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SOLUTION OF HEAT CONDUCTION PROBLEMS VIA THE GENERALIZED INTEGRAL TRANSFORM TECHNIQUE WITH DOMAIN CHARACTERIZATION THROUGH THE INDICATOR FUNCTION

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Abstract. *In this work, the use of the Generalized Integral Transform Technique (GITT), a hybrid numerical-analytical method for convection-diffusion problems, in conjunction with the Indicator Function is proposed for the solution of heat conduction problems in irregular geometries. The advantages of such approach are the error control solution and no needing of grid generation. The methodology is described and some previous results are presented. GITT is applied to a two-dimensional heat conduction problem in an irregular geometry, inside a square domain, where the Indicator Function is employed to characterize each phase or region in the problem and is built also through the use of the Integral Transform. The methodology presented here can be extended to all brands of convection-diffusion problems already solved via GITT and to any irregular geometry surrounded by a continuous border, and applied when more than one phase is present.*

Keywords: *Integral Transforms, Heat Conduction, Irregular Domain, Indicator Function*

1. INTRODUCTION

The Generalized Integral Transform Technique (GITT) has appeared in the literature (Cotta, 1993) as an alternative to conventional discrete numerical methods for partial differential equations in heat and fluid flow. Its hybrid numerical–analytical structure permits the automatic control of the global error in the simulation, which avoids the need for several computer program runs to inspect for the convergence on the final results, yielding codes that automatically work towards user prescribed accuracy targets. This method is also easy to program, since there is no need for a discretization mesh and its adaptive refinement according to the potential field and physical situation to be calculated.

The method has being constantly improved in order to solve convection-diffusion problems with increasing complexity, like the solution of the Navier-Stokes equations in internal flows inside irregular geometries (Monteiro et al, 2004, Silva et al, 2004), new eigenvalue problems with optimization of the transformed potentials (Guigon et al, 2004) and other successfully examples. However, there still a vast number of practical problems that has not being solved satisfactory by the method, considering the strong concurrence of the traditional discrete methods. One of the main difficult in GITT use for engineering problems is the need for previous algebraic treatment of the equations, which restricts its popularization, in spite of the advantages presented concerned to the error control and no mesh generation. As a consequence, some classes of problems have not being studied with this method yet, like two-dimensional phase change, complex two-phase flow and irregular domains, etc.

In several brands of engineering, the transport equations have to be solved for a combination of different phases or materials, or considering the presence of discrete source terms. In this case, the simple application of the well-known discrete numerical methods demands some specific treatment: the discontinuity must be isolated via coordinate or domain transformation or grid refinement. Both are complex numerical processes, and add some residual error to final results. In this case, the mathematical resource of the Indicator function, as defined in the Interface Tracking Method of Unverdi and Tryggvason (1992) and Juric and Tryggvason (1998) can be employed. This function is a representation of the phases or parts of the domain with the numbers 0 and 1 for each phase (in this point it is supposed exists only two phases, however this method can be used in presence of n-phases).

In this work, the use of the GITT in conjunction with the Indicator Function is proposed. The methodology is described and some previous results are presented. GITT is applied to a two-dimensional heat conduction problem in a square domain.

2. FORMULATION

Consider a square domain with unitary side where a pure conduction heat transfer occurs. The energy equation with all variables in dimensionless form will be:

$$\frac{\partial T}{\partial t} = \nabla.[\alpha(x,y).\nabla T] + Q(x,y,t) \tag{1}$$

Where Q is a source term and $\alpha(x,y)$ is the thermal diffusivity of both regions. This property is calculated using the Indicator Function – $I(x,y)$, and is given as $\alpha(x,y) = \alpha_0 + (\alpha_1 - \alpha_0)\, I(x,y)$; α_0 is the thermal diffusivity for the region where $I = 0$ and α_1 for the region where $I = 1$.

Consider also prescribed wall temperatures and initial condition equal to zero. In order to apply GITT, an associated eigenvalue problem is chosen, that has similar boundary conditions. This problem will be the Sturm-Liouville problem:

$$\Gamma_i'' = -\lambda_i^2 \Gamma_i \qquad \text{for} \qquad 0 \le x \le 1 \text{ or } 0 \le y \le 1 \tag{2.a}$$

with an analytical solution known for the eigenfunctions $\Gamma_i(x)$, eigenvalues λ_i and norms N_i. The boundary condition used is prescribed wall temperature, Eq. (2.b).

$$\Gamma_i(0) = 0 \;; \qquad \Gamma_i(1) = 0 \tag{2.b}$$

The solution allows knowing the spatial dependency of the solution analytically. Since for this case the boundary conditions are the same in the whole border, the same eingenfunctions are applied in both directions, *x* and *y*.

Next step is to define the Transformed-Inverse pair:

$$\Theta_{ij}(t) = \int_0^1 \int_0^1 \overline{\Gamma}_i(x)\overline{\Gamma}_j(y)T(x,y,t)dydx \tag{3.a}$$

$$T(x,y,t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \overline{\Gamma}_i(x)\overline{\Gamma}_j(y)\Theta_{ij}(t) \tag{3.b}$$

where $\overline{\Gamma}_i$ is the normalized eigenfunction:

$$\overline{\Gamma}_i = \Gamma_i / N_i^{1/2} \tag{4}$$

Applying the integral operator $\int_0^1\int_0^1 \overline{\Gamma}_i(x)\overline{\Gamma}_j(y)\_dydx$ over Eq. (1), substituting T by its definition in the Eq. (3.b) and taking advantage of the ortogonality of the eigenfunction, according to Mikhailov and Özisik (1984), one obtain:

$$\frac{d\Theta_{ij}(t)}{dt} = -\alpha_0(\lambda_i^2 + \lambda_j^2)\Theta_{ij}(t) + \int_0^1\int_0^1 \overline{\Gamma}_i(x)\overline{\Gamma}_j(y)[(\alpha_1 - \alpha_0).I(x,y).\nabla^2 T + Q(x,y,t)]dydx \tag{5}$$

The infinite summation will be truncated to an order NT high enough to assure the precision target. Initial conditions for transformed potentials are obtained from the transformation of the original ones.

The finite system of coupled ordinary differential equations is represented as a first order matrix system $d\boldsymbol{Y} / dt = \boldsymbol{D}(\boldsymbol{Y},t)$, where the vector $\boldsymbol{Y}$ represents the transformed potentials Θij $(i,j=1,2,...,NT)$. According to the shape of the source term Q or variation of thermal conductivity α, the system allows straight analytical solutions. This is not the case, since the spatial dependence of both terms is arbitrary, and the system is coupled.

This system is stiff, where the frequency of each solution is quite different, and has to be solved by specific computational libraries, like DIVPAG, from IMSL library (1989), based on the Gear's method. Once the transformed potentials are obtained for each time in a marching process, the original potential can be recovered through Eq. (3.b), since spatial dependence is analytically known.

The Indicator Function is defined by Poisson's equation:

$$\nabla^2 I = \nabla \boldsymbol{G} \tag{6}$$

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Vector $\boldsymbol{G}$ is the distribution of interface over the domain, and is given as:

$$\boldsymbol{G} = \int_S \boldsymbol{n}\delta(\, \mathbf{x} - \mathbf{x}_i \,)ds \tag{7}$$

where $\boldsymbol{n}$ is the normal vector along the interface, δ is the Dirac function, which is nonzero only when $\boldsymbol{x} = \boldsymbol{x_f}$ (subscript f denotes points along interface). The integral over s is done along the curve that defines the interface.

The solution of Eq. (6) is done using the common GITT approach. Function $I(x,y)$ is expanded in the Transformed-Inverse pair, as temperature in Eq. (3). The eigenfunctions $\Phi_i(\, x \, or \, y \,)$ for $I(x,y)$ are obtained from the same auxiliary problem as temperature, Eq. (2.a,c), using zero flux as boundary condition in both directions. Next step is apply the integral operator $\int_0^1 \int_0^1 \overline{\Phi}_i(\, x \,)\overline{\Phi}_j(\, y \,)\_dydx$ and again replacing I(x,y) by its expansion, yielding:

$$-(\, \phi_i^2 + \phi_j^2 \,).\bar{I}_{ij} = \int_0^1 \int_0^1 \overline{\Phi}_i(\, x \,)\overline{\Phi}_j(\, y \,)\nabla.\boldsymbol{G}.dxdy \tag{8}$$

where $\phi_{i \; or \; j}$ are the eigenvalues for each direction. From Eq. (6) and after some manipulation, including the use of the chain rule, we finally obtain an analytically expression for the transformed potential of the Indicator Function:

$$-(\, \phi_i^2 + \phi_j^2 \,).\bar{I}_{ij} = -\int_0^p [\, n_x(\, p \,).\Phi_i^{'}(\, x \,).\Phi_j(\, y \,) + n_y(\, p \,).\Phi_j^{'}(\, y \,).\Phi_i(\, x \,)]\, dp \tag{9}$$

Equation (9) provides an analytical expression for each transformed potential of the indicator function. The integral at the right hand side is done over the parameter p, that commonly can be considered the length along the curve which defines the interface, and n_x, n_y are the components of the normal vector. The length is calculated defining a beginning for the curve. One should observe that the curve must be closed, and the two regions have to be completely separated. Once the transformed potentials are known, the Indicator Function can be analytically operated, and the interface can be represented by an analytical continuous function, in a similar form as a Fourier series. The interface will have a thin thickness, which will be smaller as a higher number of terms are used in the expansion. Within this thickness, $I(x,y)$ will present values varying from 0 to 1.

3. MATHEMATICAL MODEL

The domain of calculation is showed in Fig.1. The square inside the domain corresponds to the physical domain, where the determination of the temperature field will be performed. The coordinates *x'* and *y'* will be used to compare the results for temperature profile with the ones obtained from the Finite Volume Method. A prescribed heat flux is applied in one of the faces *(x' = 0)* while the others are kept thermally insulated, which means the problem is one-dimensional in the *x'* direction. Values assumed for α were 1 inside the internal square and 0 outside it. The square side was $L = 0.5$ and the imposed heat flux was $q = 10$.

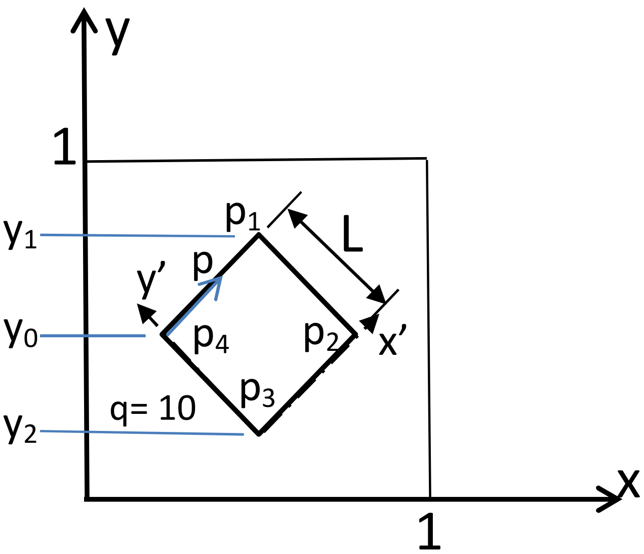

Figure 1. Schematics of the calculation and physical domain used.

The boundary heat flux at x' = *0*, is represented by a source term, employing the Dirac's Delta function:

$$Q(x,y,t) = \int_0^p q(p)\delta(x - x_f)\delta(y - y_f)dp \tag{10}$$

This integral assures that the source term will exist only over the interface between the internal square ant the external domain. The internal square is described through the combination of two functions:

$$y = \begin{cases} f_1 & for\ y_0 < y < y_1 \\ f_2 & for\ y_0 < y < y_2 \end{cases} \tag{11.a}$$

$$f_1 = \begin{cases} x(p) + b_1 & for\ 0 < p < p_1 \\ -x(p) + b_2 & for\ p_1 < p < p_2 \end{cases}$$
$$f_2 = \begin{cases} -x(p) + b_3 & for\ p_2 < p < p_3 \\ x(p) + b_4 & for\ p_3 < p < p_4 \end{cases} \tag{11.b}$$

According to this definition, the boundary heat flux can be expressed as:

$$q(p) = \begin{cases} 10 & for\ 0 < p < p_1 \\ 0 & for\ p_1 < p < p_4 \end{cases} \tag{12}$$

4. RESULTS

Applying the integral transform to the Poisson's Equation, the distribution of the indicator function is obtained for the whole domains of calculation. In order to analyze the convergence of the solution, different values for the number of terms of the expansion were used. Figures 2-4 shows the distribution of *I(x,y)* for 3 values of *n* (number of terms used).

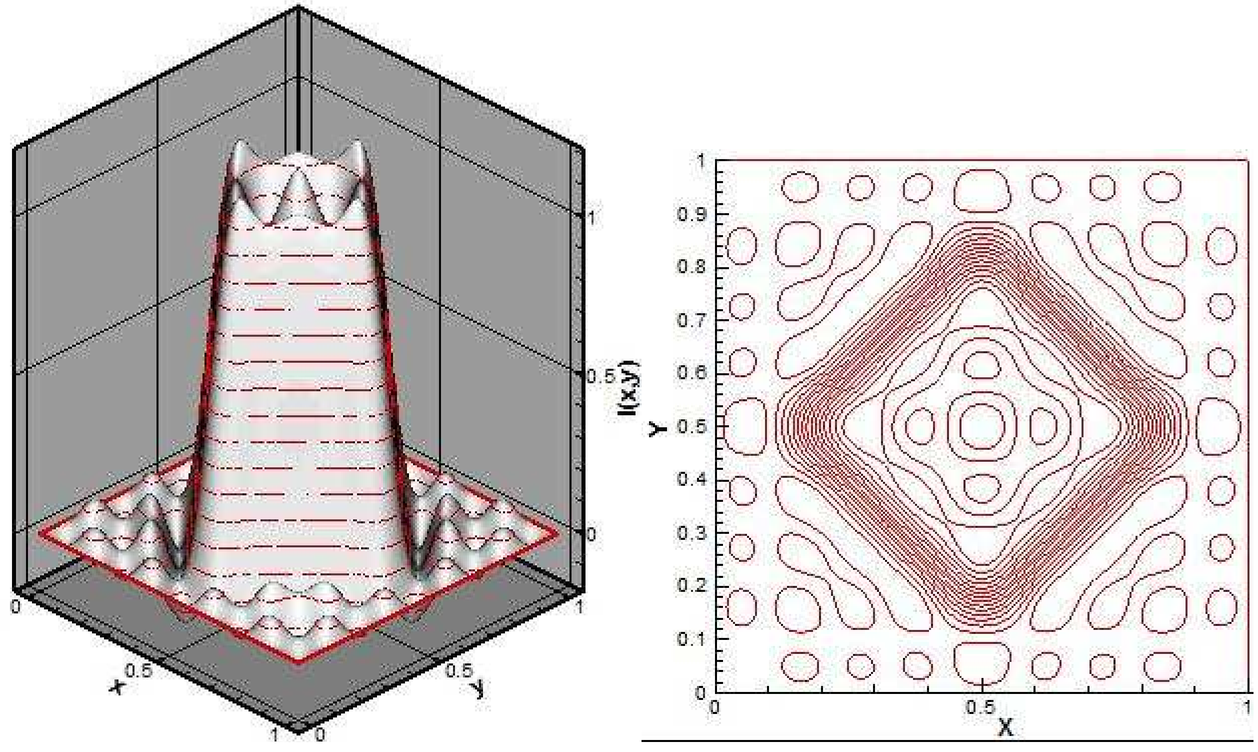

Figure 2. Distribution of Indicator Function in the domain of calculation, 3-D and 2-D views, for n = 10.

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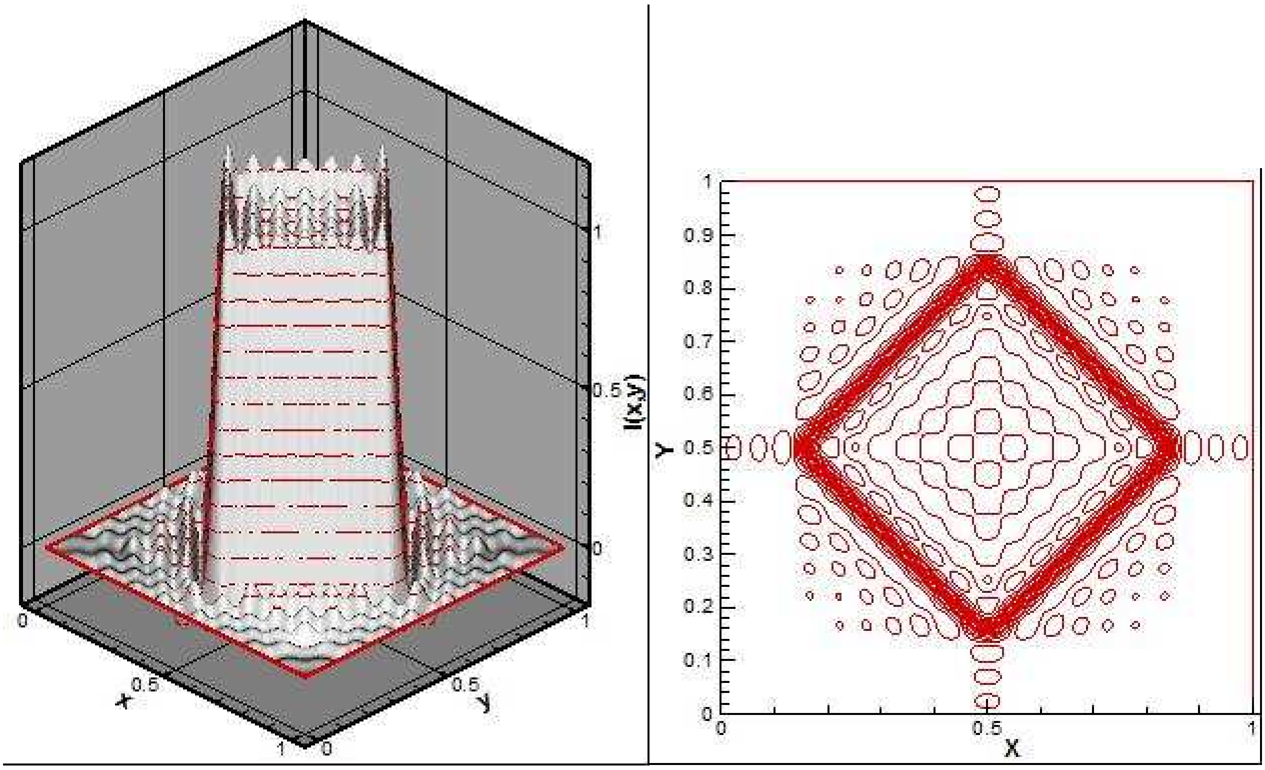

Figure 3. Distribution of Indicator Function in the domain of calculation, 3-D and 2-D views, for n = 20.

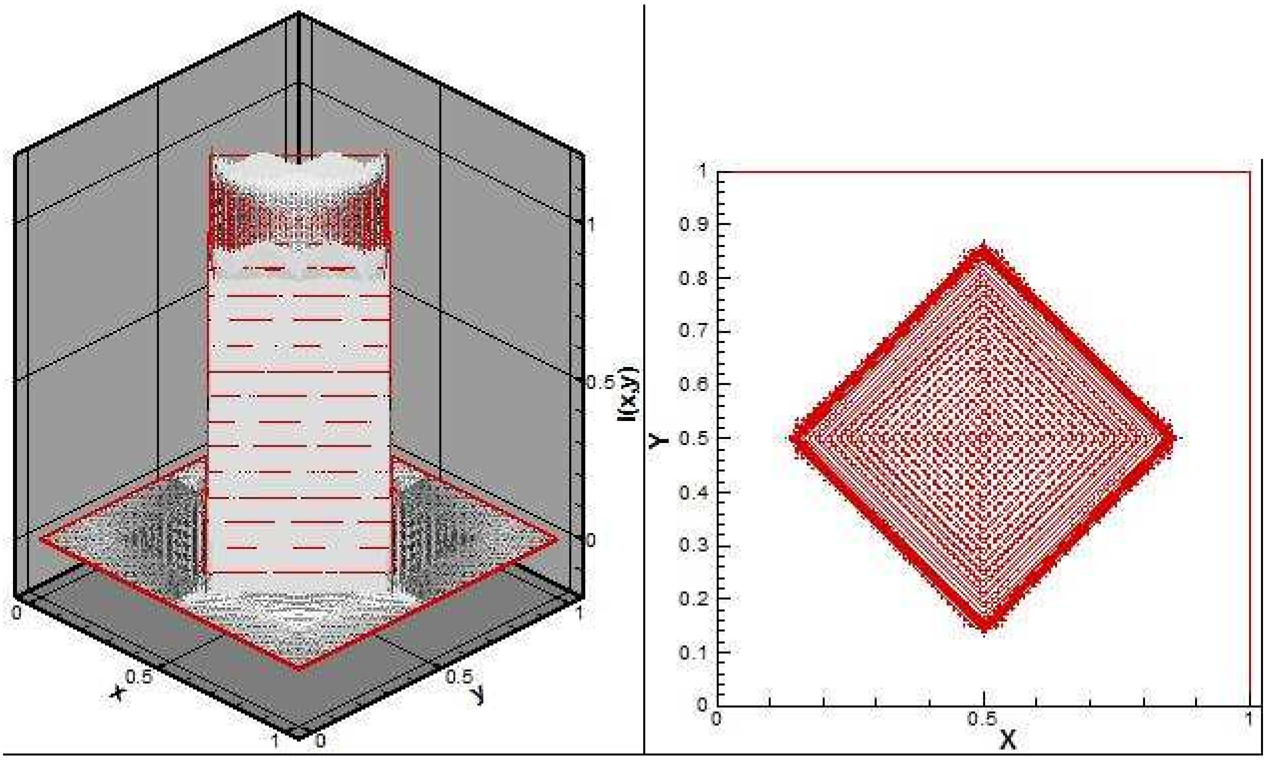

Figure 4. Distribution of Indicator Function in the domain of calculation, 3-D and 2-D views, for n = 100.

The sequence shows the thickness reduction of the interface region with the rising in the number of terms of the expansion. Figure 5 shows the direct comparison, extracted toward the diagonal of the internal square ($y = 0.5$). In this case, behind the reduction in the interface thickness (what corresponds in a reduction in the slope of the interface), the

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oscillation amplitude also reduces. However, even with 100 terms, the peaks in the interface are still high, compared to the rest of the domains. It seems that the derivatives in the interface region do not present a good precision.

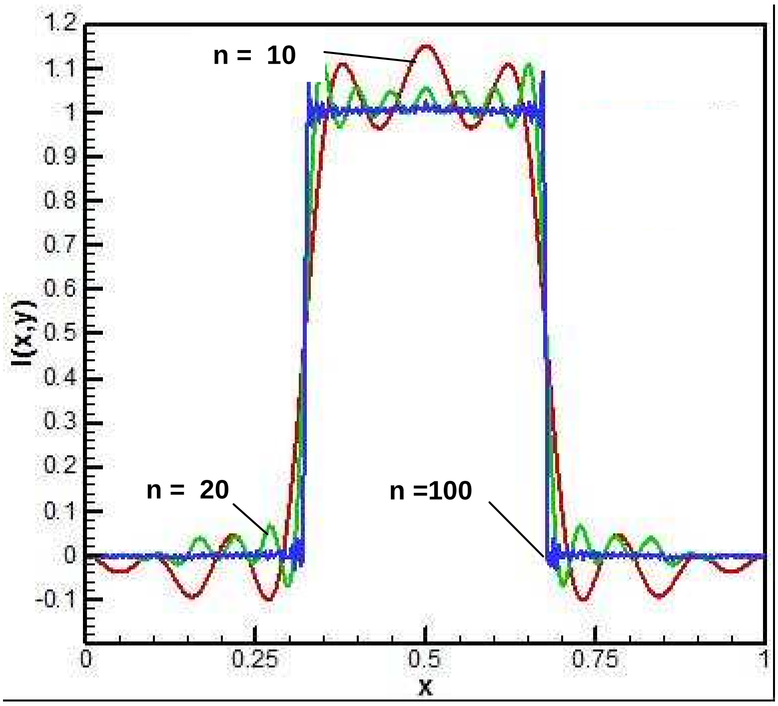

Figure 5. Convergence of Indicator Function, for y = 0.5.

The method of solution is applied to the heat conduction problem, for n = 10, 20 and 30. Results for temperature distribution are showed in Figs.6-8. The oscillation peak in the boundary where the heat flux is imposed becomes higher as the number of terms increases.

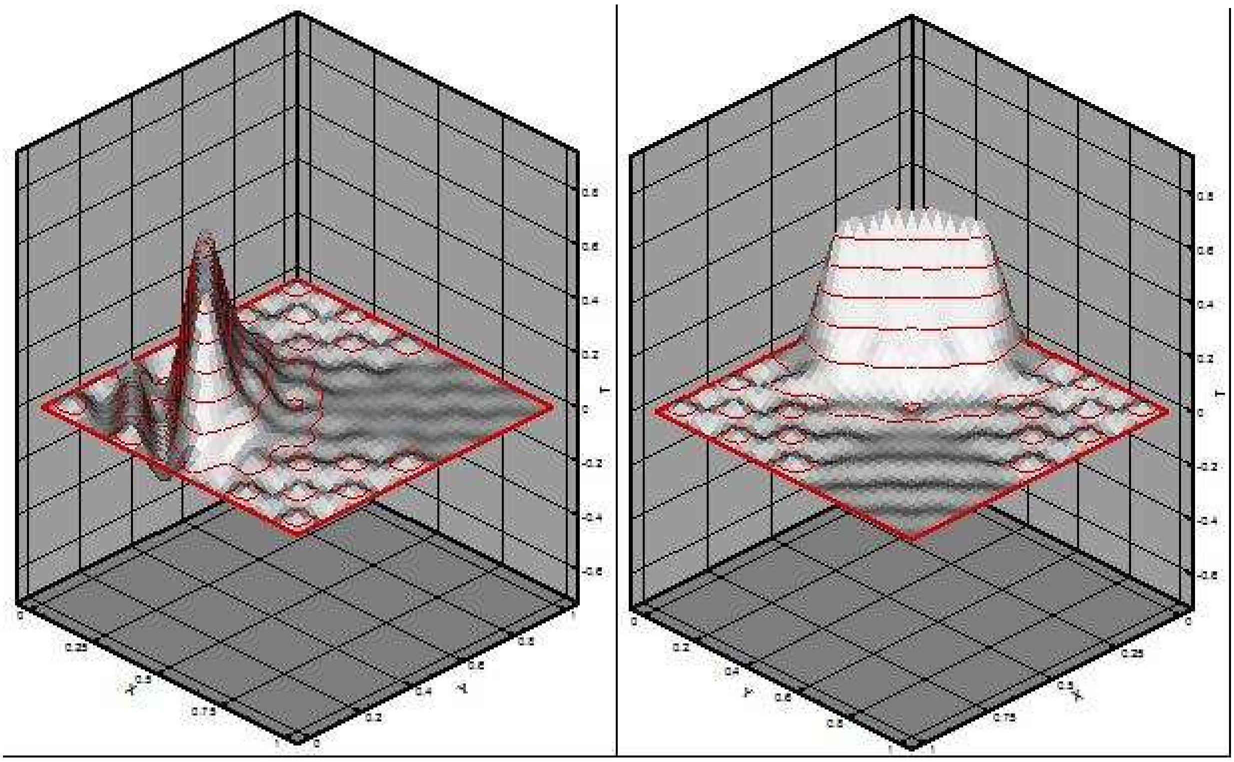

Figure 6. Temperature distribution for t = 0.01, n =10.

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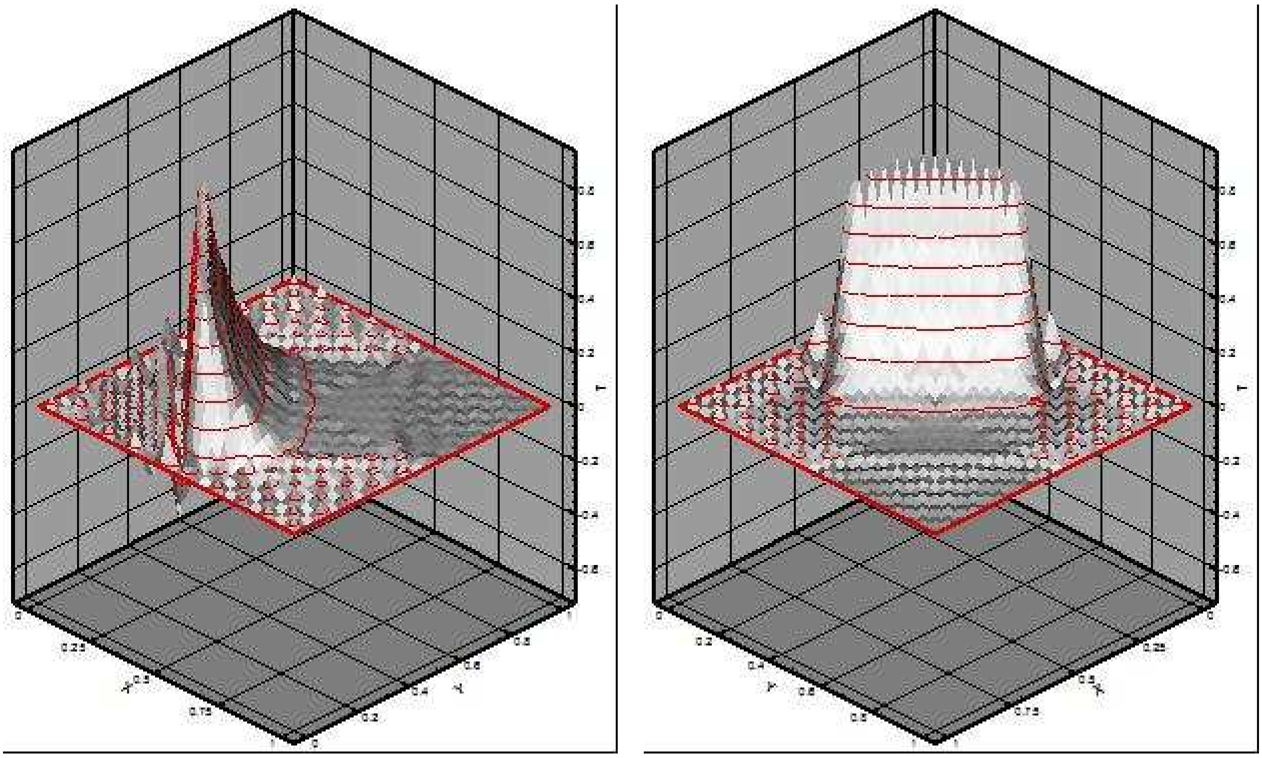

Figure 7. Temperature distribution for t = 0.01, n =20.

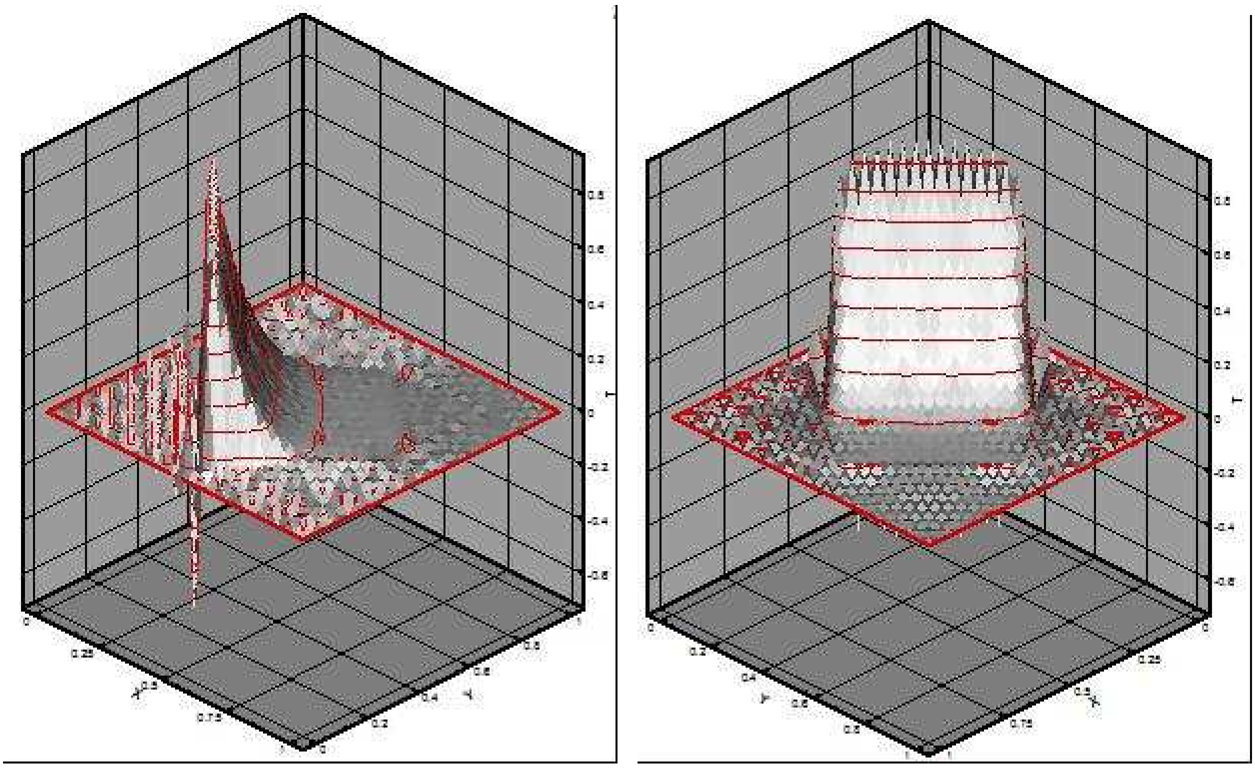

Figure 8. Temperature distribution for t = 0.01, n =30.

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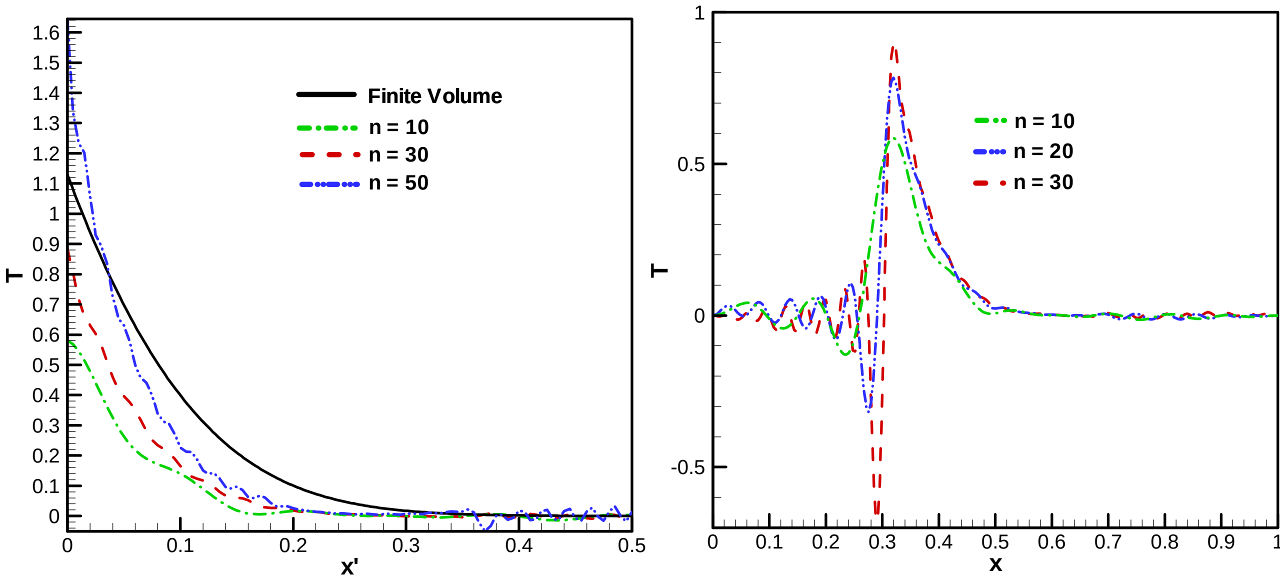
In Figure 9, the convergence of the GITT solution is showed in the *x'* and *x* coordinates at t = 0.01. In Figure 9.a, the solution is compared with results obtained through the Finite Volume Method (FVM) for the same problem. Although the results for GITT are still far from the right solution obtained via FVM, it seems to be toward this result, and presents the same behavior. In Figure 9.b, the lack of convergence and the effect of the oscillation are clear in the side where the heat flux is imposed.



(a) y' = 0.5 (b) y = 0.5

Figure 9. Convergence of temperature, for t = 0.01.

5. CONCLUSION

In this work, the GITT was employed to solve a pure heat conduction problem with prescribed heat flux in an irregular domain, represented through the use of the indicator function. The domain was well represented by the indicator function field, obtained through the integral transform. However, the solution for the temperature distribution was still far from a good result, when compared to the results obtained from the FVM. Although a good agreement was not obtained for the GITT, the qualitative behavior was considered in agreement with the right solution. An improvement in the convergence and the reduction with the oscillation would result in a very reliable method for heat conduction problems in irregular domains, considering the advantages in its implementation.

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