

# MODELING CAVITATING FLOWS OF LIQUIDS CARRYING INERT GAS IN SOLUTION

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**Abstract.** *This paper presents a consistent thermodynamics model to describe the cavitating flows of compressible liquids. The fluid is assumed to carry an inert gas in suspension. It is regarded as an homogeneous mixture of liquid, vapor and inert gas, which share the same fields of velocity and temperature. The governing equations are formed by the mass conservation equations for each constituent along with a momentum conservation equation for the mixture as a whole. The gaseous volume fraction is treated as a state variable and its restriction is properly incorporated in the constitutive relationships. The constitutive equations are derived within the framework of the Thermodynamics of Irreversible Processes by specifying two thermodynamical potentials: the Helmholtz free energy and a pseudo-potential of dissipation. The phase change transformation is consistently described as an irreversible process. In contrast to other existing models, the presence of an inert gas allows the description of the cavitation phenomenon in anywhere region of the fluid flow domain by means of a same set of equations. The capability of the model in properly describing the phase change transformation is analyzed with the aid of a numerical example. Numerical predictions of the model proposed are compared with the results of an existing cavitation model which does not take inert gas into account.*

**Keywords:** *Cavitation, inert gas, thermodynamics of irreversible processes, compressible fluids.*

## 1. INTRODUCTION

Cavitation is the formation of the vapor phase in a liquid. It is distinguished from boiling in the sense that the former is generated by pressure reduction, whereas the latter refers to vapor formation as result of heat addition (Arndt, 1981). Internal as well as external liquid flows may be subjected to vaporous cavitation under steady or unsteady regimes. The cavitation region of the fluid flow is in general restricted to localized sites and may be induced by different aspects related to the fluid flow (Rood, 1991). Among them, boundary curvature of solid surfaces in contact with the liquids (e.g. the runner blades in a turbine or pump), vortices, turbulence and transient expansion waves such as column separation in piping systems conveying liquids (Wylie and Streeter, 1993).

There are several practical engineering problems in which cavitation plays an important role and so must be regarded as a design consideration. Cavitation can affect the performance of turbo machineries by reducing their efficiency and power output in turbines (or head in pumps) as well as the thrust produced by various propulsion systems (Arndt, 1981). It can dramatically alter the cooling conditions in fast nuclear reactors (Graham, 1969). Also, it is a relevant feature in structural failure prediction of pipelines transporting liquids (Freitas Rachid and Costa Mattos, 1998) and in the analysis of the response of submerged structures excited by transient acoustic waves (Fellipa and Deruntz, 1984). Due to its importance, cavitation has been the subject of extensive research in the past years. For a broad and up-to-date review about this matter refer to Arndt (1981) and Utturkar *et al.* (2005) and references therein.

In the great majority of these problems the cavitation phenomenon is analytically treated by the traditional thermodynamics theory. The pressure is assumed to remain constant at the saturated vapor pressure and the phase change process is considered to take place without presenting any kind of dissipation. Such kind of approach simplifies considerably the solution of the fluid flow problem since the cavitation threshold does not depend on flow conditions. On the other hand, it is well accepted that this approach may lead to unsatisfactory results Utturkar *et al.* (2005), what motivates the search for more physically realistic and mathematically coherent models.

To circumvent such a deficiency, it has been proposed recently a logically consistent thermodynamic model to describe the vaporous cavitation phenomenon by means of an internal variable theory Freitas Rachid (2003). The fluid is regarded as a continuum mixture of the compressible phases (liquid and vapor), which can or can not coexist at a same material

point and time instant. The no-slip condition between the phases is assumed as a basic assumption, so that only one balance equation of momentum is considered for the whole mixture. The temperature is supposed to be the same for both phases and the cavitation process is assumed to be an isothermal irreversible transformation. To properly account for the irreversible character of the phase change transformation, the constitutive equations are derived within the framework of the Thermodynamics of Irreversible Processes. Numerical simulations have been used to demonstrate that irreversible phase change transformation can be viewed as reversible limiting cases in which the liquid-vapor transformation takes place at a constant pressure and the other in which the vapor expands and contracts in the mixture without phase change.

In spite of its ability in properly describing the cavitation phenomenon from the correct thermodynamical viewpoint, the model proposed by John presents a difficulty when practical applications are sought. Inasmuch pure liquid and pure vapor are admitted to exist at different regions in the flow domain, the set of equations to be solved changes from point-to-point, requiring the development of sophisticated and very well elaborated numerical routines.

Aiming to overcome such a difficulty, an extended version of the model developed by Freitas Rachid (2003) is presented in this paper. Instead of admitting only the existence of liquid and vapor in the mixture, an inert gas is taken as an additional constituent into account. Since it is assumed that the inert gas coexists along with the vapor throughout the flow domain, the fluid is now regarded as a two-constituent two-phase mixture. Therefore, besides the momentum conservation equation for the mixture as a whole, mass conservation equations for the liquid, the vapor and the gas are considered. As it has been done before, the constitutive equations are derived within the framework of the Thermodynamics of Irreversible Processes. Besides the Helmholtz free energy, it is assumed the existence of a pseudo-potential of dissipation from which constitutive relationships are obtained for the dissipative mechanisms. The gaseous volume fraction is considered a state variable and the constraint associated with it is treated as a material property, becoming an effective part of the constitutive laws. The macroscopic dissipative effects of the liquid-vapor transformation and the cavity growth and collapse are properly accounted for in such a way that the Second Law of Thermodynamics is always satisfied.

To evaluate the the model capability in deal with the the phase change transformation in the cavitation phenomenon, numerical experiments are carried out in which the fluid is submitted to a dynamic cycle of compression and expansion. The results obtained with the proposed model are then compared with those reported in Freitas Rachid (2003).

## 2. BALANCE EQUATIONS AND THE SECOND LAW

In contrast to general two-phase fluid flows where the phases can assume very different geometrical configurations throughout the flow region (Graham, 1969; Ishii and Hibiki, 2006), cavitation is a localized phenomenon which takes place at discrete and small regions of the fluid flow. Therefore, it is reasonable to assume that there exists no significant relative motion or slip between the liquid and gaseous phases, what is equivalent to consider that the constituents in these phases have the same velocities. In addition, if we assume that both phases have always the same temperature during liquid-vapor phase transformations, it suffices to consider the balance equations of mass, momentum along with one version of the second law for the mixture as a whole, instead of doing it for each phase separately. Such approach is equivalent to assume that the fluid can be regarded as a pseudo-mixture of the liquid and gaseous phases with average properties. Under suitable regularity assumptions, and also by restricting the analysis to isothermal transformations ( $\dot{\theta} = 0$  and  $\nabla\theta = 0$ ;  $\theta$  being the absolute temperature), only the following classical forms of the balance equations and a local version of the second law are required to describe the problem (Germain and Muller, 1995):

$$\dot{\rho} + \rho \nabla \cdot \mathbf{v} = 0 \quad (1)$$

$$\rho \dot{\mathbf{v}} = -\nabla p + \nabla \cdot \mathbf{S} + \rho \mathbf{g} \quad (2)$$

$$d = -(p + \Psi) \nabla \cdot \mathbf{v} + \mathbf{S} \cdot \mathbf{D} - \dot{\Psi} \geq 0 \quad (3)$$

The above equations, in Eulerian coordinates, represent the balance of mass, the balance of linear momentum and the Clausius-Duhem inequality, respectively. As usual, the superimposed dot stands for the material time derivative,  $\rho$  is the mass density of the mixture,  $\mathbf{v}$  is the spatial velocity field,  $\mathbf{S} = \mathbf{S}^T$  is the extra stress tensor due to motion,  $\mathbf{g}$  is the external body force per unit mass and  $\mathbf{D} = \frac{1}{2}[\nabla \mathbf{v} + (\nabla \mathbf{v})^T]$  is the rate of deformation tensor. Since all of the constituents are assumed to be compressible,  $p$  stands for the thermodynamic pressure and, finally,  $\Psi$  represents the Helmholtz free energy of the mixture per unit volume. Equation (3) is a local version of the Second Law of the Thermodynamics and requires that the rate of energy dissipation,  $d$ , be nonnegative. It establishes a distinction between possible ( $d \geq 0$ ) processes and impossible ( $d < 0$ ) ones. Processes that do not violate the second law are classed as reversible or non-dissipative ( $d = 0$ ) and irreversible or dissipative ( $d > 0$ ).

The balance equations presented so far are valid for the mixture as whole. From now on, it is assumed that the mixture is a two-phase, two-constituent homogeneous mixture. The gaseous phase is formed by the vapor of a liquid and a inert gas, whereas the liquid phase is constituted of liquid solely. The liquid and gaseous phases are assumed to coexist at every material point and time instant, To take it into account, we consider an internal variable  $\alpha$  which denotes the gaseous volume fraction. The gaseous phase volume fraction  $\alpha$ ,  $\alpha \in (0, 1)$ , is defined as being the ratio of the volume occupied by the vapor along with the inert gas and the total volume of the mixture. Thus, the mass density of the mixture can be

expressed as:

$$\varrho = (1 - \alpha)\varrho_l + \alpha(\varrho_v + \varrho_g) \quad (4)$$

in which  $\varrho_l$ ,  $\varrho_v$  and  $\varrho_g$  stand for the mass densities of the liquid, vapor and gas, respectively. Since our objective is to describe the liquid-vapor transformations, it is convenient to split the mass conservation of the whole mixture given by Eq. (1) for each phase:

$$\alpha(\dot{\varrho}_v + \dot{\varrho}_g) + (\varrho_v + \varrho_g)\dot{\alpha} + \alpha(\varrho_v + \varrho_g)\nabla \cdot \mathbf{v} - \Gamma = 0 \quad (5)$$

$$(1 - \alpha)\dot{\varrho}_l - \varrho_l\dot{\alpha} + (1 - \alpha)\varrho_l\nabla \cdot \mathbf{v} + \Gamma = 0 \quad (6)$$

in which  $\Gamma$  is a source term which represents the mass rate of phase change per unit volume. When  $\Gamma > 0$ , liquid is transformed into vapor. On the other hand, if  $\Gamma < 0$ , then vapor is transformed into liquid. Finally, if  $\Gamma = 0$ , then there is no phase change and, consequently, the liquid and vapor phases are conserved independent from each other.

To complete the problem description, we must add to the foregoing equations the constitutive relationships for the mixture in such a way that inequality (3) be satisfied no matter the external actions, initial and boundary conditions.

### 3. CONSTITUTIVE THEORY

The constitutive relations describing the macroscopic mechanical behavior of the mixture are derived in the framework of the Thermodynamics of Irreversible Processes. In this theory, once the local state of the material has been characterized by means of an appropriate choice of a set of state variables, two thermodynamical potentials - the Helmholtz free energy and a pseudo-potential of dissipation - are sufficient to derive a complete set of constitutive equations.

For this particular problem, we choose as state variables the local mass densities of the liquid  $\varrho_l$ , the vapor  $\varrho_v$  and the gas  $\varrho_g$ , the gaseous phase volume fraction  $\alpha$  and the absolute temperature  $\theta$ . This specific choice restricts the validity of the forthcoming analysis to a particular class of Newtonian and generalized Newtonian fluids, since the list of state variables might incorporate other variables not considered herein as in the case of many non-Newtonian fluids.

As we shall see, the restriction associated with  $\alpha$  is treated in this work as a physical property in the constitutive equations. This approach has already been used in the modeling of other problems in continuum mechanics.

#### 3.1 Helmholtz Free Energy

Following the classic assumption of the Thermodynamics of Irreversible Processes, the free energy per total unit volume  $\Psi$  is supposed to be a function of the state variables. Since the fluid is regarded as a mixture of the two phases, its behavior is supposed to comprise a combination of the liquid and vapor thermo mechanical properties. Thus, the following form is proposed:

$$\Psi(\varrho_l, \varrho_v, \varrho_g, \alpha, \theta) := \Psi'(\varrho_l, \varrho_v, \varrho_g, \alpha, \theta) + \bar{I}(\alpha) \quad (7)$$

in which

$$\Psi'(\varrho_l, \varrho_v, \varrho_g, \alpha, \theta) := (1 - \alpha)\varrho_l\Psi_l(\varrho_l, \theta) + \alpha\varrho_v\Psi_v(\varrho_v, \theta) + \alpha\varrho_g\Psi_g(\varrho_g, \theta) \quad (8)$$

$$\Psi_l(\varrho_l, \theta) := -C_l\theta\log\theta + K_l\log\varrho_l + M/\varrho_l \quad (9)$$

$$\Psi_v(\varrho_v, \theta) := -C_v\theta\log\theta + K_v\log\varrho_v + L \quad (10)$$

$$\Psi_g(\varrho_g, \theta) := -C_g\theta\log\theta + K_g\log\varrho_g \quad (11)$$

$$\bar{I}(\alpha) := \begin{cases} 0 & , \text{ if } \alpha \in (0, 1) \\ +\infty & , \text{ otherwise.} \end{cases} \quad (12)$$

In the above expressions,  $\Psi_\iota$ , with  $\iota \in \{l, v, g\}$ , represent the free energies per unit mass of the liquid, vapor and gas, respectively. The terms  $C_\iota$ , with  $\iota \in \{l, v, g\}$ , are the specific heats at constant volume of the liquid, the vapor and the gas, respectively. The parameters  $K_\iota = \hat{K}_\iota(\theta)$ , with  $\iota \in \{l, v, g\}$ ,  $M = \hat{M}(\theta)$  and  $L = \hat{L}(\theta)$  are positive material functions which depend on the temperature. The function  $\hat{L}$  is associated with the phase change latent heat and the functions  $\hat{K}_\iota$ , with  $\iota \in \{l, v, g\}$ , are related to the isothermal compressibility of the constituents. Since we have implicitly assumed that the vapor and the gas behaves as a perfect gases, then  $\hat{K}_v(\theta) = R_v\theta$  and  $\hat{K}_g(\theta) = R_g\theta$ ;  $R_v$  and  $R_g$  being the gas constants. The choice of the free energies  $\Psi_l$  and  $\Psi_v$  were done in order to approximate, for a fixed  $\theta$  far below from the critical temperature, the liquid and vapor behaviors, respectively, in the subcooled-liquid and superheated-vapor regions of a simple compressible substance.

The term  $\bar{I}(\alpha)$  is the indicator function of the convex set  $(0, 1) \subset \mathbb{R}$ . It is included in the free energy potential to take the internal constraint of the mixture into account as a constitutive assumption. In other words, the term  $\bar{I}(\alpha)$  prevents  $\alpha$  from getting out of its admissible interval since it would be required an infinite amount of energy to do this.

The state laws, relating the reversible components of the thermodynamic forces to the state variables, are obtained from the free energy potential as follows:

$$B^{\rho_l} := \frac{\partial \Psi'}{\partial \rho_l} = (1 - \alpha) \left( \Psi_l + \rho_l \frac{\partial \Psi_l}{\partial \rho_l} \right) = (1 - \alpha) g_l \quad (13)$$

$$B^{\rho_v} := \frac{\partial \Psi'}{\partial \rho_v} = \alpha \left( \Psi_v + \rho_v \frac{\partial \Psi_v}{\partial \rho_v} \right) = \alpha g_v \quad (14)$$

$$B^{\rho_g} := \frac{\partial \Psi'}{\partial \rho_g} = \alpha \left( \Psi_g + \rho_g \frac{\partial \Psi_g}{\partial \rho_g} \right) = \alpha g_g \quad (15)$$

$$B^\alpha := \frac{\partial \Psi'}{\partial \alpha} + h, \text{ with } h \in \partial \bar{I}(\alpha) \quad (16)$$

in which

$$p_l := \rho_l^2 \frac{\partial \Psi_l}{\partial \rho_l} \quad ; \quad p_v := \rho_v^2 \frac{\partial \Psi_v}{\partial \rho_v} \quad ; \quad p_g := \rho_g^2 \frac{\partial \Psi_g}{\partial \rho_g} \quad (17)$$

$$\frac{\partial \Psi'}{\partial \alpha} = \rho_v \Psi_v + \rho_g \Psi_g - \rho_l \Psi_l \quad (18)$$

In the above equations,  $p_i$  and  $g_i$ , with  $i \in \{l, v, g\}$  stand for the partial pressures and Gibbs specific free energies of the constituents in the mixture, respectively. The term  $\partial \bar{I}(\alpha)$  in Eq. (16) is the subdifferential of the indicator function  $\bar{I}(\alpha)$  at  $\alpha$ , which is given by the set Moreau *et al.* (1988); Ekeland and Teman (1976);

$$\partial \bar{I}(\alpha) = \{h \in \mathbb{R} \mid \bar{I}(\alpha^*) - \bar{I}(\alpha) \geq h(\alpha^* - \alpha); \quad \forall \alpha^* \in (0, 1)\} \quad (19)$$

From the above definition, we can see that  $\partial \bar{I}(0 < \alpha < 1) = \{0\}$  and  $\partial \bar{I}(\alpha) = \emptyset$  if  $\alpha \ni (0, 1)$ .

Once we have chosen a specific form for  $\Psi$  we are able to compute its material time derivative  $\dot{\Psi}$  which appears in Eq. (3):

$$\dot{\Psi} = \frac{\partial \Psi_l}{\partial \rho_l} \dot{\rho}_l + \frac{\partial \Psi_v}{\partial \rho_v} \dot{\rho}_v + \frac{\partial \Psi_g}{\partial \rho_g} \dot{\rho}_g + (\rho_v \Psi_v + \rho_g \Psi_g - \rho_l \Psi_l) \dot{\alpha} + \lim_{\substack{\Delta t \rightarrow 0 \\ \Delta t > 0}} \frac{\bar{I}(\alpha(t)) - \bar{I}(\alpha(t - \Delta t))}{\Delta t} \quad (20)$$

In computing  $\dot{\Psi}$  care should be taken since  $\Psi$  is not a smooth function. By using the definition of the subdifferential Eq. (19) it can be shown that:

$$\lim_{\substack{\Delta t \rightarrow 0 \\ \Delta t > 0}} \frac{\bar{I}(\alpha(t)) - \bar{I}(\alpha(t - \Delta t))}{\Delta t} \leq h \dot{\alpha} = 0, \quad \forall h \in \partial \bar{I}(\alpha(t)) \quad (21)$$

When the above result is used along with the state laws Eqs. (13-16), Eq. (20) can be written as:

$$\dot{\Psi} \leq B^{\rho_l} \dot{\rho}_l + B^{\rho_v} \dot{\rho}_v + B^{\rho_g} \dot{\rho}_g + B^\alpha \dot{\alpha} \quad (22)$$

Finally, inequality Eq. (22) can be used to find a lower bound  $\hat{d}$  for the dissipation  $d$  in Eq. (3) such as:

$$d \geq \hat{d} = -(p + \Psi) \nabla \cdot \mathbf{v} + \mathbf{S} \cdot \mathbf{D} - B^{\rho_l} \dot{\rho}_l - B^{\rho_v} \dot{\rho}_v - B^{\rho_g} \dot{\rho}_g - B^\alpha \dot{\alpha} \quad (23)$$

To obtain a complete set of constitutive equations, it suffices to specify a pseudo-potential of dissipation from which complementary laws are derived in such a way that the local version of the second law Eq. (3) or Eq. (23) is always verified, regardless the initial and boundary conditions.

### 3.2 Pseudo-Potential of Dissipation

To assign a dissipative character to the mixture, and also to ensure that the second law is always satisfied, we assume that the pseudo-potential of dissipation  $\Phi$  is an objective, convex and differentiable function of  $\mathbf{D}$ ,  $\Gamma$  and  $\dot{\alpha}$ , i.e.  $\Phi(\mathbf{D}, \Gamma, \dot{\alpha}; \alpha)$ , where  $\alpha$  is taken as an independent parameter, with the following properties:

$$\Phi(\mathbf{D}, \Gamma, \dot{\alpha}; \alpha) \geq 0, \quad \forall (\mathbf{D}, \Gamma, \dot{\alpha}) \quad \text{and} \quad \Phi(\mathbf{0}, 0, 0; \alpha) = 0. \quad (24)$$

The additional information associated with the dissipative behavior can be obtained from  $\Phi$  through the following complementary laws:

$$\mathbf{S} = \frac{\partial \Phi}{\partial \mathbf{D}}, \quad B^\Gamma = \frac{\partial \Phi}{\partial \Gamma}, \quad B^{\dot{\alpha}} = \frac{\partial \Phi}{\partial \dot{\alpha}} \quad (25)$$

in which  $B^\Gamma$  and  $B^\alpha$  are the thermodynamic forces associated with the phase change transformation and with the evolution of the gaseous volume fraction.

In addition, if we choose the rate of the energy dissipation  $\hat{d}$  as being given by;

$$\hat{d} = \mathbf{S} \cdot \mathbf{D} + B^\Gamma \Gamma + B^\alpha \dot{\alpha} - \bar{I}(\alpha) \mathbf{D} \cdot \mathbf{I} \quad (26)$$

for any actual evolution, with  $\mathbf{I}$  being the second-order identity tensor, then we get from the convexity property of  $\Phi$  that:

$$\hat{d} = \mathbf{S} \cdot \mathbf{D} + B^\Gamma \Gamma + B^\alpha \dot{\alpha} \geq \Phi(\mathbf{D}, \Gamma, \dot{\alpha}; \alpha) - \Phi(\mathbf{0}, 0, 0; \alpha) \geq 0, \quad \forall (\mathbf{D}, \Gamma, \dot{\alpha}) \quad \text{and} \quad \alpha \in (0, 1) \quad (27)$$

In view of Eq. (27), it is easy to see that  $\hat{d} \geq 0$  for any actual evolution and so the Second Law of Thermodynamics Eq. (3) is always satisfied. It is worth noting that the last term in the right-hand side of Eq. (26) was introduced to give the model a thermodynamically consistent character. If  $\alpha$  could be out of the interval  $(0, 1)$ , then the dissipation would become negative or infinite what is not possible. It is important to remark that  $\Phi$  does not need to be a convex function. The convexity property is only a sufficient condition in order to satisfy Eq. (27).

From the mechanical viewpoint, Eq. (26) establishes that the rate of energy dissipation is the sum of three parcels. The first parcel in the right-hand side of Eq. (26) is associated with viscous effects due to motion of the mixture as whole. The other two parcels reflect internal changes in the mixture. The second and third terms express the dissipative mechanisms associated with the evaporation-condensation process and with viscous dissipation in the liquid as a result of the expansional-compressional motion of the vapor phase.

### 3.3 Constitutive Laws

The set of constitutive equations is completed provided a specific form of  $\Phi$  is chosen. Before specifying a form for  $\Phi$ , however, it should be noticed that the thermodynamic forces  $B^{\rho_l}$ ,  $B^{\rho_v}$ ,  $B^{\rho_g}$ ,  $B^\alpha$ ,  $B^\Gamma$  and  $B^\alpha$  are not independent. There exists an intrinsic relationship among them which is a consequence of the mass balance conservation equations and the choice of the rate of energy dissipation given by Eq. (26). As we shall see, there exists an expression for the thermodynamic pressure of the mixture which is a consequence of the constitutive assumptions made so far.

In effect, for any actual evolution of the mixture with mass density given by Eq. (4), the mass balance Eqs. (5) and (6) defines a subspace of the linear space spanned by  $\mathbf{v}$ ,  $\dot{\rho}_l$ ,  $\dot{\rho}_v$ ,  $\dot{\rho}_g$  and  $\dot{\alpha}$ . So, in order that Eq. (23) be equal to Eq. (26) one must have:

$$-(p + \Psi) \nabla \cdot \mathbf{v} - B^{\rho_l} \dot{\rho}_l - B^{\rho_v} \dot{\rho}_v - B^{\rho_g} \dot{\rho}_g - B^\alpha \dot{\alpha} = B^\Gamma (\dot{\alpha} \rho_v + \alpha \dot{\rho}_v + \alpha \rho_v \nabla \cdot \mathbf{v}) + B^\alpha \dot{\alpha} - \bar{I}(\alpha) \nabla \cdot \mathbf{v} + \eta ((\rho_v - \rho_l) \dot{\alpha} + (1 - \alpha) \dot{\rho}_l + \alpha \dot{\rho}_v + ((1 - \alpha) \rho_l + \alpha \rho_v) \nabla \cdot \mathbf{v}) + \xi (\alpha \dot{\rho}_g + \rho_g \dot{\alpha} + \alpha \rho_g \nabla \cdot \mathbf{v}) \quad (28)$$

in which  $\eta$  and  $\xi$  are Lagrange multipliers. Since equation Eq. (28) holds for any independent evolution of  $\mathbf{v}$ ,  $\dot{\rho}_l$ ,  $\dot{\rho}_v$ ,  $\dot{\rho}_g$  and  $\dot{\alpha}$ , it comes that:

$$-p - \Psi - \alpha \rho_v B^\Gamma - \eta (\alpha \rho_v + (1 - \alpha) \rho_l) + \bar{I}(\alpha) - \xi \alpha \rho_g = 0 \quad (29)$$

$$-B^{\rho_l} - \eta (1 - \alpha) = 0 \quad (30)$$

$$-B^{\rho_v} - \alpha \eta - \alpha B^\Gamma - \eta \alpha = 0 \quad (31)$$

$$-B^{\rho_g} - \xi \alpha = 0 \quad (32)$$

$$-B^\alpha - B^\alpha - \rho_v B^\Gamma + \eta (\rho_l - \rho_v) - \xi \rho_g = 0 \quad (33)$$

Since  $\alpha \in (0, 1)$  for every actual evolution, the Lagrange multipliers  $\eta$  and  $\xi$  can be eliminated from the above equations giving rise to the following relationships:

$$p = (1 - \alpha) p_l + \alpha (p_v + p_g) \quad (34)$$

$$B^\alpha = p_v + p_g - p_l \quad (35)$$

$$B^\Gamma = p_l - p_v \quad (36)$$

Equations(34-36) state that the thermodynamic pressure of the mixture is the mean value between the liquid and gaseous pressures, the thermodynamic force associated with the gaseous volume fraction is the pressure difference between the gaseous and liquid pressures and that thermodynamic driving force associated with the phase change transformation is the Gibbs free energy difference between the liquid and vapor.

Without losing generality, if we assume the mixture behaves as a Newtonian fluid one possible choice for  $\Phi$  is:

$$\Phi(\mathbf{D}, \Gamma, \dot{\alpha}; \alpha) = \frac{\lambda}{2} (\mathbf{D} \cdot \mathbf{I})^2 + \mu \mathbf{D} \cdot \mathbf{D} + \frac{1}{2\beta} \Gamma^2 + \frac{\epsilon}{2} \dot{\alpha}^2 \quad (37)$$

where  $\beta = \hat{\beta}(\alpha)$  and  $\epsilon = \hat{\epsilon}(\alpha)$  are positive generic functions of  $\alpha$ .

The material parameters  $\lambda = \hat{\lambda}(\alpha)$  and  $\mu = \hat{\mu}(\alpha)$  are average properties between, respectively,  $\lambda_l$  and  $\lambda_v$  and  $\mu_l$  and  $\mu_v$  having  $\alpha$  as a weighting factor, such that  $\hat{\lambda}(\alpha = 0) = \lambda_l$ ,  $\hat{\lambda}(\alpha = 1) = \lambda_v$ ,  $\hat{\mu}(\alpha = 0) = \mu_l$  and  $\hat{\mu}(\alpha = 1) = \mu_v$ . These material parameters must be such that  $\mu \geq 0$  and  $\lambda + 2/3\mu \geq 0$  in order to satisfy (19). For this specific choice, the state and complementary laws become:

$$\mathbf{T} = -p\mathbf{I} + \lambda(\mathbf{D}\cdot\mathbf{I}) + 2\mu\mathbf{D} \quad (38)$$

$$p = (1 - \alpha)p_l + \alpha(p_v + p_g) \quad (39)$$

$$p_l = K_l \varrho_l - M \quad (40)$$

$$p_v = K_v \varrho_v \quad (41)$$

$$p_g = K_g \varrho_g \quad (42)$$

$$\Gamma = \beta B^\Gamma \quad (43)$$

$$B^\Gamma = g_l - g_v \quad (44)$$

$$\epsilon \dot{\alpha} = B^{\dot{\alpha}} \quad (45)$$

$$B^{\dot{\alpha}} = p_v + p_g - p_l \quad (46)$$

Equations (1), (2), (4-6) and (38-46) completed by suitable boundary and initial conditions describe the isothermal momentum-driven cavitating flow of a compressible Newtonian fluid carrying an inert gas. Since the constitutive relation for a Newtonian fluid is well-known, we shall from now on focus attention on the phase change transformation and the vapor volume fraction evolution.

To obtain the expression of the Gibbs free energy difference between the liquid and the vapor it is necessary to analyze the particular case of equilibrium in which  $\mathbf{v} = \mathbf{0}$ ,  $\Gamma = 0$  and  $\dot{\alpha} = 0$ . If, in addition to these assumptions, it is assumed that there is no inert gas in suspension in the liquid, then it comes out that:

$$p = p_l = p_v = p_{sv} \quad (47)$$

$$g_l = g_v \quad (48)$$

in which  $p_{sv}$  stands for the saturated vapor pressure. The Eqs.(47-48) express the classical result of the thermostatics theory which states that the pressures of the phases as well as their Gibbs free energies must be equal in equilibrium. By taking Eq. (47) into Eq. (48) the following relationship may be written:

$$K_l - K_l \frac{M}{p_{sv} + M} - C_l \theta \log \theta + K_l \log \left( \frac{p_{sv} + M}{K_l} \right) = K_v - C_v \theta \log \theta + K_v \log \frac{p_{sv}}{K_v} + L \quad (49)$$

Equation (49) establishes that, for a fixed temperature, there exists a unique saturated vapor pressure  $p_{sv}(\theta)$ . Now, by making use Eq. (49), we can finally express the Gibbs free energy difference between the liquid and vapor as:

$$g_l - g_v = K_l \log \left( \frac{p_l + M}{p_{sv} + M} \right) + K_v \log \frac{p_{sv}}{p_v} + K_l M \left( \frac{1}{p_{sv} + M} - \frac{1}{p_l + M} \right) \quad (50)$$

Since in the absence of free gas  $p_l = p_v$  in equilibrium, it can be seen from Eq. (50) that  $g_l - g_v = 0$  if and only if  $p_l = p_v = p_{sv}$ . However, when a free gas is present  $p_l = p_v + p_g$  and it can be verified based on Eq. (50) that the condition  $g_l - g_v = 0$  is satisfied for different values of  $p_l$  and  $p_v$ . In other words, there will exist more than one equilibrium state when a free gas is present in the mixture according to its partial pressure. This condition enables the simulation of the phase change transformation in cavitating flows without the need to consider the extremal situations of pure liquid  $\alpha = 0$  and pure vapor  $\alpha = 1$ , for which different set of equations must be solved when compared to those associated with the cavitating flow region ( $\alpha \in (0, 1)$ ).

#### 4. AN ANALYSIS OF THE MODEL PREDICTIONS

To investigate the ability of the proposed model in properly describing the phase change transformation that takes place in cavitating flows, numerical simulations are carried out and analyzed by comparing the theoretical predictions of the model presented herein with that of the model without the presence of free gas published in . The capability of the model in describing typical situations is illustrated by a simple quantitative example. For this purpose, consider a sample of a liquid-vapor-air mixture of pure water in equilibrium at a total pressure  $p = p_l = p_v + p_g = (1 + 0.1\%)p_{sv}$  slightly above of the saturation pressure of  $p_{sv} = 2.34kPa$  (at  $\theta = 293K$ ) having, at  $t = 0$ , a gaseous volume fraction of  $\alpha(t = 0) = 0.2$ . To simplify the interpretation of the results and to reduce the number of material parameters to be specified, we shall consider that the phase-change transformation is the only dissipative mechanism present. So, the constitutive constants of the model for the mixture air-water at  $293K$  are taken as:  $K_v = 135kJ/kg$ ,  $K_g = 84.1kJ/kg$ ,

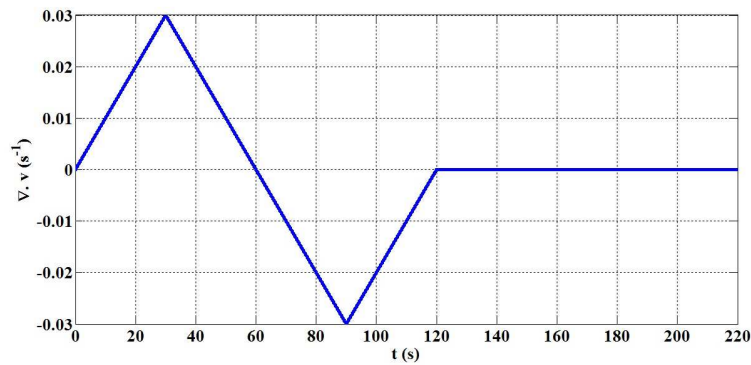


Figure 1. Input excitation history.

$K_l = 990 \text{ kJ/kg}$ ,  $M = 988 \text{ MPa}$  and  $\epsilon = 0$ . By virtue of this, we can see through Eqs. (45) and (46) that  $p = p_l = p_v + p_g$  for both equilibrium and non-equilibrium states.

The mixture is confined into a device and can have its original volume expanded or contracted along the time by means of external actions. It is further assumed that, at any time instant, the mixture is almost everywhere homogeneous throughout its volume, so that the convective terms of the balance equations of mass can be neglected. Under these circumstances, the time rate of the volume change per unit volume of the whole mixture is described by the term  $\nabla \cdot \mathbf{v}$ , which is assumed to be a prescribed input function of the time as represented in Fig. 1. The behavior of the mixture during the dynamic cycle of expansion and contraction is characterized by Eqs. (1), (5) and (6) which form a non-linear initial-value problem of ordinary differential equations for the unknowns  $\alpha$ ,  $p_l$  and  $p_v$ . This problem is numerically solved by using the Gear's method; an adaptive numerical method up to fifth-order suitable for solving stiff ordinary differential equations.

In what follows, the predictions of the model presented herein are compared with those obtained by using the model proposed in Freitas Rachid (2003) which does not take the presence of an inert gas into account. However, in both models the phase change mechanism is described by the same expressions by assuming that  $\beta = 10^{-8} \text{ kg}^2/\text{m}^3 \text{ kJ s}$ . As it has been shown in Freitas Rachid (2003), values of  $\beta$  of this order properly describe the irreversible phase change transformations for which there exists a non-null rate of energy dissipation.

The pressure histories for the vapor and liquid are presented in Fig. 2 for both models with and without inert gas. In contrast to what is observed when reversible phase changes occurs (for which the pressure of the mixture remains constant and equal to the saturated vapor pressure), it can be seen in Fig. 2 that the liquid and vapor pressures varies around the saturated vapor pressure as the phase change transformations takes place. No significant difference is observed between the two models. The pressure of the mixture decreases during the first-half period and reaches values two times greater than the saturation pressure in the second-half of the cycle. In this situation, it can be noticed the presence of a relaxation phenomenon since the pressure takes approximately 20s to come to its equilibrium value after the rate of volume change has vanished. This feature is corroborated by observing the mass rate of phase change history illustrated in Fig. 3. Once again, the responses exhibited by the models with and without inert gas are virtually the same. The pressure history of the inert gas is shown in Fig. 4 for both models with and without inert gas. The free air in solution expands and contracts during the phase change transformation in such a way with the pressure remaining below its initial pressure.

Figure 5 presents the plot of the rate of energy dissipation,  $d = B^T \Gamma$ , as a function of time for both models. The responses predicted by the models with and without free gas are superimposed. Since the energy dissipation represents the area below the curve, it becomes clear that the dissipation is greater in the second-half of the cycle than that observed in the first-half one. It can be explained by noting in Fig. 5 that the amount of liquid formed in the second period is greater than that of vapor generated in the first one. A simple calculation reveals that the overall energy dissipation is approximately  $0.75 \text{ kJ/m}^3$ . Once again, no noticeable difference between the curves with and without free gas are observed what demonstrates that the model with free gas is capable to describe the phase change transformation as if there were no free gas actually.

The evolution of the gaseous volume fraction in time is presented in Fig. 6 for the models with and without free gas. The excess of liquid generated in the second-half of the cycle reported in the past paragraph is responsible for an absolute difference between the final and initial values of the vapor volume fraction of about  $6.27 \times 10^{-10}$ . It corresponds to an increase in the final mass density of the mixture of  $1.57 \times 10^{-9}\%$  in relation to its initial value.

The physical phenomenon of change phase analyzed herein can be better visualized by plotting the total pressure against the gaseous volume fraction as shown Fig. 7. The hysteresis loops clearly demonstrate that irreversible character of the cavitation phenomenon described by the model.

Finally, it should be pointed out that although linear equation of states have been used in the simulations presented herein, the results remain unchanged whether cubic equation of states are taken into account as it has been demonstrated

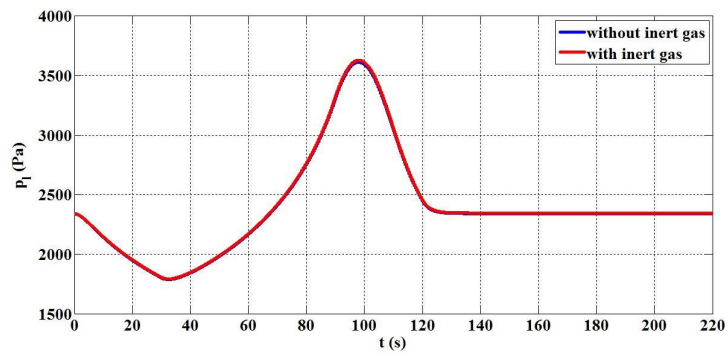


Fig. 2a

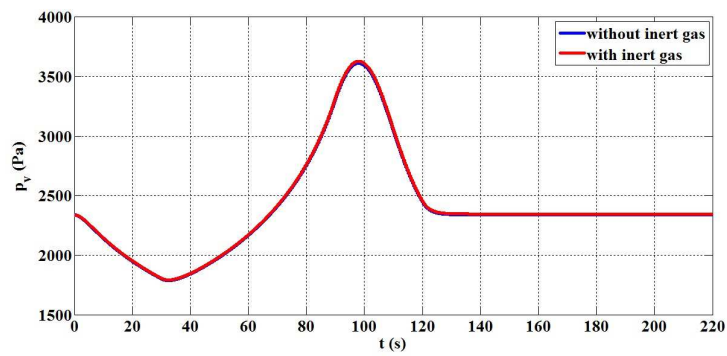


Fig. 2b

Figure 2. Pressure histories of the liquid (Fig. a) and vapor (Fig. b) for the models with and without inert gas.

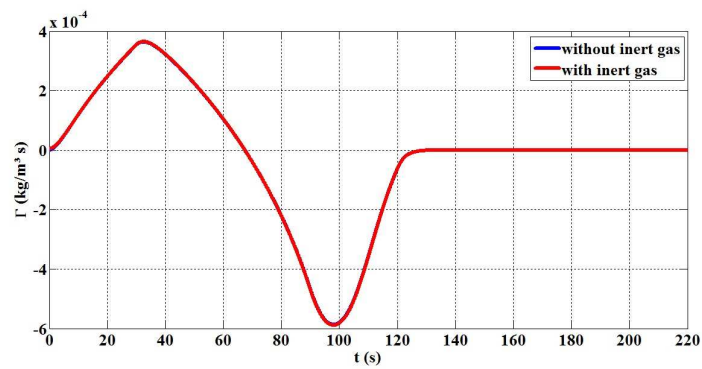


Figure 3. Mass rate of phase change for the models with and without inert gas.

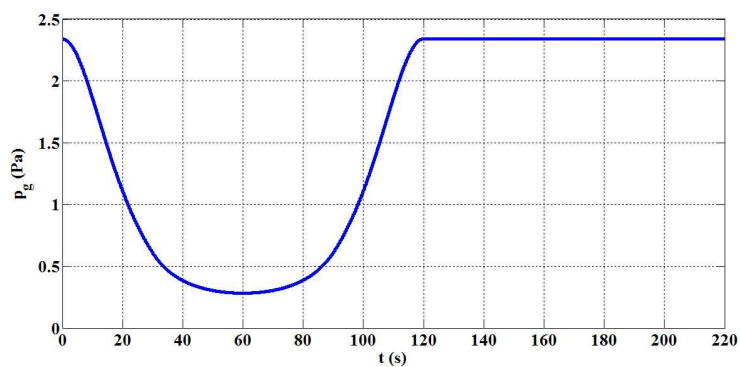


Figure 4. Pressure history of the inert gas.



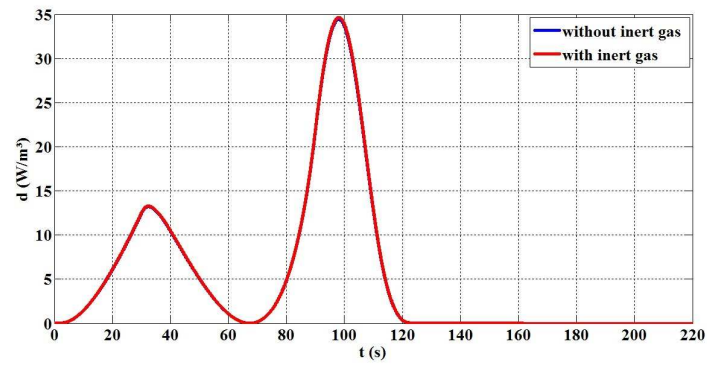


Figure 5. Rate of energy dissipation histories for the models with and without inert gas.

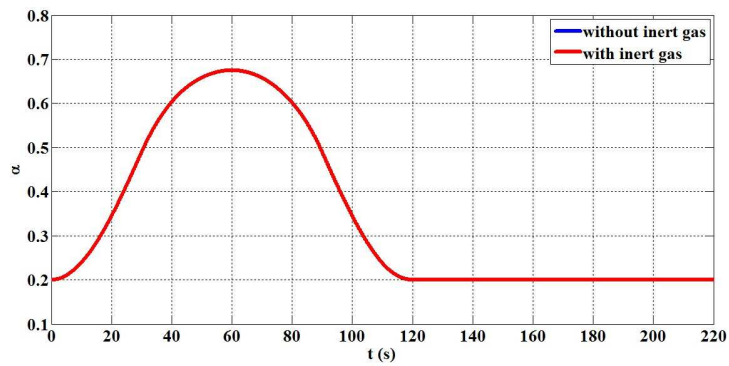


Figure 6. Gaseous volume fraction histories for the models with and without inert gas.

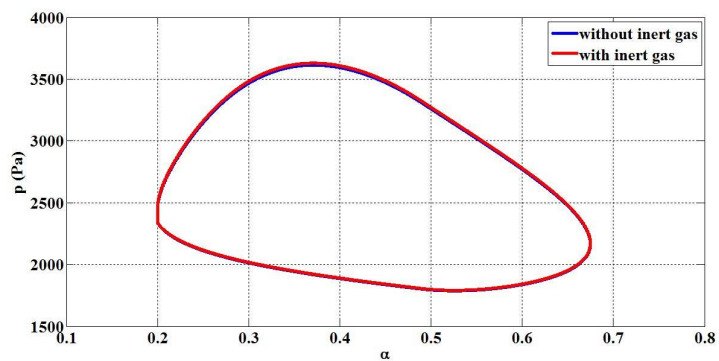


Figure 7. Total pressure versus gaseous volume fraction for the models with and without inert gas.

in Assumpção and Freitas Rachid (2008).

## 5. CONCLUDING REMARKS

A simple continuum thermodynamic model for describing isothermal cavitation in fluid flows has been presented and analyzed. The fluid is regarded as being a mixture of liquid, vapor and an inert gas, all of them treated as compressible constituents, which coexist at a same material point and time instant. The gaseous volume fraction is considered an internal variable and its constraint is treated as a material property. Dissipative effects associated with the phase change transformation as well as with the volume fraction evolution are taken into account in the model within the context of the Thermodynamics of Irreversible Processes. It is shown that by using a same set of equations for the whole flow domain, the cavitation phenomenon can be adequately described as an irreversible process to which an energy dissipation is coherently associated with. The model simplicity and its capability to properly describe real situations enable its usage as a promising tool in the study of vaporous cavitation.

## 6. ACKNOWLEDGEMENTS

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## 7. REFERENCES

The list of references must be introduced as a new section, located at the end of the paper. The first line of each reference must be aligned at left. All the other lines must be indented by 0.5 cm from the left margin. All references included in the reference list must have been mentioned in the text. References must be listed in alphabetical order, according to the last name of the first author. See the following examples:

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