12900</td>
<td>14200</td>
</tr>
<tr>
<td>Dowling</td>
<td>8770</td>
<td>14700</td>
<td>10300</td>
<td>11200</td>
<td>13600</td>
</tr>
</tbody>
</table>

Except from the results obtained from the hookean values (which are significantly non-conservative), all combinations of multiaxial damage models with multiaxial stress-strain relations resulted in predicted lives not too different, varying between 5900 and 20300 cycles. Therefore, it is reasonable to consider in proportional histories the use of simplifications such as the “highest $K_t$ method” and the $\varepsilon N$ curve applied to $\Delta \varepsilon_{Mises}/2$, despite the conservative predictions.

The hookean values result in poor estimates, overestimating $\sigma_{a1}$ and underestimating $\varepsilon_{a1}$, but it interestingly estimates quite well the product $\sigma_{a1}\varepsilon_{a1}$ (because, according to Neuber, $\sigma_{a1}\varepsilon_{a1} \cong \sigma_{a1}\varepsilon_{a1}$), therefore they resulted in good predictions when combined with SWT’s model, which is based on this product.

But in NP histories, the NP hardening can have a significant effect in the fatigue life. In addition, none of the presented $\sigma$-$\varepsilon$ models is valid in the NP case (because all of them assumed $\varphi_2$ constant). In the NP case, incremental plasticity models must be used [1].

7 Conclusions

In this work, the multiaxial damage models of Sines, Findley and Dang Van, applicable to long fatigue lives, and Brown-Miller, Fatemi-Socie and Smith-Watson-Topper (SWT), which consider plasticity, were reviewed. The Sines model is easy to compute, it considers the effect of the second principal stress $\sigma_2$ (because it uses the Mises plane), but it is only valid for proportional histories. On the other
hand, Findley’s model is hard to compute, because it requires the search for a critical plane, but for long lives it is valid for any load history, proportional or NP. Dang Van’s model is able to consider the damage in a mesoscopic scale, but it has the limitations of the stress-based models.

The strain-based models are valid for any life, short or long. Among them, the Brown-Miller and Fatemi-Socie models give more value to the shear strains $\gamma$, while SWT does it for normal strains $\varepsilon$. Brown-Miller and Fatemi-Socie combine $\Delta \gamma_{\text{max}}$ to $\Delta \varepsilon_\perp$ or to $\sigma_\perp_{\text{max}}$ normal to the direction of $\gamma_{\text{max}}$, being applicable to proportional or NP histories. SWT uses the principal strain $\varepsilon_1$. The most versatile models among the studied ones are the Fatemi-Socie and SWT, because they can include the NP hardening effect. But in order to generate a more realistic model, it is important to modify these criteria to calculate the fatigue life in the critical plane where the damage parameters of each model are maximized.

The main multiaxial stress-strain models were also reviewed and compared. It can be concluded that multiaxial stress-strain relations must be used instead of uniaxial ones, even though a few simplifications are adequate, such as the “highest $K_t$ method” for notched components. Since the critical point of a structure is usually in its surface, in general a 2D analysis (under plane stress) is enough for multiaxial fatigue design. Except for the results from the hookean values, which are significantly non-conservative, all combinations of strain-based multiaxial damage models with multiaxial stress-strain relations resulted in not too different lives (within a factor of 2) for the considered example, which has significant plastic strains (but they were not much higher than the elastic ones). The best predictions should be the ones from multiaxial models that use the critical plane idea, where the damage parameters are maximized. However, none of the studied stress-strain models is valid for NP hardening, which can have a significant influence in the fatigue lives of e.g. stainless steels.

References


