

## MODELING AND LQR CONTROL FOR A SMALL COAXIAL HELICOPTER

**Juan C. Rivera Medina**

**Fernando A. N. Castro Pinto**

COPPE - Universidade Federal do Rio de Janeiro

secexpem@coppe.ufrj.br

**Abstract.** *The control of a helicopter is a complicated task, due to the nonlinearities in his actuators and also to the presence of environmental effects that makes the system unstable. Because of these instabilities, it is necessary to use a control algorithm that helps to improve the systems performance, ensuring stability and robustness. The objective of this work is to develop a control algorithm that allows the helicopter to maintain in hover flight, controlling its attitude and altitude. In order to achieve this objective, the project was divided in three steps. In the first step, a non-linear dynamical model was developed, which describes the total behavior of the helicopter. Using aerodynamic constants from experimental data of wind tunnels were calculated parameters such as drag coefficient, thrust force, center of gravity and helicopter weight. In the second step was calculated the control law applying a LQR control strategy in order to stabilize the attitude and altitude of the helicopter. Then some simulations were made to verify the response of the linear controller and the no linear dynamic model of the helicopter, using a continuous time model. In the final step, a simulation was developed using the discretized linear controller, and the responses were compared with the continues controller model. Finally the discretized controller was implemented in an electronic board located on the helicopter. The results are presented in the simulations of the controller-system continues model, then comparisons were made with respect to the responses of the discretized controllers and the real helicopters controller.*

**Keywords:** *Helicopter Dynamics, LQR Control, Optimal Control, Coaxial Helicopter*

### 1. INTRODUCTION

Helicopters are a useful equipments because of their ability to make take-offs and landings without the use of a big infrastructure. This work will focus on the theory to develop a controller for stabilizing a coaxial helicopter. This helicopter is composed of three rotors as seen in figure 1. Two working as main rotors and a third one in the tail. The two main rotors, rotate in opposite direction to compensate the torques generated from each other. The third rotor can rotate in both directions and generates a little thrust so the helicopter can tilt on its pitch.

This document will first describe the dynamics of the helicopter, then it will summarize the LQR control design and its discretization, and finally will show some of the result of the performance of the helicopter.

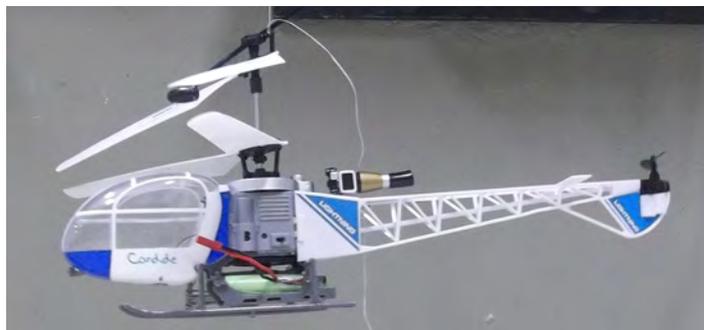


Figure 1. Helicopter

### 2. DYNAMICS

The dynamics of the helicopter are going to be affected by the behavior of the main rotor, because is who generates most of the forces and torques over the helicopter. In this work each of the contributions of the forces will be described. To calculate the dynamics of the helicopter it is necessary to define the Frames that will be used.

Three frames were defined, two on the *COG* (Center of Gravity) of the helicopter, call it the Body frame (BF) and the Spacial frame (SF) and the third one as the Inertial frame (IF). The BF will be oriented equal as the IF, and the SF will be

oriented with the movements of the helicopter.

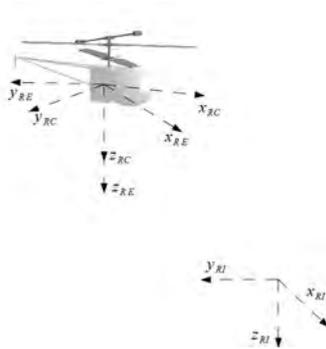


Figure 2. Frames on Helicopter Dynamics

## 2.1 Rigid Body Equations

Considering the helicopter as a rigid body it is possible to apply Newton's Second law, to calculated velocities and accelerations (Ulrik Bech Hald, 2005). To define each of the acceleration and velocities it is necessary to establish the relation between the frames, so in equation 1 it is described the relation between the BF and the IF, based on the rotation matrices shown in Bruno Siciliano (2009).

$$R_{B,I}(\phi, \theta, \psi) = \begin{pmatrix} \cos(\theta)\cos(\psi) & -\cos(\theta)\sin(\psi) & \sin(\theta) \\ \sin(\phi)\sin(\theta)\cos(\psi) + \cos(\phi)\sin(\psi) & -\sin(\phi)\sin(\theta)\sin(\psi) + \cos(\phi)\cos(\psi) & -\sin(\phi)\cos(\theta) \\ -\cos(\phi)\sin(\theta)\cos(\psi) + \sin(\phi)\sin(\psi) & \cos(\phi)\sin(\theta)\sin(\psi) + \sin(\phi)\cos(\psi) & \cos(\psi)\cos(\theta) \end{pmatrix} \quad (1)$$

Another important relation is between the euler angles rate and the angular velocity, in equation 2, that will gives the orientation of the helicopter at all time. This equation will introduce some limitation to the model because to the singularity when the pitch angle is  $\pi/2$  or  $-\pi/2$ .

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 1 & \sin(\phi)\tan(\theta) & \cos(\phi)\tan(\theta) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \frac{-\sin(\phi)}{\cos(\theta)} & \frac{\cos(\phi)}{\cos(\theta)} \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} \quad (2)$$

### 2.1.1 Accelerations

The translational in equation 4 and rotational accelerations in equation 3 are described both on the BF. The first will depend of the Forces acting on the helicopter due to the main rotors thrust and external forces and parameters as the mass  $m$ . The rotational accelerations will depend of the torques and parameters as the Inertial Momentum.

$$\dot{w} = \begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{q \cdot (I_{yy} - I_{zz}) \cdot r + L}{I_{xx}} \\ \frac{-p \cdot (I_{xx} - I_{zz}) \cdot r + M}{I_{yy}} \\ \frac{N + p \cdot q \cdot (I_{xx} - I_{yy})}{I_{zz}} \end{pmatrix} \quad (3)$$

$$\dot{V} = \begin{pmatrix} r \cdot v - q \cdot w + \frac{F_x}{m} \\ -r \cdot u + p \cdot w - \frac{F_y}{m} \\ q \cdot u - p \cdot v + \frac{F_z}{m} \end{pmatrix} \quad (4)$$

## 2.2 Forces and Torques

Four main forces were defined in the dynamics of the helicopter, first the ones generated from the two main rotors, then the tail rotor thrust and finally the gravity force.

$$F_{total} = F_{UpR} + F_{DwR} + F_{TR} + F_g \quad (5)$$

$$F_{total} = \begin{pmatrix} -T_{UpR}\sin(\beta) \\ T_{UpR}\sin(\alpha) \\ -T_{UpR}\cos(\alpha)\cos(\beta) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -T_{DwR} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -T_{TR} \end{pmatrix} + \begin{pmatrix} mg \cdot \sin(\theta) \\ -mg \cdot \sin(\phi)\cos(\theta) \\ mg \cdot \cos(\phi)\cos(\theta) \end{pmatrix} \quad (6)$$

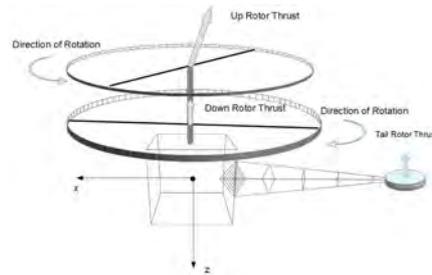


Figure 3. Helicopter Forces

To describe the total torque, this was divided in five torques (Medina, 2013). The first two torques are generated because of the rotation of the rotors. Another two torques generated in the main rotors are the drag due to the influence of the incident air to each of the blades. And finally the torque generated by the tail rotor.

$$\tau_{total} = (\tau_{DwR} - \tau_{DDw}) - (\tau_{UpR} - \tau_{DUpR}) + \tau_{TR} \quad (7)$$

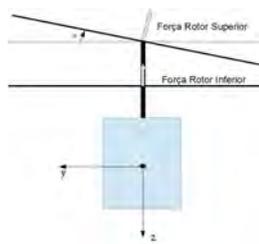
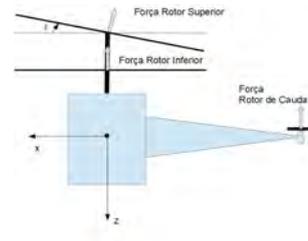
$$\tau_{total} = \left\{ \begin{pmatrix} -T_{DwR} \cdot Y_{DwR} \\ L_{DwR} \cdot T_{DwR} \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ Q_{DwR} \end{pmatrix} \right\} - \left\{ \begin{pmatrix} -\cos(\alpha)\cos(\beta)T_{UpR}Y_{UpR} - \sin(\alpha)H_{UpR}T_{UpR} \\ -\sin(\beta)H_{UpR}T_{UpR} + \cos(\alpha)\cos(\beta)L_{UpR}T_{UpR} \\ \sin(\alpha)L_{UpR}T_{UpR} + \sin(\beta)T_{UpR}Y_{UpR} \end{pmatrix} - \begin{pmatrix} Q_{UpR}\sin(\beta) \\ -Q_{UpR}\sin(\alpha) \\ Q_{UpR}\cos(\alpha)\cos(\beta) \end{pmatrix} \right\} + \begin{pmatrix} -T_{TR} \cdot Y_{TR} \\ L_{TR} \cdot T_{TR} \\ 0 \end{pmatrix} \quad (8)$$

Where the terms  $Y$ ,  $H$  and  $L$  correspond to distances from the  $COG$  to each of where the forces are acting (Medina, 2013).

### 2.3 Flybar

There is an important mechanism that part of the main rotor and is placed over the upper blades and, is called flybar. This systems will help the helicopter to make the movements slower and smoother creating a delay on the response to the control. This bar doesn't generate any thrust, but it will change the angle of the blades of the upper rotor, changing the direction of the upper rotor as seen in equation 6 from the  $F_{UpR}$  (Bouabdallah *et al.*, 2006).

To describe the movements of this bar it was necessary to suppose that the bar in movement creates a "plate" that rotates over a shat, and the position of the "plate" will depend of two angles  $\beta$  and  $\alpha$ . In figure 2.3 are the angles depending of the movement of the helicopter (Bernard Mettler, 2003).

Figure 4.  $\alpha$  AngleFigure 5.  $\beta$  Angle

This two angles were described as first order equations in 9 and 10

$$\dot{\beta}_{bar} = \frac{1}{T_{bar}}(\theta - \beta_{bar}) \quad (9)$$

$$\dot{\alpha}_{bar} = \frac{1}{T_{bar}}(\phi - \alpha_{bar}) \quad (10)$$

### 3. CONTROLLER

The Linear Quadratic Regulator was implemented to control the helicopter. To apply this theory the system was linearized over a operation point where the helicopter is in hover position, so the system can be described as equation 11.

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}\quad (11)$$

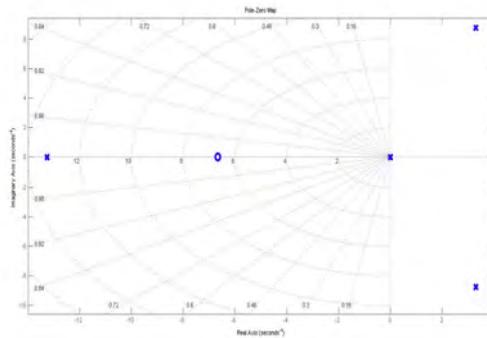
The states to be controlled:

$$x(t) = [ q(t) \quad r(t) \quad \theta(t) \quad \psi(t) \quad \beta(t) \quad w(t) \quad z(t) ] \quad (12)$$

$$u(t) = \begin{bmatrix} U_{upper}(t) \\ U_{lower}(t) \\ U_{TR}(t) \end{bmatrix} \quad (13)$$

where the first two parameters correspond to the angular velocities over  $y$  and  $z$ , the next two are the euler angles,  $\beta$  is the flybar angle described before and finally the translational speed and position altitude. The  $u$  vector are the angular velocities of rotation for each of the rotors.

The system was linearized assuming that the helicopter is in hover position and that the rotors were generating enough thrust to compensate the force of the gravity. As first step the stability of the open loop system is verified looking in to the poles over the  $A$  matrix. In figure 3, is shown the instability of the system with poles on the right side of the plane and also some poles at the origin, so it is necessary to implement a control algorithm.



can be minimized using a gain  $F$  that gives an input to system as in equation 17. This linear feedback law produces a global minimum of the performance criterium Hespanha (2007). The solution to obtain this gain is in terms of the solution of the Algebraic Riccati Equation 15  $P$  Hespanha (2007).

$$0 = A^T P + PA + Q - PBR^{-1}B^T P \quad (15)$$

$$F_{opt} = -R^{-1}B^T P \quad (16)$$

$$u(t) = -F_{opt}x(t) \quad (17)$$

Feeding back the  $u(t)$  signal to the system, the states equation will become as equation 18, creating a new  $A$  matrix and changing the position of the poles of the system.

$$\dot{x}(t) = (A - BF)x = A_c x(t) \quad (18)$$

### 3.2 Feedback Control

To calculate the  $F_{opt}$  gain the lqr algorithm from MATLAB is used. This algorithm has as inputs the  $A$  and  $B$  matrices of the linear system and the  $Q$  and  $R$  weight matrices for the cost functions, that must be positive definite.  $Q$  and  $R$  will define the behavior of the closed loop system depending on the weights for each of the states. There is not a law to calculate the right values for the  $Q$  and  $R$  matrix, although exist some rules that helps to have a start point to choose the right weights. In this case it was used the Bryston rule as start point, where the maximum value allowed for each of the states is defined and the value on the matrix is defined as in equation 19.

$$Q(i, i) = \frac{1}{x_{i,max}^2} \quad R(i, i) = \frac{1}{u_{i,max}^2} \quad (19)$$

These values can change depending on the desired response of the system. Then the final result was:

$$Q = 10^5 * \begin{bmatrix} 0.0033 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0328 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.3121 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1312 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.000001312 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (20)$$

$$R = \begin{bmatrix} 0.1111 & 0 & 0 \\ 0 & 0.1111 & 0 \\ 0 & 0 & 2.1633 * 10^{-5} \end{bmatrix}$$

using these matrices the value of  $F_{opt}$  will place the poles of the closed loop of the equation 18, as in figure 8, having all the poles at the left side of the plane, making the system stable.

### Degree of Stability

To guarantee better performance of the system, the theory of degree of stability was implemented. The degree of stability will put as condition of the closed loop that all the poles of the stable system are going to be on the left side of the chosen value (Padhi, 2013). This value will modify the cost function  $J$  adding an exponential term and giving a degree to the linear system, as shown in equation 21

$$J = \frac{1}{2} \int_{t_0}^{\infty} e^{2\alpha t} [x^T(t)Qx(t) + u^T(t)Ru(t)] dt \quad (21)$$

$$\dot{x}(t) = (A + \alpha I)x(t) + Bu(t)$$

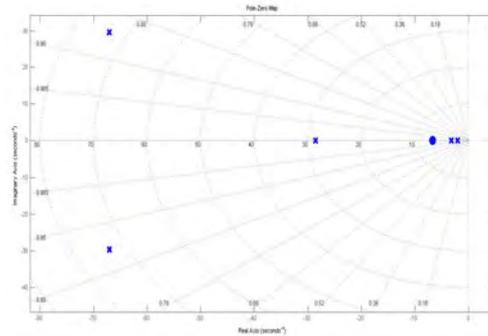


Figure 8. Closed Loop Poles of the System

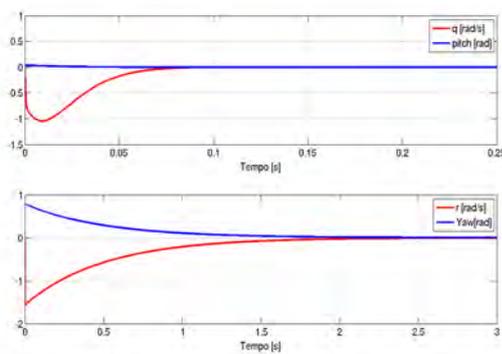


Figure 9. Response of the  $q$ ,  $\theta$ ,  $r$  and  $\psi$  states

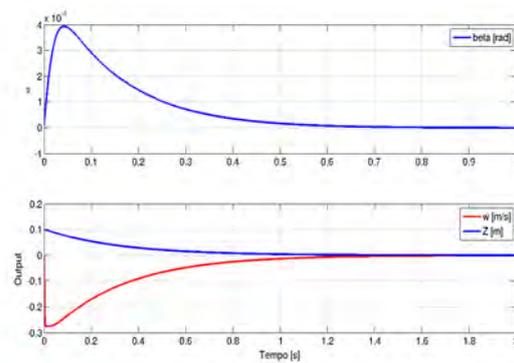


Figure 10. Response of the  $\beta$ ,  $w$  and  $Z$  states

The selected degree was  $\alpha = 0.3$ , and it was used to give a quicker response to the system. The way that this degree is calculated depends on the final desired performance and the knowledge of the system behavior.

In figures 9 and 10, the position in  $\theta$ ,  $\psi$  and  $Z$  have an initial condition, so the controller will compensate this initial value taking all the states asymptotically to zero. The states have no overshoot and a time of stabilization less than 3s, giving a good performance to the system.

### State Observer

In most of the cases when a control algorithm is developed, the gain matrix  $F$  is calculated assuming that all the stated of the system can be measured, but this is not the case in most of the implementations. In this project the states available to be measured were  $y(t) = [q(t), r(t), \theta(t), \psi(t), w(t), z(t)]^T$ , so considering these states it was calculated the Observability of the system as in (Braslavsky, 2007), allowing to design a Full State Observer as in figure 11. This observer will estimate the unmeasured states of the system based on the control inputs and the measured states, the  $\hat{y}(t)$  states will be the same states as the  $y_m(t)$  but estimated by the linear plant. Finally the observer has to follow the behavior of the real plant so the gain matrix  $F$  will be feeding back to the No Linear system. Then, final system with observer will be as in equations 22.

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) \\ \hat{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + L(y(t) - \hat{y}(t))\end{aligned}\quad (22)$$

To calculate the right Observer, its necessary to minimize the error between the measured and the estimates signals ( $e = y - \hat{y}$ ). Then substituing the error signal in to the equation 22, and after some manipulation the result will be the differential equation 23, that will converge asymptotically to zero depending of the values of  $L$ .

$$\dot{e}(t) = (A - LC)e(t)\quad (23)$$

The  $L$  matrix as in equation 12 will place the poles in the left side of the plane, but in this case it is necessary to place the poles of the observer more in to the left making the response of the error to converge faster than the poles of the controller. The equation 23 is similar case like in equation 18, so it is possible to use the same technique to calculate the Optimal value of  $L$ . Giving weights in to the  $Q_{obs}$  and  $R_{obs}$  matrices

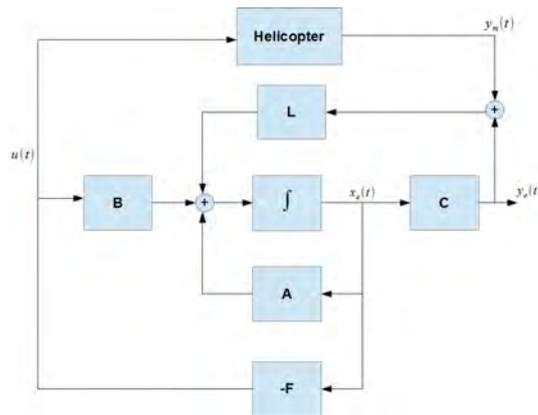


Figure 11. Observer Blocs

$$Q_{obs} = 10^9 * \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0001 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0001 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.00001 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0010 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0001 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0001 \end{bmatrix} \tag{24}$$

$$R_{obs} = 1 * 10^{-3} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

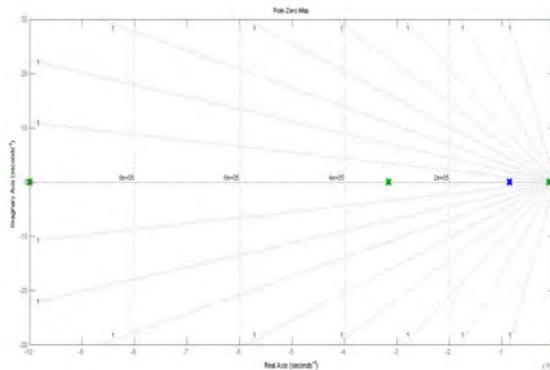


Figure 12. System and Observer Poles

After giving weights to the  $Q_{obs}$  and  $R_{obs}$  matrices, in figure 12, the poles of the observer (green) are more in the left side of the plane than the poles of the system (blue), this will let the observer to track the behavior of the helicopter in the operation point.

As seen in figure 13 and 14, the observer estimates the states of the plant using as inputs the measured signals from the helicopter  $y_m$  and computing the error with the  $y_e$ , stabilizing the helicopter. Although it was used a control tracking technique as in (Padhi, 2013), allowing the helicopter to follow a reference input signal. Because of its configuration this helicopter was able to change its orientation over the  $\psi$  angle and its altitude  $z$ .

To verify the robustness of the controller, in these simulations some of the parameters as the mass of the helicopter were changed, obtaining a good performance.

Juan C. Rivera Medina, Fernando A. N. Castro Pinto  
Modeling and LQR Control for a Small Coaxial Helicopter

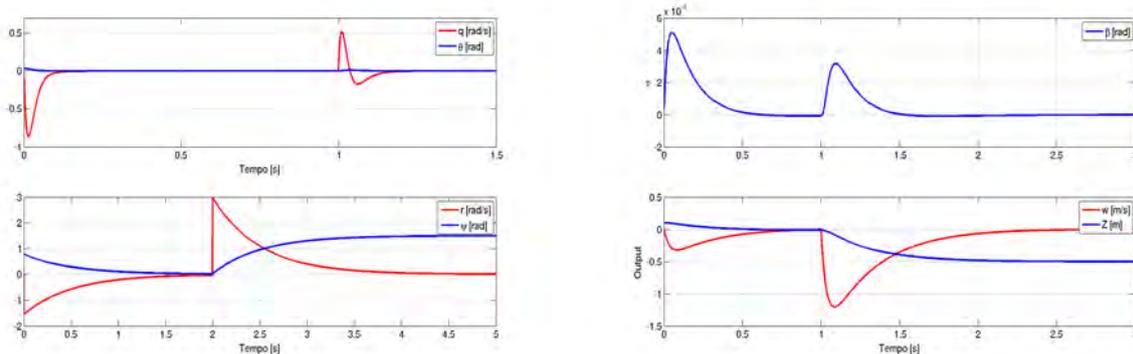


Figure 13. Response of the  $q$ ,  $\theta$ ,  $r$  and  $\psi$  states, using an Optimal Observer  
Figure 14. Response of the  $\beta$ ,  $w$  and  $Z$  states, using an Optimal Observer

#### 4. CONCLUSIONS

The development of the dynamic model took in to account most the main forces and torques that are acting over the helicopter, allowing to have a good description of its final movements and showing the nonlinearities because of coupled forces. The use of a linear control algorithm as LQR shows, is a good solution for MIMO systems, getting a good stabilization and improving the final behavior of the helicopter as shown in the different simulations. For future projects the LQR controller can be implemented in a electronic board to control the real plant and verifying the real performance of the controller.

#### 5. REFERENCE

- Bernard Mettler, Takeo Kanade, M.B.T., 2003. "System identification modeling of a model-scale helicopter".
- Bouabdallah, S., Siegwart, R. and Caprari, G., 2006. "Design and control of an indoor coaxial helicopter". In *Proc. of The IEEE International Conference on Intelligent Robots (IROS)*.
- Braslavsky, J.H., 2007. "Controlabilidad y observabilidad, notas caut2-124". URL <http://iaci.unq.edu.ar/materias/control2/web/clases/Cap6.pdf>.
- Bruno Siciliano, L.S., 2009. *Robotics Modelling Planning and Control*.
- Hespanha, J.P., 2007. "Undergraduate lecture notes on lqg/lqr controller design". URL [http://www.ece.ucsb.edu/hespanha/ece147c-me106a/web/lqr\\_lqgnotes.pdf](http://www.ece.ucsb.edu/hespanha/ece147c-me106a/web/lqr_lqgnotes.pdf). Notas de Aula.
- Medina, J.C.R., 2013. *Modelagem e Controle de um Helicoptero Coaxial em Escala Reduzida*. Master's thesis.
- Padhi, D.R., 2013. "Lqr design lecture 28, indian institute of science".
- Ulrik Bech Hald, Mikkel Vandborg Hesselbaek, M.S., 2005. *Autonomous helicopter - Modelling and Control*. Master's thesis, Aalborg University, Institute of Electronic Systems.