OPTIMAL TWO-IMPULSE TRAJECTORIES FOR EARTH-MOON FLIGHT

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Abstract. *In this paper, a systematic study of optimal trajectories for Earth-Moon flight of a space vehicle is presented. The optimization criterion is the total characteristic velocity and three dynamical models are used. The optimization problem has been formulated using the patched-conic approximation and two versions of the planar circular restricted three-body problem (PCR3BP). In all cases, the problem has been solved using a gradient algorithm in conjunction with Newton-Raphson method.*

Keywords*: Earth-Moon flight, optimal trajectories, minimum delta-V.*

1. INTRODUCTION

In this paper, the problem of transferring a space vehicle from a circular low Earth orbit (LEO) to a circular low Moon orbit (LMO) with minimum fuel consumption is studied. The class of two impulse trajectories is considered: a first accelerating velocity impulse tangential to the space vehicle velocity relative to Earth is applied at a circular low Earth orbit and a second braking velocity impulse tangential to the space vehicle velocity relative to Moon is applied at a circular low Moon orbit (Miele and Mancuso, 2001). The minimization of fuel consumption is equivalent to the minimization of the total characteristic velocity which is defined by the arithmetic sum of velocity changes (Marec, 1979).

Three dynamical models are used to describe the motion of the space vehicle: the well-known patched-conic approximation (Bate et al, 1971) and two versions of the planar circular restricted three-body problem (PCR3BP). One version of PCR3BP assumes the Earth is fixed in space; this version will be referred as simplified version of PCR3BP and it is same one used by Miele and Mancuso (2001). The second version of PCR3BP assumes the Earth moves around the center of mass of the Earth-Moon system (Szebehely, 1967; Roy, 2005). In all cases, the optimization problem has one degree of freedom and can be solved by means of an algorithm based on gradient method (Miele et at, 1969) in conjunction with Newton-Raphson method (Stoer and Bulirsch, 2002). The analysis of optimal trajectories is carried out considering several final altitudes of a clockwise or counterclockwise circular low Moon orbit for a specified altitude of a counterclockwise circular low Earth orbit which corresponds to the altitude of the Space Station. The results are compared to the ones obtained by Miele and Mancuso (2001) who used the sequential gradient-restoration algorithm for solving the optimization problem (Miele et at, 1969).

2. OPTIMIZATION PROBLEM BASED ON PATCHED-CONIC APPROXIMATION

In this section, the optimization problem based on the patched-conic approximation is formulated. A detailed presentation of the patched-conic approximation can be found in Bate et al (1971). The following assumptions are employed:

1. The Earth is fixed in space;
2. The eccentricity of the Moon orbit around Earth is neglected;
3. The flight of the space vehicle takes place in the Moon orbital plane;
4. The gravitational fields of Earth and Moon are central and obey the inverse square law;
5. The trajectory has two distinct phases: geocentric and selenocentric trajectories. The geocentric phase corresponds to the portion of the trajectory which begins at the point of application of the first impulse and extends to the point of entering the Moon's sphere of influence. The selenocentric phase corresponds to the portion of trajectory in the Moon's sphere of influence and ends at the point of application of the second impulse. In each one of these phases, the space vehicle is under the gravitational attraction of only one body, Earth or Moon;
6. The class of two impulse trajectories is considered. The impulses are applied tangentially to the space vehicle velocity relative to Earth (first impulse) and Moon (second impulse).

An Earth-Moon trajectory is completely specified by four quantities: r_0 - radius of circular LEO; v_0 - velocity of the space vehicle at the point of application of the first impulse after the application of the impulse; φ_0 - flight path angle at the point of application of the first impulse and γ_0 - phase angle at departure. These quantities must be determined such that the space vehicle is injected into a LMO with specified altitude after the application of the second impulse. It is particularly convenient to replace γ_0 by the angle λ_1 which specifies the point at which the geocentric trajectory crosses the Moon's sphere of influence.

Equations describing each phase of an Earth-Moon trajectory are briefly presented in what follows. It is assumed that the geocentric trajectory is direct and that lunar arrival occurs prior to apoapsis of the geocentric orbit. Figure 1 shows the geometry of the geocentric phase.

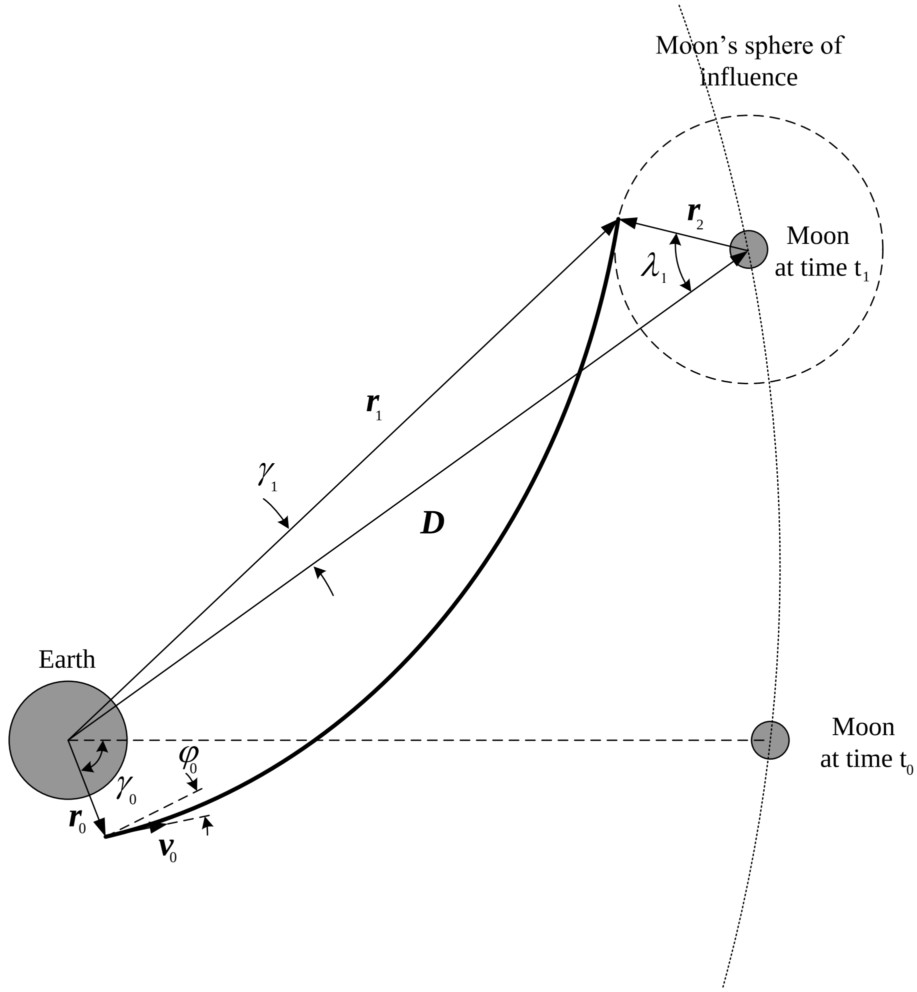

Figure 1 – Geometry of the geocentric phase

For a given set of initial conditions (r_0, v_0, φ_0), energy and angular momentum of the geocentric trajectory can be determined from the equations

$$\varepsilon = \frac{1}{2}v_0^2 - \frac{\mu_E}{r_0}, \tag{1}$$

$$h = r_0 v_0 \cos\varphi_0 , \tag{2}$$

where μ_E is Earth gravitational parameter.

From the geometry of the geocentric phase (Fig. 1), one finds

$$r_1 = \sqrt{D^2 + R_S^2 - 2DR_S\cos\lambda_1}\ , \tag{3}$$

$$\sin\gamma_1 = R_S \sin\lambda_1 / r_1 , \tag{4}$$

where D is the distance from the Earth to the Moon, R_S is the radius of the Moon's sphere of influence. Subscript 1 denotes quantities of the geocentric trajectory calculated at the edge of the Moon's sphere of influence.

From energy and angular momentum of the geocentric trajectory, one finds

$$v_1 = \sqrt{2(\varepsilon + \mu_E / r_1)}\,, \tag{5}$$

$$\cos\varphi_1 = \frac{h}{r_1 v_1}\,. \tag{6}$$

The selenocentric phase begins at the point at which the geocentric trajectory crosses the Moon's sphere of influence. Figure 2 shows the geometry of the selenocentric phase for a clockwise arrival to LMO. Thus,

$$r_2 = R_S\,, \tag{7}$$

$$\boldsymbol{v_2} = \boldsymbol{v_1} - \boldsymbol{v}_M\,, \tag{8}$$

where $\boldsymbol{v}_M$ is the velocity vector of the Moon relative to the center of the Earth. Subscript 2 denotes quantities of the selenocentric trajectory calculated at the edge of the Moon's sphere of influence.

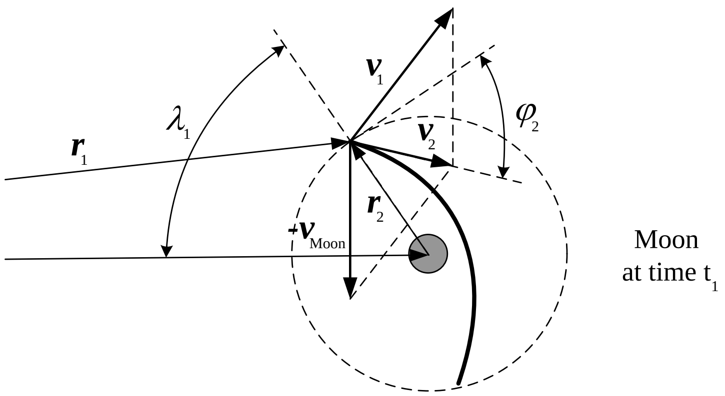

Figure 2 – Geometry of the selenocentric phase

From Eq. (8), one finds

$$v_2 = \sqrt{v_1^2 + v_M^2 - 2 v_1 v_M \cos(\varphi_1 - \gamma_1)}\,, \tag{9}$$

$$\tan(\lambda_1 \pm \varphi_2) = -\frac{v_2 \sin(\varphi_1 - \gamma_1)}{v_M - v_2 \cos(\varphi_1 - \gamma_1)}\,. \tag{10}$$

The upper sign refers to clockwise arrival to LMO and the lower sign refers to counterclockwise to LMO.

The semi-major axis a_f and eccentricity e_f of the selenocentric trajectory are given by

$$a_f = \frac{r_2}{2 - Q_2}\,, \tag{11}$$

$$e_f = \sqrt{1 + Q_2 (Q_2 - 2)\cos^2\varphi_2}\,, \tag{12}$$

where $Q_2 = r_2 v_2^2 / \mu_M$ and μ_M is Moon gravitational parameter.

The second impulse is applied at the periselenium of the selenocentric trajectory such that the terminal conditions, before the impulse, are defined by

$$r_{p_M} = a_f\left(1 - e_f\right), \tag{13}$$

$$v_{p_M} = \sqrt{\frac{\mu_M\left(1+e_f\right)}{a_f\left(1-e_f\right)}} . \tag{14}$$

Equations (1) – (14) lead to the following two-point boundary value problem: For a specified value of λ_1 and a given set of initial parameters r_0 and $\varphi_0 = 0$ (the impulse is applied tangentially to the space vehicle velocity relative to Earth) determine v_0 such that the final condition $r_{p_M} = r_f$ is satisfied, where r_0 is the radius of LEO and r_f is the radius of LMO (both orbits, LEO and LMO, are circular). This boundary value problem can be solved by means of Newton-Raphson method (Stoer and Bulirsch, 2002).

After computing v_0, the velocity changes at each impulse can be determined

$$\Delta v_1 = v_0 - \sqrt{\frac{\mu_E}{r_0}} , \tag{15}$$

$$\Delta v_2 = \sqrt{\frac{\mu_M\left(1+e_f\right)}{a_f\left(1-e_f\right)}} - \sqrt{\frac{\mu_M}{r_f}} . \tag{16}$$

The total characteristic velocity is then given by

$$\Delta v_{Total} = \Delta v_1 + \Delta v_2 . \tag{17}$$

Note that the total characteristic velocity is a function of λ_1 for a given set of parameters $\left(r_0, \varphi_0 = 0, r_f\right)$. Accordingly, the following optimization problem can be formulated: Determine λ_1 to minimize Δv_{Total}. This minimization problem has been solved using a classic gradient method (Miele et at, 1969). The results are presented in Section 5.

The total flight time of an Earth-Moon trajectory is given by

$$T = \Delta t_E + \Delta t_M , \tag{18}$$

where Δt_E is the flight time of the geocentric trajectory and Δt_M is the flight time of the selenocentric trajectory. These times are calculated from the well-known equations times of flight of two-body dynamics as follows

$$\Delta t_E = \sqrt{\frac{a_0^3}{\mu_E}}\left(E_1 - e_0 \sin E_1\right), \tag{19}$$

$$\Delta t_M = \sqrt{\frac{\left(-a_f\right)^3}{\mu_E}}\left(e_f \sinh F_2 - F_2\right), \tag{20}$$

with eccentric anomaly E_1 and hyperbolic eccentric anomaly F_2 obtained from

$$\cos E_1 = \frac{1}{e_0}\left(1 - \frac{r_1}{a_0}\right), \qquad \cosh F_2 = \frac{1}{e_f}\left(1 - \frac{r_2}{a_f}\right).$$

Since lunar arrival occurs prior to apoapsis of the geocentric trajectory, $0 < E_1 \le 180°$. The semi-major axis a_0 and eccentricity e_0 of the geocentric trajectory are given by

$$a_0 = \frac{r_0}{2 - Q_0}, \qquad e_0 = \sqrt{1 + Q_0 (Q_0 - 2)\cos^2 \varphi_0},$$

where $Q_0 = r_0 v_0^2 / \mu_E$. Recall that the impulses are applied at the periapses of the geocentric and selenocentric trajectories.

3. OPTIMIZATION PROBLEM BASED ON THE SIMPLIFIED VERSION OF PCR3BP

In this section, the optimization problem based on the simplified PCR3BP is formulated. A detailed presentation of this problem can be found in Miele and Mancuso (2001). The following assumptions are employed:

1. The Earth is fixed in space;
2. The eccentricity of the Moon orbit around Earth is neglected;
3. The flight of the space vehicle takes place in the Moon orbital plane;
4. The space vehicle is subject to only the gravitational fields of Earth and Moon;
5. The gravitational fields of Earth and Moon are central and obey the inverse square law;
6. The class of two impulse trajectories is considered. The impulses are applied tangentially to the space vehicle velocity relative to Earth (first impulse) and Moon (second impulse).

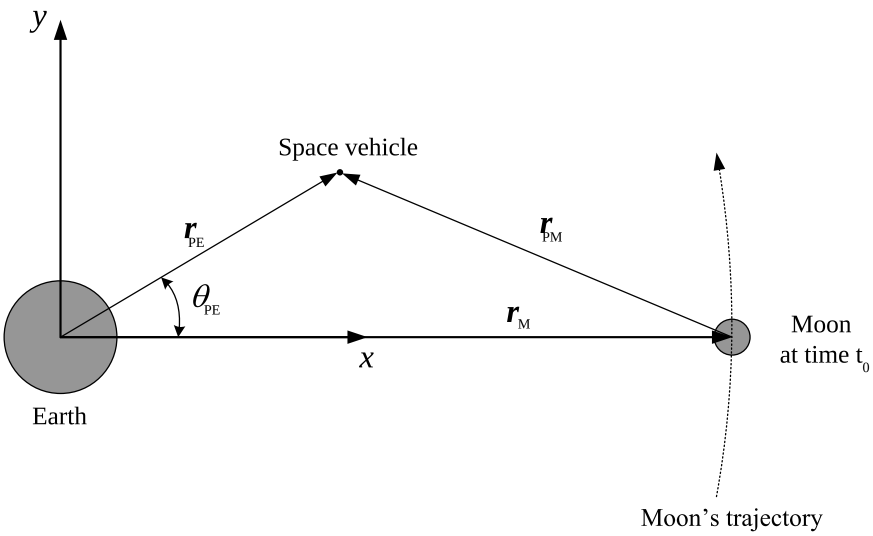

Figure 3 – Inertial reference frame E*xy*

Consider an inertial reference frame E*xy* contained in the Moon orbital plane: its origin is the Earth center; the *x*-axis points towards the Moon position at the initial time $t_0 = 0$ and the *y*-axis is perpendicular to the *x*-axis. Figure 3 shows the inertial reference frame E*xy*.

In the E*xy* reference frame, the motion of the space vehicle (P) is described by the following differential equations:

$$\frac{dx_P}{dt} = u_P$$

$$\frac{dy_P}{dt} = v_P$$

$$\frac{du_P}{dt} = -\frac{\mu_E}{r_{PE}^3} x_P - \frac{\mu_M}{r_{PM}^3}(x_P - x_M)$$

$$\frac{dv_P}{dt} = -\frac{\mu_E}{r_{PE}^3} y_P - \frac{\mu_M}{r_{PM}^3}(y_P - y_M), \tag{21}$$

where r_{PE} and r_{PM} are, respectively, the radial distances of space vehicle from Earth (E) and Moon (M), that is, $r_{PE}^2 = (x_P - x_E)^2 + (y_P - y_E)^2$ and $r_{PM}^2 = (x_P - x_M)^2 + (y_P - y_M)^2$. Because the origin of the inertial reference frame E*xy* is the Earth center, the position vector of the Earth is defined by $\boldsymbol{r}_E = (0,0)$. The position vector of the Moon in the inertial reference frame E*xy* is defined by $\boldsymbol{r}_M = (x_M, y_M)$. Since the eccentricity of the Moon orbit around Earth is neglected, the Moon inertial coordinates are given by

$$x_M(t) = D\cos(\omega_M t) \qquad y_M(t) = D\sin(\omega_M t), \tag{22}$$

where ω_M is the angular velocity of the Moon, it is assumed constant and is defined by $\omega_M = \sqrt{\mu_E / D^3}$.

The initial conditions of the system of differential equations correspond to the position and velocity vectors of the space vehicle after the application of the first impulse. The initial conditions ($t_0 = 0$) can be written as follows:

$$x_P(0) = x_{PE}(0) = r_{PE}(0)\cos\theta_{PE}(0), \tag{23}$$

$$y_P(0) = y_{PE}(0) = r_{PE}(0)\sin\theta_{PE}(0), \tag{24}$$

$$u_P(0) = u_{PE}(0) = -\left[\sqrt{\frac{\mu_E}{r_{PE}(0)}} + \Delta v_1\right]\sin\theta_{PE}(0), \tag{25}$$

$$v_P(0) = v_{PE}(0) = \left[\sqrt{\frac{\mu_E}{r_{PE}(0)}} + \Delta v_1\right]\cos\theta_{PE}(0), \tag{26}$$

where Δv_1 is the velocity change at the first impulse, $r_{PE}(0) = r_{PE_0}$ and $\theta_{PE}(t)$ is the angle defining the position of the space vehicle in the inertial reference frame E*xy* at time *t*, more precisely the angle which the position vector $\boldsymbol{r}_P$ forms with *x*-axis. It should be noted that $\boldsymbol{r}_{PE}(0)$ and $\boldsymbol{v}_{PE}(0)$ or, equivalently, $\boldsymbol{r}_P(0)$ and $\boldsymbol{v}_P(0)$ are orthogonal, because the impulse is applied tangentially to LEO, assumed circular.

The final conditions of the system of differential equations correspond to the position and velocity vectors of the space vehicle before the application of the second impulse. The final conditions ($t_f = T$) can be written as follows:

$$x_P(T) = x_{PM}(T) + x_M(T) = r_{PM}(T)\cos\theta_{PM}(T) + x_M(T), \tag{27}$$

$$y_P(T) = y_{PM}(T) + y_M(T) = r_{PM}(T)\sin\theta_{PM}(T) + y_M(T), \tag{28}$$

$$u_P(T) = u_{PM}(T) + u_M(T) = \pm\left[\sqrt{\frac{\mu_M}{r_{PM}(T)}} + \Delta v_2\right]\sin\theta_{PM}(T) + u_M(T), \tag{29}$$

$$v_P(T) = v_{PM}(T) + v_M(T) = \mp\left[\sqrt{\frac{\mu_M}{r_{PM}(T)}} + \Delta v_2\right]\cos\theta_{PM}(T) + v_M(T), \tag{30}$$

where Δv_2 is the velocity change at the second impulse, $r_{PM}(T) = r_{PM_f}$ and $\theta_{PM}(t)$ is the angle which the position vector $\boldsymbol{r}_{PM}$ forms with *x*-axis. The upper sign refers to clockwise arrival to LMO and the lower sign refers to counterclockwise to LMO. Since the eccentricity of the Moon orbit around Earth is neglected, it follows from Eq. (22) that the components of the Moon inertial velocity at time *T* are given by the

$$u_M(T) = -D\omega_M\sin(\omega_M T) \qquad v_M(T) = D\omega_M\cos(\omega_M T).$$

The angle $\theta_{PM}(T)$ is free and can be eliminated. After the problem has been solved, the angle $\theta_{PM}(T)$ can be calculated from Eqns (27) and (28). So, combining Eqns (27) – (30), the final conditions can be put in the form:

$$(x_P(T) - x_M(T))^2 + (y_P(T) - y_M(T))^2 = (r_{PM}(T))^2, \tag{31}$$

$$\left(u_P(T)-u_M(T)\right)^2+\left(v_P(T)-v_M(T)\right)^2=\left[\sqrt{\frac{\mu_M}{r_{PM}(T)}}+\Delta v_2\right]^2, \tag{32}$$

$$\left(x_P(T)-x_M(T)\right)\left(v_P(T)-v_M(T)\right)-\left(y_P(T)-y_M(T)\right)\left(u_P(T)-u_M(T)\right)=\mp r_{PM}(T)\left[\sqrt{\frac{\mu_M}{r_{PM}(T)}}+\Delta v_2\right]. \tag{33}$$

The upper sign refers to clockwise arrival to LMO and the lower sign refers to counterclockwise to LMO. It should be noted that constraint defined by Eq. (33) is derived from the angular momentum considering a direct (counterclockwise arrival) or a retrograde (clockwise arrival) orbit around the Moon.

The problem defined by Eqns (21) – (33) involves four unknowns Δv_1, Δv_2, T and $\theta_{PE}(0)$ that must be determined in order to satisfy the three final conditions (31) – (33). Since this problem has one degree of freedom, an optimization problem can be formulated as follows: Determine Δv_1, Δv_2, T and $\theta_{PE}(0)$ which satisfy the final constraints (31) – (33) and minimize the total characteristic velocity $\Delta v_{Total}=\Delta v_1+\Delta v_2$. This problem was solved by Miele and Mancuso (2001) using the sequential gradient-restoration algorithm for mathematical programming problems developed by Miele et al (1969).

In this paper, the optimization problem described above is solved by means of an algorithm based on gradient method (Miele et at, 1969) in conjunction with Newton-Raphson method (Stoer and Bulirsch, 2002), similarly to the one described in the previous section for the problem based on the patched-conic approximation. The angle $\theta_{PE}(0)$ has been chosen as the iterative variable in the gradient phase with Δv_1, Δv_2 and T calculated through Newton-Raphson method. The results are presented in the Section 5.

4. OPTIMIZATION PROBLEM BASED ON THE CLASSICAL VERSION OF PCR3BP

In this section, the optimization problem based on the classical version of PCR3BP is formulated. The assumptions employed in this formulation are the same ones previously presented in Section 3, except the Assumption 1 which must be replaced by the following one: Earth moves around the center of mass of the Earth-Moon system.

Consider an inertial reference frame G*xy* contained in the Moon orbital plane: its origin is the center of mass G of the Earth-Moon system; the *x*-axis points towards the Moon position at the initial time $t_0=0$ and the *y*-axis is perpendicular to the *x*-axis. Figure 4 shows the inertial reference frame G*xy*.

In the G*xy* reference frame, the motion of the space vehicle (P) is described by the following differential equations:

$$\frac{dx_P}{dt}=u_P$$

$$\frac{dy_P}{dt}=v_P$$

$$\frac{du_P}{dt}=-\frac{\mu_E}{r_{PE}^3}\left(x_P-x_E\right)-\frac{\mu_M}{r_{PM}^3}\left(x_P-x_M\right)$$

$$\frac{dv_P}{dt}=-\frac{\mu_E}{r_{PE}^3}\left(y_P-y_E\right)-\frac{\mu_M}{r_{PM}^3}\left(y_P-y_M\right), \tag{34}$$

where r_{PE} and r_{PM} are, respectively, the radial distances of space vehicle from Earth (E) and Moon (M), that is, $r_{PE}^2=\left(x_P-x_E\right)^2+\left(y_P-y_E\right)^2$ and $r_{PM}^2=\left(x_P-x_M\right)^2+\left(y_P-y_M\right)^2$. Because the origin of the inertial reference frame G*xy* is the center of mass of Earth-Moon system, the position vectors of the Earth and the Moon are, respectively, defined by $\boldsymbol{r}_E=\left(x_E,y_E\right)$ and $\boldsymbol{r}_M=\left(x_M,y_M\right)$. Since the eccentricity of the Moon orbit around Earth is neglected, the Earth and Moon inertial coordinates are given by

$$x_E(t)=-\mu x_M(t) \qquad y_E(t)=-\mu y_M(t)$$

$$x_M(t) = \frac{D}{1+\mu}\cos(\omega_M t) \qquad y_M(t) = \frac{D}{1+\mu}\sin(\omega_M t), \tag{35}$$

where $\mu = \mu_M / \mu_E$. Here, the angular velocity of the Moon is defined by $\omega_M = \sqrt{(\mu_E + \mu_M)/D^3}$.

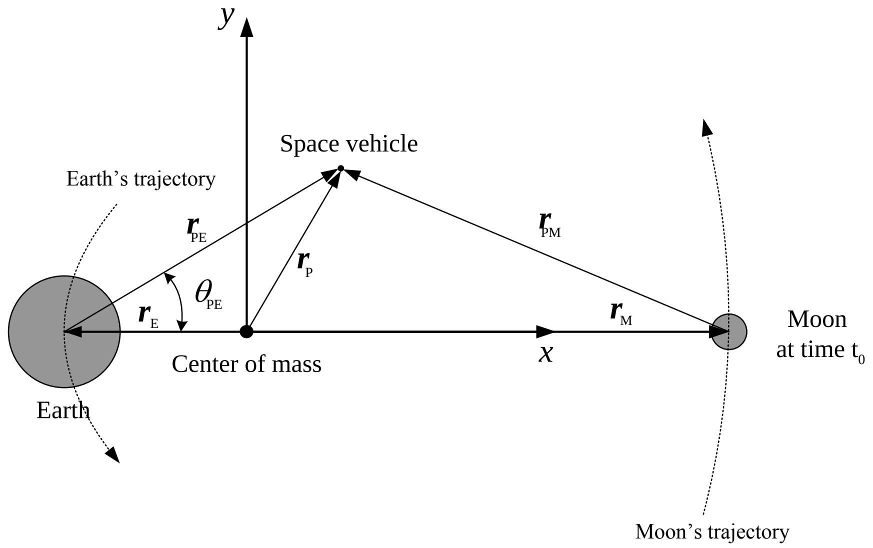

Figure 4 – Inertial reference frame G*xy*

The initial conditions of the system of differential equations correspond to the position and velocity vectors of the space vehicle after the application of the first impulse. The initial conditions ($t_0 = 0$) can be written as follows:

$$x_P(0) = x_{PE}(0) + x_E(0) = r_{PE}(0)\cos\theta_{PE}(0) + x_E(0), \tag{36}$$

$$y_P(0) = y_{PE}(0) + y_E(0) = r_{PE}(0)\sin\theta_{PE}(0) + y_E(0), \tag{37}$$

$$u_P(0) = u_{PE}(0) + u_E(0) = -\left[\sqrt{\frac{\mu_E}{r_{PE}(0)}} + \Delta v_1\right]\sin\theta_{PE}(0) + u_E(0), \tag{38}$$

$$v_P(0) = v_{PE}(0) + v_E(0) = \left[\sqrt{\frac{\mu_E}{r_{PE}(0)}} + \Delta v_1\right]\cos\theta_{PE}(0) + v_E(0), \tag{39}$$

where Δv_1, $r_{PE}(0)$ and $\theta_{PE}(t)$ have the same meaning previously defined in Section 3 and, from Eq. (35), $x_E(0) = -\frac{\mu D}{1+\mu}$, $y_E(0) = 0$, $u_E(0) = 0$, $v_E(0) = -\frac{\mu D}{1+\mu}\omega_M$. It should be noted that $\boldsymbol{r}_{PE}(0)$ and $\boldsymbol{v}_{PE}(0)$ are orthogonal, because the impulse is applied tangentially to LEO, assumed circular.

The final conditions of the system of differential equations correspond to the position and velocity vectors of the space vehicle before the application of the second impulse, and, they are given by Eqs (27) – (30) with the final position and velocity vectors of Moon obtained from Eq. (35), that is, given by:

$$x_M(T) = \frac{D}{1+\mu}\cos(\omega_M T), \qquad y_M(T) = \frac{D}{1+\mu}\sin(\omega_M T),$$

$$u_M(T) = -\frac{D\omega_M}{1+\mu}\sin(\omega_M T), \qquad v_M(T) = \frac{D\omega_M}{1+\mu}\cos(\omega_M T).$$

Accordingly, the final conditions can be put in the same form defined by Eqs (31) – (33).

Therefore, we have the same optimization problem defined in Section 3. The results are presented in the next section.

5. RESULTS

In this section, results are presented for lunar missions using the three approaches previously described and are compared to the results obtained by Miele and Mancuso (2001). The following data are used:

$\mu_E = 3.986\times10^5\ \text{km}^3/\text{s}^2$, $\mu_M = 4.903\times10^3\ \text{km}^3/\text{s}^2$,
$D = 384400\,\text{km}$ (distance from the Earth to the Moon),
$a_E = 6378\,\text{km}$ (Earth radius), $a_M = 1738\,\text{km}$ (Moon radius),
$h_0 = 463\,\text{km}$ (altitude of LEO), $h_f = 100, 200, 300\,\text{km}$ (altitude of LMO).

Table 1 shows the results for lunar missions with clockwise arrival at LMO and Table 2 shows the results for lunar missions with counterclockwise arrival at LMO. The departure from LEO is counterclockwise for all missions. The major parameters that are presented in these tables are the velocity changes Δv_1 and Δv_2 at each impulse, the total characteristic velocity $\Delta v_{Total} = \Delta v_1 + \Delta v_2$, the flight time of lunar mission *T* and the initial position of the space vehicle in the inertial reference frame E*xy* at the initial time $t_0 = 0$ defined by the angle $\theta_s(0)$.

Results in Tables 1 and 2 show good agreement. It should be noted the excellent results obtained using the patched-conic approximation model. In all missions, the patched-conic approximation model yields very accurate estimate for the first impulse in comparison to the results obtained using the PCR3BP models. For the second impulse, there exists a small difference between the results given by the patched-conic approximation model and the PCR3BP models.

Tables also show a small difference in the flight time *T* and in the angle $\theta_s(0)$ calculated by the three approaches. We suppose that the difference between the values obtained in this paper and the values presented by Miele and Mancuso (2001) for the flight time *T* and the angle $\theta_s(0)$ calculated using the simplified PCR3BP model should be related to the accuracy in the integration of differential equations and in the solution of the terminal constraints. The algorithm based on gradient algorithm in conjunction with Newton-Raphson method, described in this paper, uses a Runge-Kutta-Fehlberg method of order 4 and 5, with step-size control, and, relative error tolerance of 10^{-10} and absolute error tolerance of 10^{-11} (Forsythe et al, 1977), and, the terminal constraints are satisfied with an error lesser than 10^{-8}. In all simulations, canonical units are used: 1 distance unit = a_E and 1 time unit = $\sqrt{a_e^3/\mu_E}$, such that $\mu_E = 1.0\ \text{d.u.}^3/\text{t.u.}^2$. The paper by Miele and Mancuso does not describe the accuracy used in the calculations.

Table 1 – Lunar mission, clockwise LMO arrival, major parameters

LMO altitude *km*	***Model***	Δv_{Total} *km/s*	Δv_1 *km/s*	Δv_2 *km/s*	*T* *days*	$\theta_s(0)$ *deg*
100	*Patched-conic*	3.8528	3.0683	0.7845	4.936	−113.681
	Simplified PCR3BP	3.8811	3.0677	0.8134	4.750	−114.215
	Classical PCR3BP	3.8829	3.0686	0.8143	4.763	−113.795
	Miele and Mancuso (2001)	3.882	3.068	0.814	4.50	−116.88
200	*Patched-conic*	3.8379	3.0683	0.7696	4.941	−113.638
	Simplified PCR3BP	3.8670	3.0677	0.7993	4.757	−114.187
	Classical PCR3BP	3.8688	3.0686	0.8002	4.769	−113.742
	Miele and Mancuso (2001)	3.868	3.068	0.800	4.50	−116.88
300	*Patched-conic*	3.8243	3.0683	0.7560	4.944	−113.608
	Simplified PCR3BP	3.8541	3.0678	0.7863	4.760	−114.116
	Classical PCR3BP	3.8559	3.0687	0.7872	4.771	−113.716
	Miele and Mancuso (2001)	3.855	3.068	0.787	4.50	−116.88

Table 2 – Lunar mission, counterclockwise LMO arrival, major parameters

LMO altitude *km*	***Model***	Δv_{Total} *km/s*	Δv_1 *km/s*	Δv_2 *km/s*	*T* *days*	$\theta_s(0)$ *deg*
100	*Patched-conic*	3.8482	3.0655	0.7827	4.794	–115.723
	Simplified PCR3BP	3.8758	3.0649	0.8109	4.564	–116.800
	Classical PCR3BP	3.8777	3.0658	0.8119	4.573	–116.410
	Miele and Mancuso (2001)	3.876	3.065	0.811	4.37	–118.98
200	*Patched-conic*	3.8331	3.0654	0.7677	4.794	–115.750
	Simplified PCR3BP	3.8614	3.0648	0.7966	4.562	–116.832
	Classical PCR3BP	3.8634	3.0658	0.7976	4.571	–116.451
	Miele and Mancuso (2001)	3.862	3.065	0.797	4.37	–119.00
300	*Patched-conic*	3.8194	3.0654	0.7540	4.793	–115.777
	Simplified PCR3BP	3.8483	3.0648	0.7835	4.560	–116.881
	Classical PCR3BP	3.8502	3.0657	0.7845	4.569	–116.491
	Miele and Mancuso (2001)	3.849	3.065	0.784	4.37	–119.03

Finally, note that:

1. Lunar missions with clockwise LMO arrival spend more fuel than lunar missions with counterclockwise LMO arrival;
2. The flight time is nearly the same for all lunar missions with clockwise LMO arrival and for lunar missions with counterclockwise LMO arrival, independently on the model used in the analysis.
3. The first change velocity Δv_1 is nearly independent of the LMO altitude.
4. The second change velocity Δv_2 decreases with the LMO altitude.
5. The flight time for lunar missions with clockwise LMO arrival is larger than the flight time for lunar missions with counterclockwise LMO arrival.
6. For the PCR3BP and patched-conic approximation models, the angle $\theta_s(0)$ varies with the LMO altitude for all lunar missions.

Some of these general results are quite similar to the ones described by Miele and Mancuso (2001).

6. CONCLUSION

In this paper a systematic study of optimal trajectories for Earth-Moon flight of a space vehicle is presented. The optimization criterion is the total characteristic velocity. The optimization problem has been formulated using the patched-conic approximation or two versions of the planar circular restricted three-body problem (PCR3BP) and has been solved using a gradient algorithm in conjunction with Newton-Raphson method. Results are presented for the same lunar missions described by Miele and Mancuso (2001). All models show that lunar missions with clockwise LMO arrival spend more fuel than lunar missions with counterclockwise LMO arrival. Finally, note that all optimal trajectories presented in this paper involve short flight time (about 4.5 to 5.0 days) and they corresponds to local minima of the total characteristic velocity; but, other local minima exist with larger flight time (Yagasaki, 2004) that can lead to different conclusions.

7. ACKNOWLEDGEMENTS

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