DETERMINATION OF THERMAL PROPERTIES IN INVERSE HEAT CONDUCTION PROBLEMS USING DIFFERENTIAL EVOLUTION

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Abstract. *The inverse analysis is encountered in various branches of science and engineering. In the field of heat transfer, inverse analysis is used for the estimation of surface conditions such as temperature and heat flux. Also, the determination of thermal properties such as thermal conductivity and heat capacity of solids by utilizing transient temperature measurements collected from points within the medium finds numerous practical applications. The standard heat conduction problems are in general well posed, i.e., the solution exists, the solution is unique, and the solution is stable with respect to small changes in the input data. On the other hand, the inverse heat conduction problems are sensitive to measurement errors: as a result, they are ill posed. Traditionally, three alternatives are proposed for the solution of parameter identification problems by using optimization techniques: the deterministic, the non-deterministic and the hybrid approaches. Among the existing methodologies, the Differential Evolution Algorithm has been applied with success both to mono and multi-objective contexts. In this work, an inverse analysis is presented to determine the thermal conductivity and heat capacity in heat conduction problems. The results show that the methodology used represents an interesting alternative to the treatment of the inverse problem as compared with those obtained by using the Multi-Particle Collision Algorithm.*

Keywords: *Inverse heat conduction problem, thermal properties, Differential Evolution Algorithm.*

1. INTRODUCTION

The problem of parameter estimation characterizes a typical inverse problem (IP) in engineering. It arises from the difficulty in building theoretical models that are able to represent satisfactorily physical phenomena under real operating conditions. Considering the possibility of using more complex models along with the information provided by experimental data, the parameters obtained through an inverse problem approach may be further used to simulate the behavior of the system for different operation conditions.

The increasing interest on IP is due to the large number of practical applications in scientific and technological areas such as tomography (Kim and Charette, 2007), environmental sciences (Hanan, 2001) and parameter estimation (Souza et al., 2007), to mention only a few.

In the field of heat transfer, the use of inverse analysis for the estimation of surface conditions such as temperature and heat flux, or the determination of thermal properties such as thermal conductivity and heat capacity of solids by utilizing transient temperature measurements taken within the medium, lead to several practical applications (Alifanov and Artyukhin, 1975; Huang and Özişik, 1991; Özişik, 1993; Huang and Wu, 1994; Dantas and Orlande, 1996). Many studies involve the estimation of boundary conditions, especially surface heat flux (Alifanov and Artyukhin, 1975; Huang and Wu, 1994), and the estimation of the temperature dependence of thermo-physical properties by means of parameter estimation (Huang and Özişik, 1991) or function estimation (Dantas and Orlande, 1996). The latter is a more difficult problem since no prior information is available on the functional form of the unknown variable. The standard heat conduction problems are well posed, i.e., the solution exists, the solution is unique, and the solution is stable to small changes in the input variables. On the other hand, inverse heat conduction problems are sensitive to measurement errors, and hence they are ill posed (Özişik, 1993).

Traditionally, this kind of problem has been treated by using either classical or deterministic optimization techniques (Baltes et al., 1994; Cazzador and Lubenova, 1995). In recent years however, the use of non-deterministic techniques or the coupling of these techniques with classical ones, thus forming a hybrid methodology, became very popular due to the simplicity and robustness of evolutionary techniques (Wang et al., 2001; Silva Neto and Soeiro, 2002; Silva Neto and Soeiro, 2003, Silva Neto and Silva Neto, 2003).

This work is organized as follows. Section 2 presents the mathematical formulation of the direct and inverse heat conduction problems. A review about the Differential Evolution (DE) algorithm is presented in Section 3. Section 4 shows the corresponding computational scheme. The results obtained with the DE algorithm compared with Multi-Particle Collision algorithm (MPC) are presented in Section 5, and the conclusions are outlined in Section 6.

2. MATHEMATICAL FORMULATION AND SOLUTION OF THE DIRECT PROBLEM

In the present contribution, a heat conduction problem with thermal conductivity $k(x)$ and heat capacity $C(x)$, both dependent on the spatial coordinate, is considered.

Figure 1 shows the geometry and the coordinates of the plate of thickness L, which is initially at a temperature equal to zero. When $t > 0$, a known constant heat flux q_o (W/m^2) is applied to the surface at $x = 0$ and the boundary surface at $x = L$ is kept insulated. Temperature sensors are installed at multiple positions ($x_i = 0, 1, 2, \ldots, N$) and the temperature measurements are taken at the times t_j ($j = 0, 1, 2, \ldots, M$).

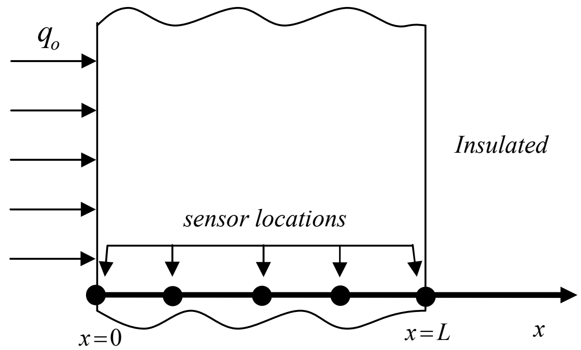

Figure 1: Geometry and coordinates of the heat conduction problem.

Mathematically, this problem is formulated as (Özişik, 1973):

$$C(x)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial T}{\partial x}\right) \quad \text{in } 0 \le x \le L,\ t > 0 \tag{1}$$

$$-k(0)\frac{\partial T}{\partial x} = q_o \quad \text{at } x = 0,\ t > 0 \tag{2}$$

$$\frac{\partial T}{\partial x} = 0 \quad \text{at } x = L,\ t > 0 \tag{3}$$

$$T = 0 \quad \text{at } t = 0,\ 0 \le x \le L \tag{4}$$

where T is the temperature, and $k(x)$, $C(x)$ and q_o are considered as known.

In order to solve the direct problem, the Method of Lines (MOL) (Zafarullah, 1970; Carver, 1978; Schiesser, 1991) is used. This method is a general technique for solving partial differential equations by typically using finite difference relationships for the spatial derivatives and ordinary differential equations for the time derivatives. This system of equations is solved by using a convenient ordinary differential equation solver. According to Sadiku and Obiozor (2000), the MOL has the following properties that justify its use: computational efficiency, numerical stability, reduced programming effort, and affordable computational time.

3. MATHEMATICAL FORMULATION AND SOLUTION OF THE INVERSE PROBLEM

The inverse heat conduction problem consists in the determination of the thermal conductivity and the heat capacity profiles ($k(x)$ and $C(x)$, respectively), through the minimization of the difference between the experimental (T^{exp}) and calculated (T^{cal}) values, given by Eqs. (5):

$$Q = \sum_{i=1}^{N}\sum_{j=1}^{M}\left(T_{ij}^{cal}\left(k(x_i), C(x_i)\right) - T_{ij}^{exp}\right)^2 \tag{5}$$

where $N \times M$ is the number of experiments considered.

As real experimental data was not available, sets of synthetic experimental data were generated from Eq. (6), where $\boldsymbol{T}^{cal}$ represents the calculated values of the temperature, by using the exact values of the thermal properties, $\vec{Z}_{exact}$, which are not available in a real application (to be determined through the inverse problem solution), $\cdot\kappa$ stands for the standard deviation of the measurement errors, and ξ is a pseudo-random number in the range [−1, 1].

$$T_{ij}^{exp} = T_{ij}^{cal}\left(\vec{Z}_{exact}\right) + \kappa\zeta \quad (6)$$

To solve the inverse heat conduction problem, the Differential Evolution (DE) algorithm, proposed by Storn and Price (Price and Storn, 1997; Storn et al., 2005), is used. This approach is an optimization technique that belongs to the family of evolutionary computation, which differs from other evolutionary algorithms in the mutation and recombination schemes. Basically, DE executes its mutation by adding a weighted difference vector between two individuals to a third one. Then, the mutated individuals will perform discrete crossover and greedy selection with the corresponding individuals from the last generation to produce the offspring.

The canonical DE algorithm is presented in the following (Storn et al., 2005), where *NP* is the population size, *P* is the population of the current generation, and *P'* is the population to be formed for the next generation.

```
Differential Evolution

  Initialize and evaluate population P
  while (not done) {
    for (i = 0; i < NP; i++) {
      Create candidate C[i]
      Evaluate C[i]
      if (C[i] is better than P[i])
            P'[i] = C[i]
      else
            P'[i] = P[i]
    }
    P = P'
  }
```

The routine **Create candidate** $C[i]$ is illustrated below, where $C[i]$ is the candidate solution with population index i, $C[i][j]$ is the j-th entry in the solution vector of $C[i]$ and r is a random number between 0 and 1.

```
Create candidate C[i]

  Randomly select parents P[i1], P[i2], and P[i3],
where i, i1, i2, and i3 are different.
  Create initial candidate C1[i] = P[i1] + F ×(P[i2]-
P[i3]).
  Create final candidate C[i] by crossing over the
genes of P[i] and C1[i] as follows:
  for (j = 0; j < NP; j++) {
    if (r < CR)
      C[i][j] = C1[i][j]
    else
      C[i][j] = P1[i][j]
  }
```

The key control parameters for DE are the population size (*NP*), the crossover constant (*CR*), and the weight (*F*) applied to the random differential (scaling factor). According to Price and Storn [25, 26], *NP* should be about 5 to 10 times the dimension (the number of parameters in a vector) of the problem, and *CR* should be between 0.1 and 1 and *F* from 0.1 to 2.

The Differential Evolution (DE) algorithm has been successfully applied to various fields, such as the estimation of drying parameters in rotary dryers (Lobato et al., 2008), solution of inverse radiative transfer problems (Lobato et al., 2009b; Lobato et al., 2010), solution of coupled inverse conduction-radiation problem (Lobato et al., 2009a), apparent thermal diffusivity estimation in the drying of fruits (Mariani et al., 2008), solution of multi-objective optimal control problems with index fluctuation applied to a fermentation process (Chiou and Wang, 1999), digital filter design (Storn,1995), and other applications (Storn et al., 2005).

4. COMPUTATIONAL SCHEME

The computational scheme for solving the inverse heat conduction problem consists first in defining the variables of the optimization problem and the DE parameters. For each evaluation of the objective function a system of ordinary differential equations (direct problem) is solved by using the MOL, as described in section 2. Figure 2 illustrates this procedure.

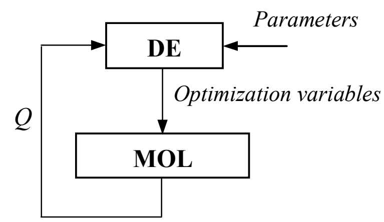

Figure 2: Flow chart for the proposed procedure.

5. RESULTS AND DISCUSSION

In order to evaluate the performance of the DE strategy, two case studies are considered, each one with a different material: a metal-like and an insulating material such as fiberglass, respectively. In all cases, a linear variation of $k(x)$ and $C(x)$ with respect to x is considered, according to Eqs. (7) and (8) as in Özişik (1993):

$$k(x) = \left(K_f - K_o\right)\frac{x}{L} + K_o \tag{7}$$

$$C(x) = \left(C_f - C_o\right)\frac{x}{L} + C_o \tag{8}$$

where K_o, K_f, C_o and C_f are the four parameters that depend on the material used by the inverse analysis. We consider a plate of thickness $L = 1$ m, initially at zero temperature. The boundary at $x = 0$ for $t > 0$ is subjected to a constant heat flux q_o (W/m^2), whereas the boundary surface at $x = L$ is kept insulated. Temporal temperature measurements are taken with sensors installed at six locations (i.e., $x = 0$, $x = 0.2$, $x = 0.4$, $x = 0.6$, $x = 0.8$ and $x = 1$ m) over a period of 300 seconds (with a time interval of 20 seconds).

In order to examine the accuracy of the considered inverse analysis methodology for the estimation of the thermal parameters, two test cases with ($\kappa = 0.02$, corresponding to an error of 5 %) and without ($\kappa = 0$) noise have been studied.

The parameters used in the DE are the following: population size equal to 20, crossover and perturbation rate equal to 0.8, and strategy DE/1/rand to generate potential candidates (Price and Storn, 1997; Storn et al., 2005). To evaluate the performance of the DE to estimate the values of the coefficients and heat fluxes will be used the Multi-Particle Collision (MPC) algorithm with following parameters (Luz et al., 2008; Lobato and Steffen Jr, 2010): population size equal to 20, maximum number of iterations equal to 250. The stopping criterion used in each algorithm is the maximum number of generations. This work considered 250 generations, and hence there are 5020 evaluations of the objective function in each run of both algorithms. Each case study was run 10 times in order to obtain average resulting values. Details about the MPC algorithm can be found in Luz et al., (2008) and Lobato and Steffen Jr (2010).

5.1. Case A

Table 1 presents the exact values of the coefficients and heat fluxes considered in this first case study (Özişik, 1993).

Table 1. Coefficients and heat fluxes for case A, with a plate of metal-like material.

q_o (W/m^2)	K_o (W/m^{o}C)	K_f (W/m^{o}C)	C_o (kJ/m^{3o}C)	C_f (kJ/m^{3o}C)
25000	50	59	3600	4500

Considering noiseless data ($\kappa = 0$, see Table 2), both DE and MPC were able to estimate the parameters satisfactorily as demonstrated by the values obtained for the objective function. When noise is added ($\kappa = 0.02$, see Table 2) good estimates are also obtained.

Table 2. Results obtained for case A by using the DE and MCP algorithms.

Error	Algorithm		K_o	K_f	C_o	C_f	Q
0 %	DE	*Best*	49.99	59.00	3599.92	4500.16	8.46E-8
		Average	49.99	58.99	3600.91	4499.00	1.14E-7
		Worst	49.99	58.98	3600.94	4499.01	2.05E-7
0 %	MPC	*Best*	49.89	59.02	3599.80	4500.01	5.67E-7
		Average	49.89	59.02	3599.80	4500.03	6.90E-7
		Worst	49.87	59.01	3599.79	4500.03	7.01E-7
5 %	DE	*Best*	50.00	59.78	3541.75	4560.90	1.31E-2
		Average	49.99	59.77	3541.49	4560.19	1.30E-2
		Worst	49.89	59.75	3540.54	4559.87	1.32E-2
5 %	MPC	*Best*	49.70	59.55	3540.22	4559.98	3.98E-2
		Average	49.73	59.58	3540.20	4559.89	4.02E-2
		Worst	49.73	59.59	3540.33	4559.77	4.12E-2

Figures 3 and 4 show the exact and the estimated values of $k(x)$ and $C(x)$ with a thermal conductivity given by 50 W/(m^{o}C), as a function of the position (x) by using the DE algorithm.

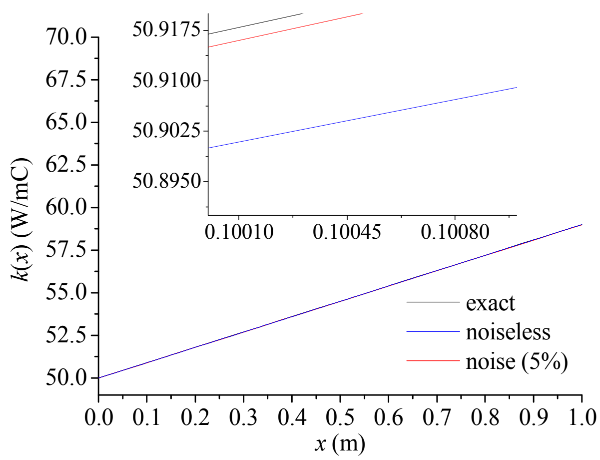

Figure 3: Estimated thermal conductivity $k(x)$ for a plate of metal-like iron by using the DE algorithm.

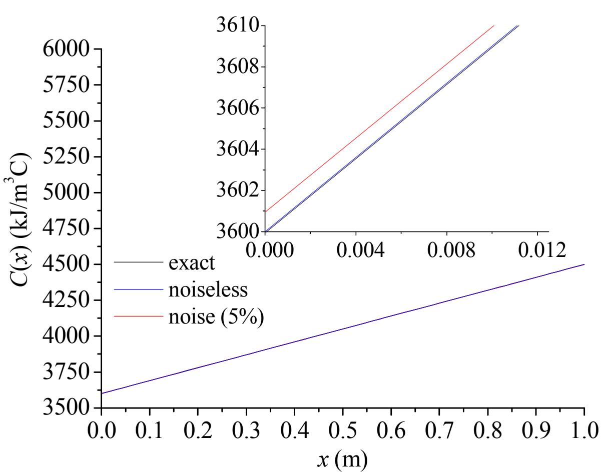

Figure 4: Estimated heat capacity $C(x)$ for a plate of metal-like material by using the DE algorithm.

5.2. Case B

The following example considers the parameters listed in Table 3 (Özişik, 1993).

Table 3. Coefficients and heat fluxes for case B, with a plate of insulating material (fiberglass).

q_o (W/m^2)	K_o (W/m^{o}C)	K_f (W/m^{o}C)	C_o (kJ/m^{3o}C)	C_f (kJ/m^{3o}C)
100	0.04	0.07	13	14

Table 4 presents the results obtained by using DE considering both noiseless ($\kappa = 0$) and noisy ($\kappa = 0.02$) data.

Table 4. Results obtained for case B by using the DE and MCP algorithms.

Error	Algorithm		K_o	K_f	C_o	C_f	Q
0 %	DE	*Best*	0.04	0.07	13.01	13.99	2.25E-8
		Average	0.04	0.07	12.99	14.00	6.51E-8
		Worst	0.03	0.06	13.02	13.99	5.99E-7
0 %	MPC	*Best*	0.39	0.07	13.00	13.92	4.59E-8
		Average	0.39	0.07	13.02	13.89	5.19E-8
		Worst	0.39	0.07	13.11	13.80	9.59E-7
5 %	DE	*Best*	0.03	0.07	13.08	13.87	3.51E-3
		Average	0.04	0.06	12.04	13.84	4.19E-3
		Worst	0.05	0.06	12.99	13.25	5.44E-3
5 %	MPC	*Best*	0.03	0.07	13.12	13.77	4.15E-3
		Average	0.03	0.07	13.20	13.75	5.03E-3
		Worst	0.03	0.07	13.25	13.30	7.11E-3

As in the first case study, for the noiseless case, both DE and MPC were able to estimate the parameters satisfactorily. When noise is added the corresponding results represent good estimates to the thermal parameters.

Figures 5 and 6 show the exact and estimated values of $k(x)$ and $C(x)$ for a plate of insulating material such as fiberglass by using the DE algorithm.

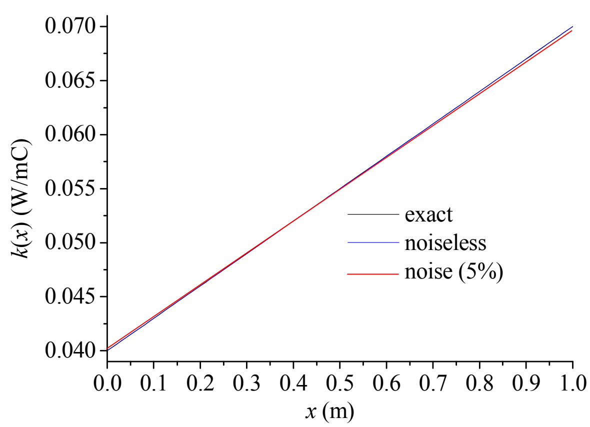

Figure 5: Estimated thermal conductivity $k(x)$ for a plate of insulating material (fiberglass) by using the DE algorithm.

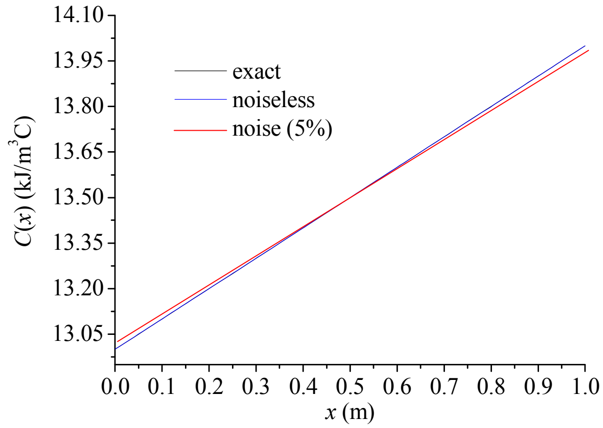

Figure 6: Estimated heat capacity $C(x)$ for a plate of insulating material (fiberglass) by using the DE algorithm.

6. CONCLUSIONS

In this paper, the Differential Evolution algorithm and Multi-Particle Collision algorithm were used to estimate the conductivity and the heat capacity with space-dependency from two representative case studies previously proposed by Özişik (1993). In association with these algorithms, the Method of Lines was used to solve a direct heat conduction problem. In general terms, both DE and MPC were able to estimate the parameters satisfactorily. When noise is added the results obtained can be considered as good estimates to the thermal parameters. Further research will be focused on the influence of the parameter values required by DE on the solution of the inverse problem and the reduction of the number of evaluations of objective function.

7. ACKNOWLEDGMENTS

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