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BOUNDARY ELEMENT FORMULATION FOR THE CLASSICAL LAMINATED BEAM THEORY

Angelo Vieira Mendonça
Paulo Cavalcante do Nascimento Júnior
Universidade Federal da Paraiba
paulocavalcante_jr@hotmail.com
mendonca@ct.ufpb.br

Abstract. *In this article a new solution based on boundary element technique for classical laminated beam theory is established. Discussions on mathematical steps to write down both integral equations and fundamental solutions for laminated beam problem are properly made. Only in-plane bending is taken into account and numerical results for typical cases of rectangular cross-section beam are presented as well.*

Keywords: *BEM, Laminated beam, Fundamental solution, Integral Equation.*

1. INTRODUCTION

Composite laminates are generally engineered materials made by assembling of oriented layers and they are usually manufactured to maximize strength/weight ratio, especially in engineering applications. Laminated beams have been mathematically represented by beam theories associated with E*quivalent Single-Layer (ESL)* models and *Discrete-Layer(DL) approaches* Reddy (1997). The Classical Laminated Beam Theory (CLBT) is the simplest model of ESL family and it can seen as an adaptation of the hypotheses of Euler-Bernoulli beam theory to laminated composite materials.

Structural analysis of beams based on CLBT has been developed using analytical and numerical solutions. For the first case, closed solutions have been appeared for special cases, for instance Khdeir and Reddy (1997), Chandrashekhara and Krishnamurthy (1990), Kargarnovin et al. (2013), Han et al. Lu (2010). When laminated beam problems is solved by numerical solutions, they are generally done by Finite Element Modeling (FEM), see for instance, the review article by Hajianmaleki and Qatu (2013).

For many engineering problems as Boundary Element Method (BEM) is an alternative numerical technique to FEM, but this has not been verified for laminated beam problems. In fact, BEM solutions have been applied only to static and dynamic analysis of beams and frames made of homogenous materials, for instance, Banerjee (1981), Antes (2003), Beskos and Providakis (1986), Cruz (2012), Antes et al. (2004).

In this paper a direct BEM formulation is established for classical laminated beam theory, so that integral equations, fundamental solutions, and algebraic system are properly derived. Only in-plane bending problem is taken into account and the BEM results are compared to other solutions available in literature.

2.REAL AND FUNDAMENTAL PROBLEMS

The classical laminated beam theory is based on the following hypotheses: planar shape conversation of the cross section is assumed from the undeformed state until the deformed configuration; The normals to cross section rotate such that they keep their orthogonalities after the deformation; displacement, rotation, and strain are assumed to be smooth (small) fields. Then, axial and transverse displacement of the beam can be written as follows

$$u(x,z) = u(x) - z\frac{dw(x)}{dx} \tag{1}$$

$$w(x,z) = w(x) \tag{2}$$

Where u, w are axial and transverse displacements, respectively. z is the depth of point with respect to neutral line.

In addition, axial strain is given by

$$\varepsilon_x = \frac{du}{dx} - z\frac{d^2w}{dx^2} \tag{3}$$

Axial stress is associated with axial strain by a constitutive relation given by

$$\sigma_x = \bar{Q}_{11}\varepsilon_x \tag{4}$$

Where $\bar{Q}_{11}$ is the transformed stiffness in x-direction, which can related to plane stress reduced stiffnesses Q_{ij} in the principal axes (x_1 , x_2) as follows

$$\bar{Q}_{11} = Q_{11}cos^4\theta + Q_{22}sin^4\theta + 2(Q_{12} + 2Q_{66})sin^2\theta cos^2\theta. \tag{5}$$

With

$$Q_{11} = E_1^2/(E_1 - \nu_{12}^2 E_2),$$
$$Q_{22} = E_1E_2/(E_1 - \nu_{12}^2 E_2),$$
$$Q_{12} = \nu_{12}Q_{22},$$
$$Q_{66} = G_{12}$$
$$(6a\text{-}d)$$

where E_i are Young moduli for x_1- and x_2 directions as well as G_{12} is shear modulus and ν_{12} is Poisson's ratio. θ is the angle between x and x_1, see Figure 1.

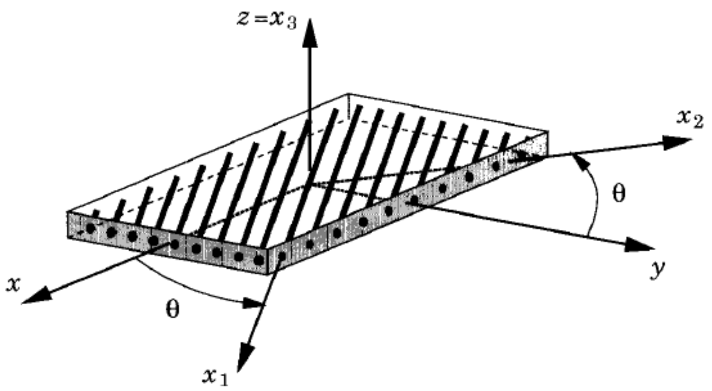

Figure1. Laminate fiber orientation.(Reddy, 1997)

Force and moment resultants can be obtained from constitutive relation Eq. (4) and Eq. (3), yielding to:

$$N(x) = A_{11}\frac{du(x)}{dx} - B_{11}\frac{d^2w(x)}{dx^2}$$

$$M_y(x) = B_{11}\frac{du(x)}{dx} - D_{11}\frac{d^2w(x)}{dx^2}$$

$$V_z(x) = B_{11}\frac{d^2u(x)}{dx^2} - D_{11}\frac{d^3w(x)}{dx^3} \tag{7a-c}$$

Where A_{11}, B_{11} and D_{11} are the rigidity moduli given by:

$$(A_{11}, B_{11}, D_{11}) = b\int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{Q}_{11}(1, z, z^2)dz \tag{8}$$

Applying the equilibrium conditions for forces and moments shown in Figure 2, following relations can be written

$$\frac{dN}{dx} + p_x = 0,$$

$$\frac{dQ}{dx} + p_z = 0\,,$$
$$\frac{dM}{dx} + Q = 0$$
$$(9a\text{-}c)$$

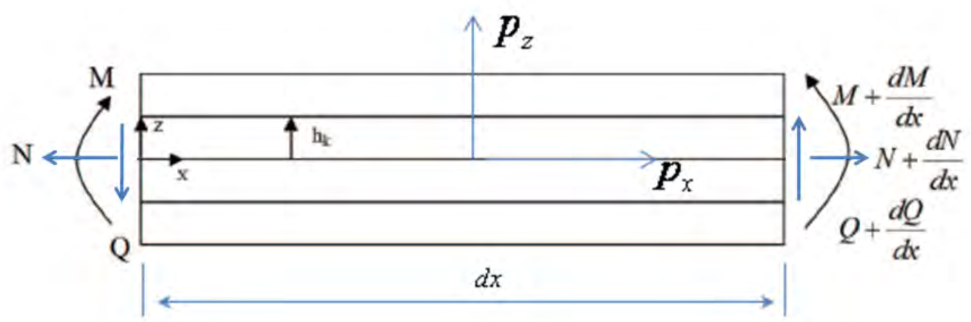

Figure 2- Forces and moments of the beam

Substituting Eq. (7) into Eq. (9), the governing equations in terms of displacements can be finally written

$$A_{11}\frac{d^2u(x)}{dx^2} - B_{11}\frac{d^3w(x)}{dx^3} = -p_x$$

$$B_{11}\frac{d^3u(x)}{dx^3} - D_{11}\frac{d^4w(x)}{dx^4} = -p_z \qquad (10a\text{-}b)$$

The fundamental problem of the classical composite laminated beam is associated with an infinite domain member under point loads (p_x^*, p_z^*) and governed by same relations applied to real problem. Hence, fundamental governing equations are analogous to Eq. (10), resulting in:

$$[B][G] = [f] \qquad (11)$$

where

$$[B] = \begin{bmatrix} A11\frac{d^2}{dx^2} & -B_{11}\frac{d^3}{dx^3} \\ B_{11}\frac{d^3}{dx^3} & -D_{11}\frac{d^4}{dx^4} \end{bmatrix},\ [G] = \begin{bmatrix} u_F^*(x,\hat{x}) & u_P^*(x,\hat{x}) \\ w_F^*(x,\hat{x}) & w_P^*(x,\hat{x}) \end{bmatrix},$$

$$[f] = -\begin{bmatrix} \delta(x,\hat{x}) & 0 \\ 0 & \delta(x,\hat{x}) \end{bmatrix}.$$

Where (u_F^*, w_F^*) are the displacement fundamental solutions when only $p_x^*(x,\hat{x}) = \delta(x,\hat{x})$ is applied. (u_P^*, w_P^*) are solution counterparts for $p_z^*(x,\hat{x}) = \delta(x,\hat{x})$ activation only.

The solution of Eq. (11) can be found using Hormander(1963)' method, which is a decoupling technique where the solution is written in terms of a scalar parameter ψ, yielding to

$$[G] = [B^{cof}]^T\psi \qquad (12)$$

If Eq. (12) is substituted into Eq. (11), gives

$$\det[B]\psi(x,\hat{x}) = -\delta(x,\hat{x}) \qquad (13)$$

After algebraic manipulation, Eq. (12) can be written as follows

$$\frac{d^6\psi}{dx^6} = -\frac{\delta(x,\hat{x})}{(B_{11}^2 - A11D11)} \tag{14 }$$

The solution here proposed for the Eq. (14) is:

$$\psi(r) = -\frac{1}{240(B_{11}^2 - A11D11)}r^5 \tag{15}$$

Where $r = |x - \hat{x}|$.

When Eq. (15) is substituted into Eq. (12), yields to explicit form of displacement fundamental solutions:

$$u_F^*(x,\hat{x}) = \frac{D_{11}r}{2(B_{11}^2 - A11D11)}$$

$$u_P^*(x,\hat{x}) = -\frac{B_{11}}{4(B_{11}^2 - A11D11)}r^2 sgn(x,\hat{x})$$

$$w_F^*(x,\hat{x}) = \frac{B_{11}}{4(B_{11}^2 - A11D11)}r^2 sgn(x,\hat{x})$$

$$w_P^*(x,\hat{x}) = -\frac{B_{11}}{12(B_{11}^2 - A11D11)}r^3 \tag{16a-d}$$

where $sgn(x,\hat{x})$ is the sign function.

Force fundamental solutions can be obtained substituting Eq. (16) into fundamental counterparts of the Eq. (7), resulting:

$$N_F^*(x,\hat{x}) = -\frac{1}{2}sgn(x,\hat{x})$$

$$M_{Fy}^*(x,\hat{x}) = V_{Fz}^*(x,\hat{x}) = N_P^*(x,\hat{x}) = N_{P,\hat{x}}^*(x,\hat{x}) = 0$$

$$M_{Py}^*(x,\hat{x}) = -\frac{1}{2}r$$

$$V_{Pz}^*(x,\hat{x}) = -\frac{1}{2}sgn(x,\hat{x})$$

$$V_{Pz,\hat{x}}^*(x,\hat{x}) = \delta(x,\hat{x})$$

$$M_{Py,\hat{x}}^*(x,\hat{x}) = \frac{1}{2}sgn(x,\hat{x}) \tag{17a-f}$$

3. INTEGRAL AND ALGEBRAIC EQUATIONS

If Eq. (10) is weighted by fundamental solution Eq. (9), the method of weighted residuals states:

$$\int_0^L \left\{ \begin{bmatrix} A11\frac{d^2}{dx^2} & -B_{11}\frac{d^3}{dx^3} \\ B_{11}\frac{d^3}{dx^3} & -D_{11}\frac{d^4}{dx^4} \end{bmatrix} \begin{Bmatrix} u(x) \\ w(x) \end{Bmatrix} + \begin{Bmatrix} p_x \\ p_z \end{Bmatrix} \right\}^T \begin{bmatrix} u_F^*(x,\hat{x}) & u_P^*(x,\hat{x}) \\ w_F^*(x,\hat{x}) & w_P^*(x,\hat{x}) \end{bmatrix} dx = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}^T \tag{18}$$

After applying conveniently successive integrations by parts of Eq. (18) and then with help of Eq. (7), Eq. (11) and the property of the Dirac delta, one obtains the integral equations for both axial and transverse displacements:

$$u(\hat{x}) + [N_F^*(x,\hat{x})u(x)]_0^L + \ [V_{zF}^*(x,\hat{x})w(x)]_0^L - \left[M_{yF}^*(x,\hat{x})\frac{dw(x)}{dx}\right]_0^L$$

$$= \ [N(x)u_F^*(x,\hat{x})]_0^L \ + [V_z(x)w_F^*(x,\hat{x})]_0^L - \ \left[M_y(x)\frac{dw_F^*(x,\hat{x})}{dx}\right]_0^L +$$

$$+ \int_0^L [p_x u_F^*(x,\hat{x}) + p_z w_F^*(x,\hat{x})]dx$$

$$w(\hat{x}) + [N_P^*(x,\hat{x})u(x)]_0^L + [V_{zP}^*(x,\hat{x})w(x)]_0^L - \left[M_{yP}^*(x,\hat{x})\frac{dw(x)}{dx}\right]_0^L$$

$$= [N(x)u_P^*(x,\hat{x})]_0^L + \ [V_z(x)w_P^*(x,\hat{x})]_0^L - \ \left[M_y(x)\frac{dw_P^*(x,\hat{x})}{dx}\right]_0^L$$

$$+ \int_0^L [p_x u_P^*(x,\hat{x}) + p_z w_P^*(x,\hat{x})]dx \qquad (19a\text{-}b)$$

The classical laminated beam problems require three unknowns at boundary to be determined. Hence, an additional equation is necessary to be established in order to get the problem solvable. Then, this remaining equation can be associated with the derivative of Eq. (19b) at source point $dw(\hat{x})/d\hat{x}$, yielding to slope integral equation:

$$\frac{dw(\hat{x})}{d\hat{x}} + [N_{P,\hat{x}}^*(x,\hat{x})u(x)]_0^L + [V_{zP,\hat{x}}^*(x,\hat{x})w(x)]_0^L - \left[M_{yP,\hat{x}}^*(x,\hat{x})\frac{dw(x)}{dx}\right]_0^L =$$

$$[N(x)u_{P,\hat{x}}^*(x,\hat{x})]_0^L + \ [V_z(x)w_{P,\hat{x}}^*(x,\hat{x})]_0^L - \ \left[M_y(x)\frac{dw_{P,\hat{x}}^*(x,\hat{x})}{dx}\right]_0^L +$$

$$+ \int_0^L [p_x u_{P,\hat{x}}^*(x,\hat{x}) + p_z w_{P,\hat{x}}^*(x,\hat{x})]dx \qquad (19c)$$

By collocating the source at the edges of the bar, ie for $\hat{x} = \lim_{\xi\to 0}(0+\xi)$ and $\hat{x} = \lim_{\xi\to 0}(L-\xi)$, an algebraic representation in terms of boundary quantities for displacements (see Fig.3b) and for forces (see Fig.3a) can be written as follows

$$\{u\} + [H]\{u\} = [G]\{p\} + \{f\} \qquad (20)$$

Where $[H]$, $[G]$ and $\{f\}$ are the influence matrices and load vector. $\{u\}$ and $\{p\}$ are the displacement and force vectors, which their explicit forms are (see Figure 3):

$$\{u\} = \left[u_i \quad w_i \quad \frac{dw_i}{dx} \quad u_j \quad w_j \quad \frac{dw_j}{dx}\right]^T$$

$$\{p\} = [N_i \quad V_i \quad M_i \quad N_j \quad V_j \quad M_j]^T$$

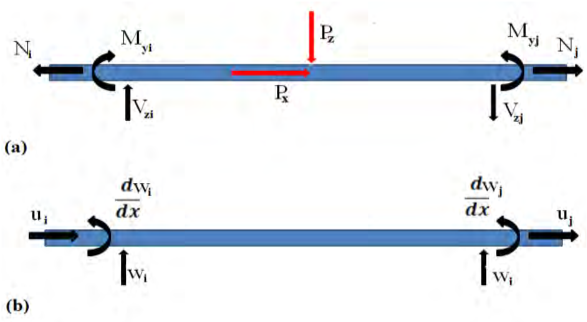

Figure 3. Boundary forces (a) and displacements (b).

The explicit forms of the influence matrices in Eq. (20) are:

$$[H] = \begin{bmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{L}{2} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{L}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} \quad (21)$$

$$[G] = \frac{1}{2(B_{11}^2 - A_{11}D_{11})} \begin{bmatrix} 0 & 0 & 0 & D_{11}L & \frac{B_{11}}{2}L^2 & -B_{11}L \\ 0 & 0 & 0 & -\frac{B_{11}}{2}L^2 & -\frac{A_{11}}{6}L^3 & \frac{A_{11}}{2}L^2 \\ 0 & 0 & 0 & B_{11}L & \frac{A_{11}}{2}L^2 & -A_{11}L \\ -D_{11}L & \frac{B_{11}}{2}L^2 & B_{11}L & 0 & 0 & 0 \\ -\frac{B_{11}}{2}L^2 & \frac{A_{11}}{6}L^3 & \frac{A_{11}}{2}L^2 & 0 & 0 & 0 \\ -B_{11}L & \frac{A_{11}}{2}L^2 & A_{11}L & 0 & 0 & 0 \end{bmatrix} \quad (22)$$

If both loads px and pz are uniform along beam domain, the load vector {f} becomes:

$$\{f\} = \frac{1}{2(B_{11}^2 - A_{11}D_{11})} \begin{Bmatrix} \frac{p_x D_{11}}{2}L^2 + \frac{p_z B_{11}}{6}L^3 \\ -\frac{p_x B_{11}}{6}L^3 - \frac{p_z A_{11}}{24}L^4 \\ \frac{p_x B_{11}}{2}L^2 + \frac{p_z A_{11}}{6}L^3 \\ \frac{p_x D_{11}}{2}L^2 - \frac{p_z B_{11}}{6}L^3 \\ \frac{p_x B_{11}}{6}L^3 - \frac{p_z A_{11}}{24}L^4 \\ \frac{p_x B_{11}}{2}L^2 - \frac{p_z A_{11}}{6}L^3 \end{Bmatrix} \quad (23)$$

After determination of the boundary unknowns, axial displacement $u(\hat{x})$, transverse displacement $w(\hat{x})$, and slope $dw(\hat{x})/d\hat{x}$ at any point of the beam domain can be evaluated using Eq. (19).

4.NUMERICAL RESULTS

A rectangular cross section beam having 1 m length, 0.025 m width, and 0.05 m height was considered here. In addition, the beam has mechanical properties (E_1 = 180 GPa, E_2 = 8.96 GPa; G_{12}= 7.1 GPa; ν_{12} =0.3) and it is subjected to a uniform load 250 kN/m and under two sets of the boundary conditions (simply supported or clamped beam). The maximum deflections of the beam are analysed for cross-ply and angle-ply laminates. The results for simply supported beam are shown in Table 1 while clamped beam responses are shown in Table 2. The BEM responses are compared to the Euler Approach results given by Hajianmaleki and Qatu (2011).

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Table 1. Maximum deflection of the simply supported bam.

	$w_{max}(m)$	
Laminate	Euler Approach	BEM
$(0)_4$	0.0901	0.0901
$(0/90)_s$	0.1019	0.1020
$(45)_4$	0.2753	0.2753

Table 2. Maximum deflection of the clamped beam.

	$w_{max}\,(m)$	
Laminate	Euler Approach	BEM
$(0)_4$	0.01801	0.01801
$(0/90)_s$	0.02039	0.02039
$(45)_4$	0.05506	0.05506

5. CONCLUSIONS

In this paper a boundary element modelling was established to composite laminated beam problems under hypotheses of the classical laminated beam theory. Only in-plane bending is considered and the results suggest the correctness and effectiveness of the formulation here presented.

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