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DISCUSSION OF A NEW RIGID BODIES COLLISION MODEL

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Abstract. *In general the motion of a body takes place in a confined environment and collisions of the body with the containing wall are possible. In order to predict the dynamics of a body in these conditions one must know what happens in a collision. Therefore, the problem is: if one knows the pre-collision dynamics of the body and the properties of the body and the wall one wants to predict the post-collision dynamics. This problem is quite old and it appeared in the literature in 1668. Up to 1984 it seemed that Newton's model was enough to solve the problem. But it was found that this was not the case and a renewed interest in the problem appeared. The aim of this paper is to treat the problem of plan collisions of rigid bodies, to classify the different models found in the literature and to present a new model, called C-S model, that is a generalization of most of these models.*

Keywords: *Collisions, Dynamics, Modelling, Simulation, Rigid Bodies*

1. INTRODUCTION

Form the simplest observation, we can say that the dynamics of a body, or of a system with more than one particle, can be modeled properly if collisions are taken into account. In the works of Galileu and Descartes there are references to the collision between particles, but the first model of this problem seems to be due to John Wallis and Christopher Wren, independently, in 1668. Some great scientists such as Newton, Huygens, Coriolis, Darboux, Routh, Apple, Carnot and Poisson have also treated the problem. At the beginning of this century the problem generated some discussions, as we can see in the works of Painlevé (1905) and Klein (1910). But, up to 1984, all of these works used the theory developed by Newton or by Poisson and the dificulty was to include friction in the modelling, as was pointed out by Painlevé in his paper "Sur les lois de frottement de glissement".

In 1984, Kane (1984) published a work, in a journal with limited circulation, where he pointed out na apparent paradox: the application of Newton's theory with Coulomb's friction, universally accepted, in a problem of collisions of a double pendulum, conducted to generation of energy. What was wrong ?

In 1986, Keller (1986) presented a solution to Kane's paradox, but the solution was not easy to generalize. Keller's work was published in a journal with large circulation and arose widespread interest. In these thirteen years the interest has increased and there are some books totally dedicated to this topic, as the ones written by Glocker-Pfeiffer (1996), Brach(1991), Brogliato (1996) and Monteiro-Marques (1993).

Brach (1989) presented a model with linear equations containing some nondimensional parameters that characterize the collision and he defined "ratio between impulses" instead of coefficient of friction. However, his consideration did not give clear solutions to the problem when one considers *reverse sliding* during the collision. Stronge (1990) suggested a coefficient of restitution relating the energy during the compression phase to the energy during the expansion phase. Smith (1991,1992) presented a model with nonlinear equations. Wang-Mason (1992) applied the Routh's technique (1877) and compared the coefficients of restitution given by Newton and by Poisson. Sabine Durand (1996) studied the dynamics of systems with unilateral restirctions and included some ystems related to the collisions. Chatterjee (1997) presented new laws based in simple algorithms. He has not used many parameters and he obtained good results. Stoianovici and Hermuzlu (1996) have shown the limits of validity of some rigid bodies collision models. As their main interest was in Robotics, they focused in collisions of slender bodies at low velocities. Cathérine Cholet (1998) developed a new theory of rigid bodies collisions that satisfies the Principles of the Mechanics. Her work was based in he ideas introduced by Michel Frémond: a system formed by a set of rigid bodies is deformable because the relative positions between each pair of bodies vary. They discussed the theory and showed that it is coherent from the mathematical point of view and also experimentally validated.

2. MOTION EQUATIONS

The collision is modeled as instantaneous. we consider the generalized position of the system in the instant t defined by $q = (q_1, q_2, \ldots, q_n)^t$. We consider the contact between two bodies C_1 and C_2 and let R be the force of reaction exerted by C_1 on C_2. Then we write $R = (R_N \ R_T)^t$.
The dynamics of the system is given by the Lagrangean equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q_i} = Q_i + r_i \qquad (1)$$

with Q_i the contribution of the external generalized forces, r_i the generalized force due to the reaction in the contact and T the kinetic energy of the system. We should observe that r_i is only present when there is contact, otherwise it is null.

Considering only a planar situation, we have n parameters of position and two reactions in the contact (R_N and R_T) also unknown. Then, we need, not only the n equations obtained from Lagrange's equations but also two equations more, given by the collision laws that will be discussed later.

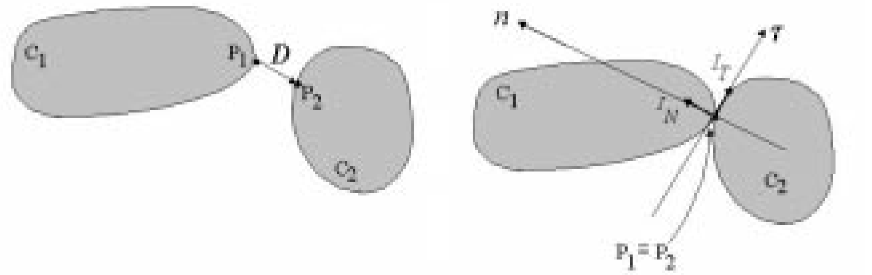

Figure 1. Collision between two bodies.

We consider P_1 and P_2 the points of the bodies C_1 and C_2, respectively, that will be in contact in the collision. We denote by $\mathbf{D}$ the vector that represents the relative displacement between the two bodies and by $\dot{\mathbf{D}}$ the vector that represents the relative velocity between the bodies, as shown in the Fig. (1).

In the point of contact we represent the impulses in the normal and tangential directions by I_N and I_T. We use and $\mathbf{u}_n$ and $\mathbf{u}_\tau$, the unitary vectors of the normal direction (given by n) and tangential direction (given by τ), in a frame which we will call collision frame, shown in Fig. (1).

Evaluating the relative velocity between the contact points we have,

$$\dot{D} = \sum_{i=1}^{n} \frac{\partial P_2}{\partial q_i}\dot{q}_i + \frac{\partial P_2}{\partial t} - \sum_{i=1}^{n} \frac{\partial P_1}{\partial q_i}\dot{q}_i - \frac{\partial P_1}{\partial t} = \sum_{i=1}^{n}\left(\frac{\partial P_2}{\partial q_i} - \frac{\partial P_1}{\partial q_i}\right)\dot{q}_i + \frac{\partial P_2}{\partial t} - \frac{\partial P_1}{\partial t}$$

We use the notations,

$$W_T^i = \left(\frac{\partial P_2}{\partial q_i} - \frac{\partial P_1}{\partial q_i}\right)\cdot\mathbf{u}_\tau \quad , \quad W_N^i = \left(\frac{\partial P_2}{\partial q_i} - \frac{\partial P_1}{\partial q_i}\right)\cdot\mathbf{u}_n \ , \quad \tilde{\omega}_T = \left(\frac{\partial P_2}{\partial q_i} - \frac{\partial P_1}{\partial q_i}\right)\cdot\mathbf{u}_\tau \quad , \quad \tilde{\omega}_N = \left(\frac{\partial P_2}{\partial q_i} - \frac{\partial P_1}{\partial q_i}\right)\cdot\mathbf{u}_n$$

We consider, then, $\mathbf{W}_T$ the column vector in which the components are W_T^i and $\mathbf{W}_N$ the column vector in which the components are W_N^i.
We can write the normal $(\dot{D}_N)$ and tangential $(\dot{D}_T)$ components of $\dot{D}$ as
$\dot{D}_N = \mathbf{W}_N^t\,\dot{q} + \tilde{\omega}_N$
and $\dot{D}_T = \mathbf{W}_N^t\,\dot{q} + \tilde{\omega}_T$.

Or we can write $\dot{\mathbf{D}} = [W]^t + \tilde{\omega}$, with

$$\dot{\mathbf{D}} = \begin{pmatrix}\dot{D}_N \\ \dot{D}_T\end{pmatrix}, \ [W]^t = \begin{pmatrix}\mathbf{W}_N^t \\ \mathbf{W}_T^t\end{pmatrix} \text{a matrix (2,n) and } \tilde{\boldsymbol{\omega}} = \begin{pmatrix}\tilde{\omega}_N \\ \tilde{\omega}_T\end{pmatrix}.$$

The generalized force $\mathbf{r}$ can be written as

$$\mathbf{r} = (\mathbf{W}_N \quad \mathbf{W}_T)\begin{pmatrix}R_N \\ R_T\end{pmatrix} \quad \text{or} \quad \mathbf{r} = [W]\mathbf{R}.$$

Integrating Eq. (1) in the interval $(t-\varepsilon, t+\varepsilon)$, with t the instant of collision and after some algebric manipulations we have the following equation

$$[M](\dot{\mathbf{q}}_E - \dot{\mathbf{q}}_A) = [W]\mathbf{I} = (\mathbf{W}_N \quad \mathbf{W}_T)\begin{pmatrix}I_N \\ I_T\end{pmatrix}. \qquad (2)$$

Our problem is to find $\dot{\mathbf{q}}_E$ and $\mathbf{I}$ given $[M]$, $[W]$ and $\dot{\mathbf{q}}_A$. Then, there are n equations and we want to find n+2 unknowns. Therefore, we need two more equations. These two equations are given by the restitution laws discussed later.

In some cases we can consider also an impulse of moment denoted by $\mathbf{I}_\theta$. In this case, the equation will be given by

$$[M](\dot{\mathbf{q}}_E - \dot{\mathbf{q}}_A) = [W]\mathbf{I} = (\mathbf{W}_N \quad \mathbf{W}_T \quad \mathbf{W}_\theta)\begin{pmatrix} I_N \\ I_T \\ I_\theta \end{pmatrix}. \tag{3}$$

In this case, we have n equations to find n+3 unknowns. We need three more equations. These three equations will be given by the restitution laws.

We construct a collision model when we join the n equations that describe the motion of the system with the equations given by the restitution laws.

In order to solve the problem we use a strategy that consists in defining a process called *virtual process*. It is not related to time. We show a scheme in the Fig. (2) to ilustrate this idea.

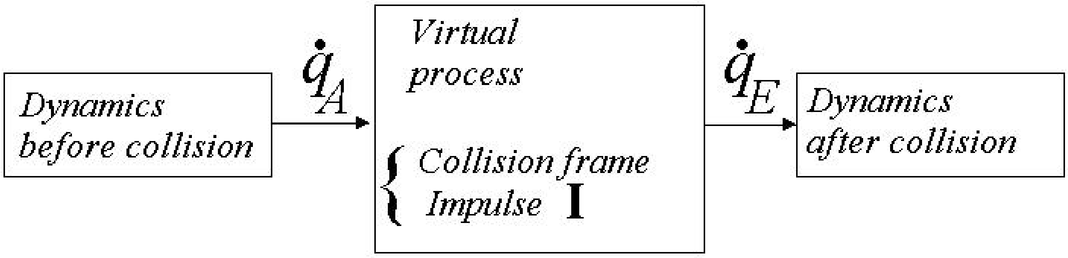

Figure 2. Virtual process sheme.

2.THE LOCAL MATRIX MASS

Instead of writing the equations in terms of $\dot{\mathbf{q}}$ we can use $\dot{\mathbf{D}}$. The vector $\mathbf{D}$ was shown in Fig. (1) and it is important because it monitors when the collision occurs.
We can write

$$\dot{\mathbf{D}} = [W]^t \dot{\mathbf{q}} + \tilde{\boldsymbol{\omega}} = \begin{pmatrix} W_N^t \\ W_T^t \end{pmatrix} \dot{\mathbf{q}} + \begin{pmatrix} \tilde{\omega}_N \\ \tilde{\omega}_T \end{pmatrix} \tag{4}$$

Then, $\dot{D}_E - \dot{D}_A = [W]^t [M]^{-1} [W] I = [\tilde{M}_L] I \Rightarrow I = [M_L](\dot{D}_E - \dot{D}_A)$

when $[\tilde{M}_L]$ is invertible and $[\tilde{M}_L]^{-1} = [M_L]$. We call $[M_L]$ the local matrix mass.

3. COMPRESSION PHASE AND EXPANSION PHASE

In order to describe some of the collision models we will think, formally, that the change between the pre-collision velocity to the post-collision velocity occurs in two phases: the compression phase and the expansion phase. The virtual process will be composed by these two phases as it is shown schematically in Fig. (3).

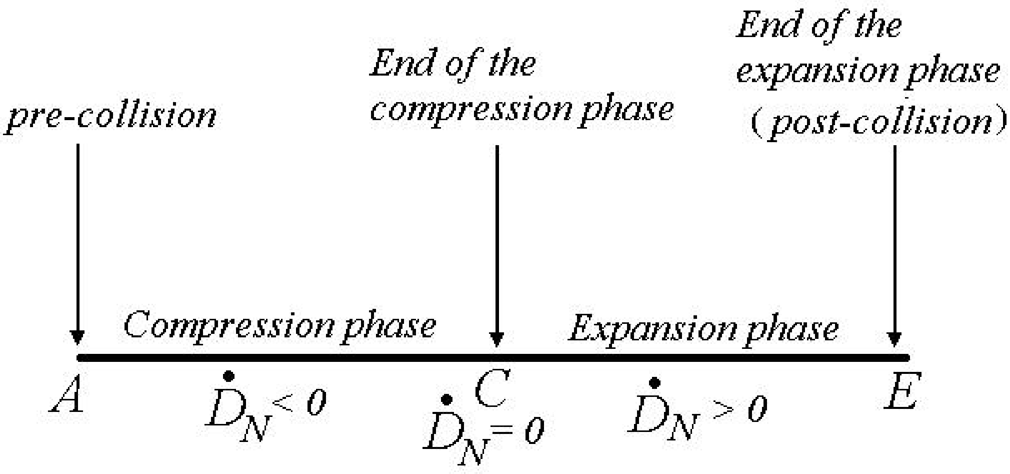

Figure 3. Compression phase and expansion phase.

4. A NEW COLLISON MODEL: THE C-S MODEL

We present a new collision model that tries to generalize some of the models from the literature and it also can predict some behavior that those models cannot.

The equations used are given in the following:

In the compression phase:

$$[M](\dot{\mathbf{q}}_C \quad \dot{\mathbf{q}}_A) = (\mathbf{W}_T \quad \mathbf{W}_N \quad \mathbf{W}_\theta)\begin{pmatrix} I_{NC} \\ I_{TC} \\ I_{\theta C} \end{pmatrix} \tag{5}$$

$$\begin{pmatrix} \dot{D}_{NC} \\ \dot{D}_{TC} \\ \dot{D}_{\theta C} \end{pmatrix} = \begin{pmatrix} \mathbf{W}_N^t \\ \mathbf{W}_T^t \\ \mathbf{W}_\theta^t \end{pmatrix} (\dot{\mathbf{q}}_C \quad \dot{\mathbf{q}}_A) + \begin{pmatrix} \dot{D}_{NA} \\ \dot{D}_{TA} \\ \dot{D}_{\theta A} \end{pmatrix} \tag{6}$$

To the restitution in the tangential direction we use the Coulomb's law in the form

$$\begin{cases} |I_{TC}| < \mu I_{NC} \Rightarrow \dot{D}_{TC} = 0 \\ I_{TC} = \mu I_{NC} \Rightarrow \dot{D}_{TC} \le 0 \\ I_{TC} = -\mu I_{NC} \Rightarrow \dot{D}_{TC} \ge 0 \end{cases} \tag{7}$$

We use the coefficient of moment in the compression phase given by

$$e_{mC} I_{\theta C} = -(1 + e_{mC})\,\bar{J}\,\dot{D}_{\theta C} \tag{8}$$

$$\bar{J} = \frac{J_1 J_2}{J_1 + J_2} \tag{9}$$

with J_1 and J_2 the moment of inertia related to the center of mass of each body.

In the expansion phae:

$$[M](\dot{\mathbf{q}}_E \quad \dot{\mathbf{q}}_C) = (\mathbf{W}_T \quad \mathbf{W}_N \quad \mathbf{W}_\theta)\begin{pmatrix} I_{NE} \\ I_{TE} \\ I_{\theta E} \end{pmatrix} \tag{10}$$

$$\begin{pmatrix} \dot{D}_{NE} \\ \dot{D}_{TE} \\ \dot{D}_{\theta E} \end{pmatrix} = \begin{pmatrix} \mathbf{W}_N^t \\ \mathbf{W}_T^t \\ \mathbf{W}_\theta^t \end{pmatrix} (\dot{\mathbf{q}}_E \quad \dot{\mathbf{q}}_C) + \begin{pmatrix} \dot{D}_{NC} \\ \dot{D}_{TC} \\ \dot{D}_{\theta C} \end{pmatrix} \tag{11}$$

We use the coefficient of moment in the compression phase given by

$$e_{mE} I_{\theta E} = -(1 + e_{mE}) \bar{J} \dot{D}_{\theta E} \tag{12}$$

$$\bar{J} = \frac{J_1 J_2}{J_1 + J_2} \tag{13}$$

To the tangential restitution we consider

$$I_{TS} = \frac{1}{2}[\mu \nu I_{NE} \mathrm{sign}(I_{TC}) + e_n e_t I_{TC}] \ , \ 0 \le \nu \ , \ e_t \le 1 \tag{14}$$

and then we use

If $\quad I_{TC} \ge 0 \ , \ I_{TS} \ge 0 \Rightarrow -\mu I_{NE} + 2 I_{TS} \le I_{TE} \le \mu I_{NE}$

$$\begin{cases} -\mu I_{NE} + 2 I_{TS} < I_{TE} < \mu I_{NE} \Rightarrow \dot{D}_{TE} = 0 \\ I_{TE} = +\mu I_{NE} \Rightarrow \dot{D}_{TE} \le 0 \\ I_{TE} = -\mu I_{NE} + 2 I_{Ts} \Rightarrow \dot{D}_{TE} \ge 0 \end{cases}$$

If $\quad I_{TC} \le 0 \ , \ I_{TS} \le 0 \Rightarrow -\mu I_{NE} \le I_{TE} \le \mu I_{NE} + 2 I_{TS}$

To the normal restitution we use the Poisson's coefficient given by $e_{np} = \frac{I_{NE}}{I_{NC}}$.

I_{NC} is the normal impulse in the of the compression phase and I_{NE} is the normal impulse in the end of the expansion phase.

4. PARTICULAR CASES

As we had said, the C-S model generalizes some of the models from the literature. We will describe briefly three of these models and we will show what should be done, in the C-S model, to particularize the respective model.

4.1. Newton's model

This is the simplest model. It considers the coefficient of restitution given by Newton; that is, $e_n = -\frac{\dot{D}_{NE}}{\dot{D}_{NA}}$.

It does not consider friction (there will be notangential restitution) and it does not consider the impulse of moment. If in the C-S model we consider $e_{mC} = e_{mE} = -1$, $e_t = \nu = 0$ and $\mu = 0$ then we will obtain the Newton's model. It is important to observe that when we do not consider friction the Newton's coefficient of restitution and the Poisson's coefficient are equivalent. It can be seen in Cataldo and Sampaio(1999,2000).

4.2. Wang-Mason's model (considering the Poisson's coefficient of restitution)

This model uses the coefficient of restitution in the normal direction given by Poisson; that is, $e_{np} = \frac{I_{NE}}{I_{NC}}$. It considers friction given by Coulomb's law (modified, as we have presented) and it does not consider the impulse of moment.. If in the C-S model we consider $e_{mC} = e_{mE} = -1$ and $e_t = \nu = 0$ then we will obtain Wang-Mason's model.

4.3. Glocker-Pfeiffer's model

This model uses the coefficient of restitution in the normal direction given by Poisson. It considers friction and also the possible reversible portions of the tangential impulse. It does not consider the impulse of moment. If in the C-S model we consider $e_{mC} = e_{mE} = -1$ then we will obtain the Glocker-Pfeiffer's model.

4. NEW RESULTS USING THE C-S MODEL

Using the C-S model we can observe some behaviors that couldn't be described using other models. As na example we consider the problem of a ball coliding with two barriers as shown in the Fig. (4).

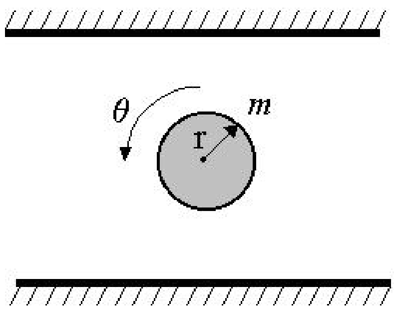

Figure 4. Collision of a ball with two barriers.

We consider the following values to the parameters and inital conditions: mass of the ball=1kg, $\theta_0 = 0$, $\dot{\theta}_0 = 0$, $x_0 = 0$, $\dot{x}_0 = 1$, $y_0 = 0.9$ m, $\dot{y}_0 = -1$ m/s, $e_{np} = 1$, distance between the barriers=1.01m, $\mu = 1$ and $r = 0.1$. If we consider $e_{mC} = -1$, $e_{mE} = -1$, $e_t = 0$ and $\nu = 0$ we obtain the same prediction obtained from Wang-Mason's model or Glocker-Pfeiffer's model. It is the behavior of a ball used, for example, in a table tennis game as shown in Fig. (5).

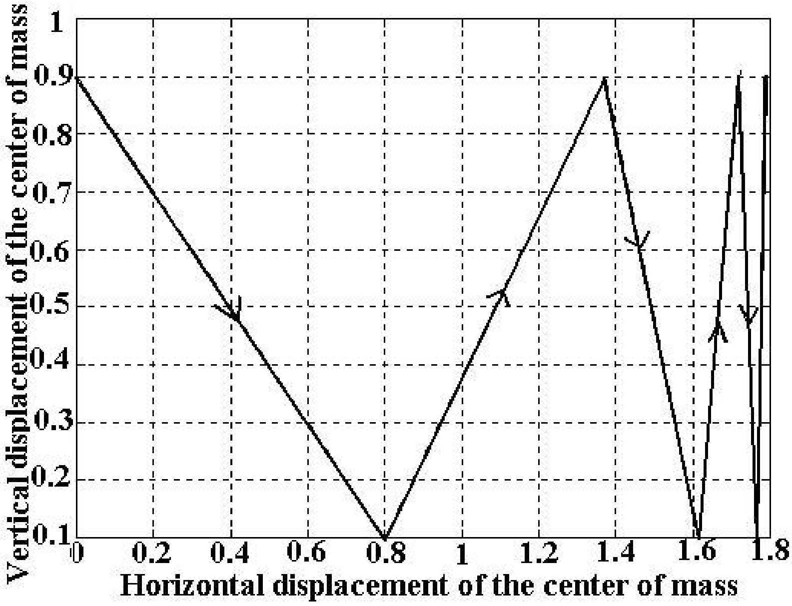

Figure 5. The C-S model using $e_{mC} = e_{mE} = -1$, $e_t = 0$ and $\nu = 0$. Units (m).

If we consider $e_{mC} = e_{mE} = -1$, $e_t = 1$ and $\nu = 1$ we obtain the same prediction obtained from Glocker-Pfeiffer's model in the case called superball-like behavior. It is the behavior of a ball made of steel and not hollow. Its trajectory is shown in the Fig. (6).

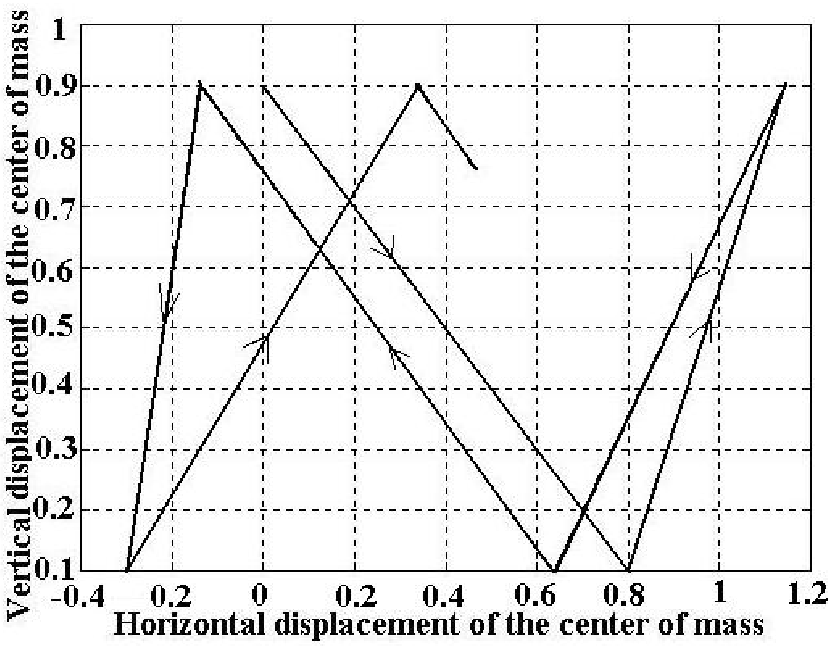

Figure 6. The C-S model using $e_{mC} = e_{mE} = -1$, $e_t = 1$ and $\nu = 1$. Units: m.

If we consider $e_{mC} = -1$, $e_{mE} = -0.5$, $e_t = 1$ and $\nu = 1$ we obtain a new behavior that couldn't be observed if we had used other models. We show the trajectory of the center of mass in the Fig. (7).

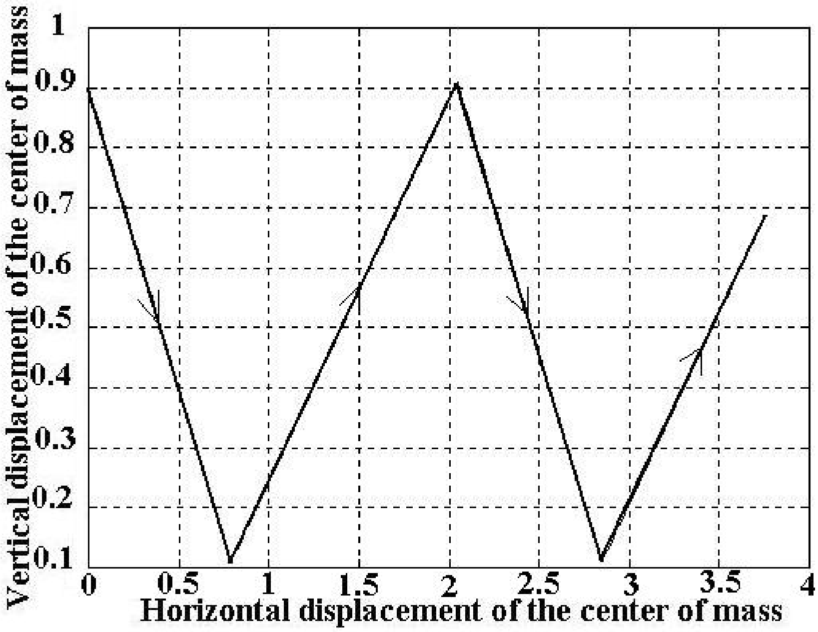

Figure 7. The C-S model using $e_{mC} = e_{mE} = -1$, $e_t = 1$ and $\nu = 1$. Units (m).

5. CONCLUSIONS

We studied rigid body collisions considering that these collisions are instantaneous. After making a systematic study of some rigid body collisions models we could formulate a new model: the C-S model. Using this model we could show some comparisons between models and we could present some behaviors that couldn't be obtained using other models. We showed simulations and animations using the C-S model in a way that would make us understand what was happening.

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