ANALYTICAL SOLUTION OF THE RTE EQUATION COUPLED WITH THE WEIGHTED-SUM-OF-GRAY-GASES

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Abstract. *This work presents an analytical solution to the RTE equation applied to solve the radiative heat exchange in isothermal, purely absorbing homogenous participating media. The spectral integration of the radiative equation is performed with the weighted-sum-of-gray-gases (WSGG) technique, which models the entire spectrum with few bands having uniform absorption coefficients, each band corresponding to a gray gas. The solution was implemented to a one-dimensional geometry having black walls and containing a mixture of water vapor, carbon dioxide and inert gas. In addition to an expected gain in computational time, an advantage of the analytical approach is that the resulting solution is valid for any value of the angular variable, instead of being restrict to the choice of a quadrature set. The error will be mainly due to the integration in the wavelength spectrum. Therefore, the proposed method can be a useful tool for the evaluation of spectral models.*

Keywords*: participating media, RTE equation, WSGG model.*

1. INTRODUCTION

Radiation heat transfer in participating media is often the dominant heat transfer mode in applications that involve absorbing media at high temperatures, such as in furnaces and engine chambers. The solution of this kind of problem may be a hard task because to obtain the temperature field, a set of integro-differential equations must be solved. Furthermore, due highly irregular dependence of the radiative properties with the wavelength, special methods are necessary to the spectral integration.

The integration in the wavelength spectrum can be accurately solved by line-by-line (LBL) integration, which considers the emission and absorption of each spectral line. However this method is computational expensive and, therefore, is used only to obtain benchmark solutions. To avoid this drawback many other models have been proposed. These models are combined with arbitrary methods of solution of the radiative transfer equation.

Denison and Webb (1995a) used absorption-line blackbody distribution function (ALBDF) to develop mathematical correlations to CO_2. These results were used to solve one-dimensional medium with ordinates discrete methods coupled with the Weighted-Sum-of-Gray-Gases (WSGG). Maurente *et al.* (2008) considered participating media constituted by water vapor and carbon dioxide to represent combustion of methane and octane. Monte Carlo method was applied with the absorption-line blackbody distribution function (MC-ALBDF). Mossi *et al*. (2007) applied the Cumulative Wavenumber (CW) along with the discrete ordinates method to deal with one-dimensional problems. The results were compared with the LBL integration and MC-ALBDF model. Denison and Webb (1995b) applied Spectral-Line Weighted-sum-of-gray-gases (SLW) model for non-isothermal and non-homogeneous H_2O/CO_2 mixtures. The discrete ordinates method was used to angular integration. Solovjov and Webb (2008) used discrete ordinates with CW and SLW models to solve non-isothermal gaseous medium. This last model was also implemented by Solovjov and Webb (2000) in multicomponent gas mixtures.

The aforementioned papers solved only one-dimensional medium. The following works solved more difficult geometries, but considering only a gray medium. Parthasarathy *et al.* (1995) showed solutions for two-dimensional irregular geometries with absorbing, emitting, and isotropically scattering medium using Monte Carlo method. Jendoubi *et al.* (1993) applied discrete ordinates method to a finite cylinder.

Results for one and two-dimensional inhomogeneous gas mixtures with varying temperature and mole fraction fields are presented in Zhang and Modest (2002) using a *k*-distribution based model coupled with P-1 approximation.

Spectral integration is still a challenge because the models are either quite complex or have significant restrictions. In this context, the aim of this work is to develop an analytical tool for evaluation of spectral models. The RTE equation is solved to one dimensional slab. It is considered an isothermal and homogeneous medium with 20% H_2O and 10% CO_2. The spectral integration was performed with the WSGG model.

2. PHYSICAL AND MATHEMATICAL MODELING

2.1. The weighted sum of gray gases

In this method the non-gray gas is replaced by a number of gray gases, for which the heat transfer rates are calculated independently. The total heat flux is then found by adding the heat fluxes of the gray gases after multiplication with certain weight factors.

The weighted-sum-of-gray-gases (WSGG) model consists of the division of the spectrum into I regions where the absorption coefficient K_i is assumed constant. Figure 1 shows a representation of a medium with three gray gases on wide bands and transparent windows. According to this model, the total emittance of the medium along a path of length s is given by,

$$\varepsilon = \sum_{i=0}^{I} C_{e,i}(T)\left[1 - e^{-K_i s}\right] \tag{1}$$

where $C_{e,i}$ is the weighting factor for the ith gray gas, which is considered to be solely dependent of the gas temperature, T.

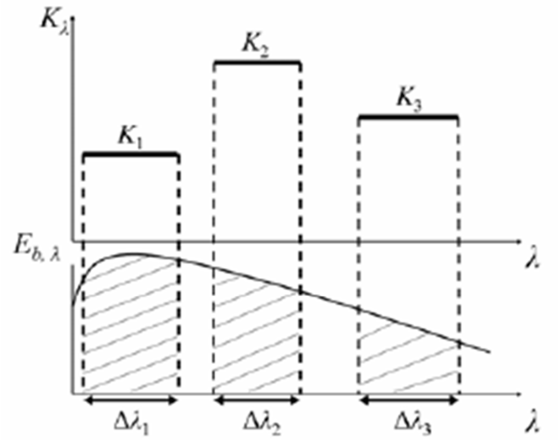

Figure 1. Graphic representation of a medium composed of three gray gases for the WSGG model.

Smith *et al.* (1982) represented the weighting factors by a polynomial function of order J-1 of the temperature, which is usually employed with I = 3 and J = 4.

$$C_{e,i}(T) = \sum_{j=1}^{J} c_{e,i,j} T^{j-1} \tag{2}$$

where $c_{e,i,j}$ corresponds to temperature polynomial coefficients of the gas emittance.

2.2. Mathematical Formulation

Modest (1991) coupled RTE equation with WSGG model allowing the total radiative transfer calculus in gases using arbitrary solution methods. Thus, in the absence of scattering, the WSGG form of the RTE is written,

$$\frac{dI_i}{ds} = K_i\left(C_{e,i}(T)\frac{\sigma T^4}{\pi} - I_i\right) \tag{3}$$

where I_i is the radiation intensity associated with the ith gray gas and σ is the Stefan-Boltzmann constant.

A non-scattering and uniform medium is shown in Fig. 2. The two boundaries surfaces, separated by distance L, are infinite and isothermal.

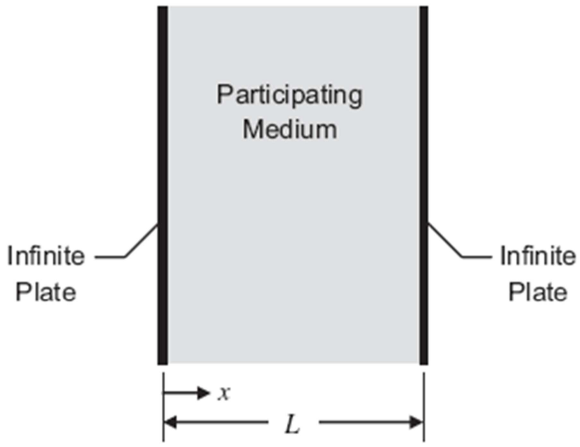

Figure 2. One-dimensional medium between isothermal plates.

If the entire medium, represented in Fig. 2, has uniform temperature, Eq. (3) becomes,

$$\mu \frac{\partial I_i(x,\mu)}{\partial x} + K_i I_i(x,\mu) = S_i(T) \tag{4}$$

where $\mu \in [-1, 1]$, and, for simplicity, it was introduced

$$S_i(T) = \frac{C_{e,i}(T)\sigma T^4}{\pi} \tag{5}$$

The problem is subjected to the following boundary conditions:

$$\begin{cases} I_i(0,\mu) = F_1(\mu) & \mu > 0 \\ I_i(L,-\mu) = F_2(\mu) & \mu > 0 \end{cases} \tag{6}$$

where, here,

$$F_\alpha(\mu) = \varepsilon_\alpha \frac{\sigma T_\alpha{}^4}{\pi} \quad \alpha = 1, 2 \tag{7}$$

where T_1 and T_2 are the boundary temperatures; ε_1 and ε_2, the emissivities.

We follow Barichello and Siewert (2001) and we integrate Eq. (4) to obtain the spectral intensities as a function of location and direction in the plane layer. The solution, for $\mu > 0$, is expressed by Eq. (8) e and Eq. (9):

$$I_i(x,\mu) = \frac{S_i}{K_i}\left[1 - e^{-K_i x/\mu}\right] + F_1(\mu) e^{-K_i x/\mu} \tag{8}$$

$$I_i(x,-\mu) = \frac{S_i}{K_i}\left[1 - e^{-K_i (L-x)/\mu}\right] + F_2(\mu) e^{-K_i (L-x)/\mu} \tag{9}$$

Analogously, the net spectral flux, $q_{r,i}$, is calculated in terms of the intensities in the positive and negative directions, given in Eqs. (8) and (9), such that:

$$q_{r,i}(x) = 2\pi \int_0^1 [I_i(x,\mu) - I_i(x,-\mu)]\mu d\mu \tag{10}$$

$$q_{r,i}(x) = 2\pi \left[\frac{S_i}{K_i} - F_2 \right] E_3[K_i(L-x)] + 2\pi \left[F_1 - \frac{S_i}{K_i} \right] E_3(K_i x) \tag{11}$$

The Equation (11) was written in terms of exponential integral that is defined for positive real arguments as:

$$E_n(x) \equiv \int_0^1 \mu^{n-2} e^{-x/\mu} d\mu \tag{12}$$

And at each location x the total radiative flux is the summation over all gray gas.

$$q_r(x) = \sum_{i=1}^{I} q_{r,i}(x) \tag{13}$$

Differentiating the Eq. (11) yields to a quantity – Eq. (14) – that after the summation over all gray gas becomes the divergence of the radiative flux or radiative heat source.

$$\frac{dq_{r,i}(x)}{dx} = 2\pi[S_i - K_i F_2] E_2[K_i(L-x)] - 2\pi[K_i F_1 - S_i] E_2(K_i x) \tag{14}$$

$$\frac{dq_r(x)}{dr} = \sum_{i=1}^{I} \frac{dq_{r,i}(x)}{dr} \tag{15}$$

The additional recurrence relation was used (Siegel and Howell, 2002),

$$\frac{d}{dx} E_n(x) = -E_{n-1}(x) \qquad n \geq 2 \tag{16}$$

In the foregoing solutions, two expressions were applied to the exponential integral calculus. The first represented by Eq. (17); the second, by Eq. (19).

The general series expansion to the exponential integrals (Siegel and Howell, 2002) is:

$$E_n(x) = \frac{(-x)^{n-1}}{(n-1)!}[-\ln x + \psi(n)] - \sum_{\substack{m=0 \\ (m \neq n-1)}}^{\infty} \frac{(-x)^m}{(m-n+1)m!} \tag{17}$$

here $\psi(1) = -\gamma$, where $\gamma = 0.577216$ is the Euler's constant and

$$\psi(n) = -\gamma + \sum_{m=1}^{n-1} \frac{1}{m} \qquad n \geq 2 \tag{18}$$

However to large values another expression to exponential integral must be used (Siegel and Howell, 2002).

$$E_n(x) = \frac{e^{-x}}{x} \left[1 - \frac{n}{x} + \frac{n(n+1)}{x^2} - \frac{n(n+1)(n+2)}{x^3} + \dots \right] \tag{19}$$

It is interesting to remark that the analytical solution, wrote in terms of the exponential integral, does not depend nodes of the quadrature scheme.

3. RESULTS AND DISCUSSION

The complete combustion process of fossil fuels yields gas constituted by water vapor and carbon dioxide. This wok models a hydrocarbon fuel as an isothermal and homogeneous mixture de gases at 1780 K. A one dimensional medium filled with spatially constant 20% H_2O and 10% CO_2 between infinite parallel walls is considered. The black walls are apart 1 m and the temperatures are equals to 300 K.

In this work it was used some correlations obtained by Galarça *et al.* (2008), which are valid for gaseous mixtures generated in the stoichiometric combustion of methane. The Table 1 depicts the coefficients to the mixture water $(Y_{\mathrm{H_2O}} = 0{,}2)$, carbon dioxide $(Y_{\mathrm{CO_2}} = 0{,}1)$ and air (or any other inert gas, such as nitrogen). From ideal gas mixture theory, the ratio between the partial pressure of each species to the total pressure is equivalent to its molar concentration. Thus, the absorption coefficients, are $K_i = k_i (Y_{\mathrm{H_2O}} + Y_{\mathrm{CO_2}})$.

Table 1. Coefficients for emissivity (Galarça *et al.*, 2008).

i	$k_i^{(1)}$	$c_{e,i,1} \times 10^1$	$c_{e,i,2} \times 10^4$	$c_{e,i,3} \times 10^7$	$c_{e,i,4} \times 10^{11}$
1	0.517	2.801	3.244	-1.299	0.712
2	9.559	2.003	1.869	-2.394	5.454
3	161.988	1.24	-1.086	0.283	-0.114

(1): in units of 1/(atm.m)

The analytical solution was compared with the results presented by Maurente (2007), where the Monte Carlo method was applied with the absorption-line blackbody distribution function (MC-ALBDF). There is no significant difference between the radiative heat flux on the walls: 119.59 KW/m³ (present work) against 120.43 KW/m³ (Maurente, 2007). The volumetric radiative heat source (Fig. 3) also shows good agreement with Maurente (2007).

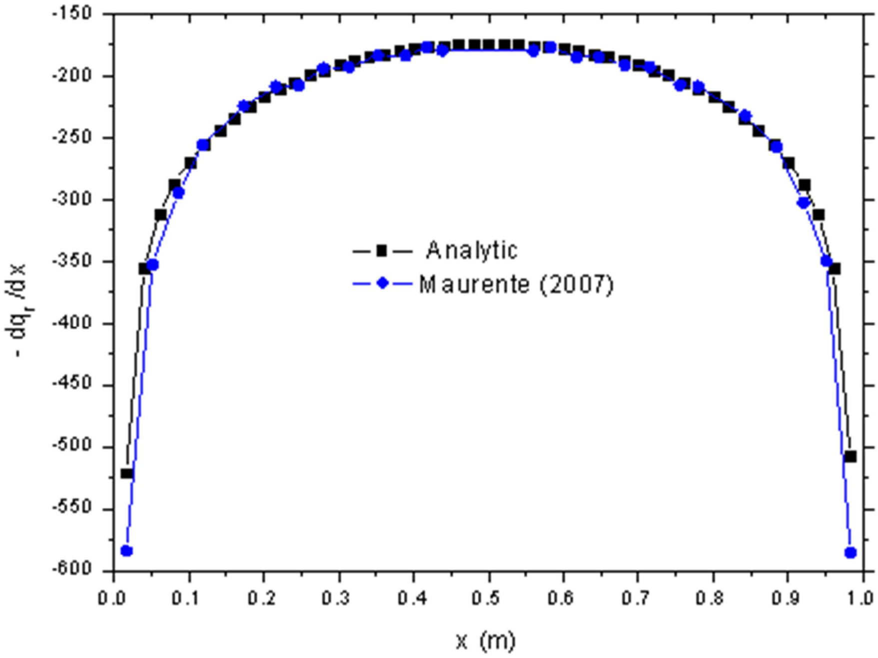

Figure 3. Radiative heat source for 20% H_2O and 10% CO_2 at 1780 K between black infinite walls at 300 K.

Figure 4 and Figure 5 depict the radiative heat flux and the radiative heat source, respectively, for 20% H_2O and 10% CO_2 to different temperature between black infinite walls at 300 K. As expected, the increase in the medium temperature raise the radiative heat transfer rates. Close to the walls the divergence change rapidly with position because in this region the effect of the wall temperature is higher and, therefore, the radiative heat source enhance.

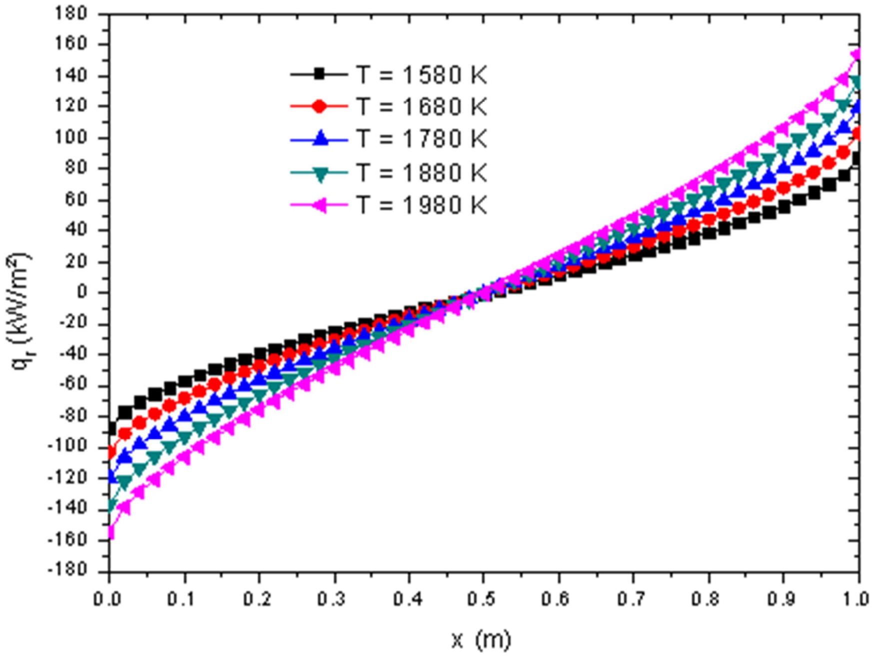

Figure 4. Radiative heat flux for 20% H_2O and 10% CO_2 to different temperature between black infinite walls at 300 K.

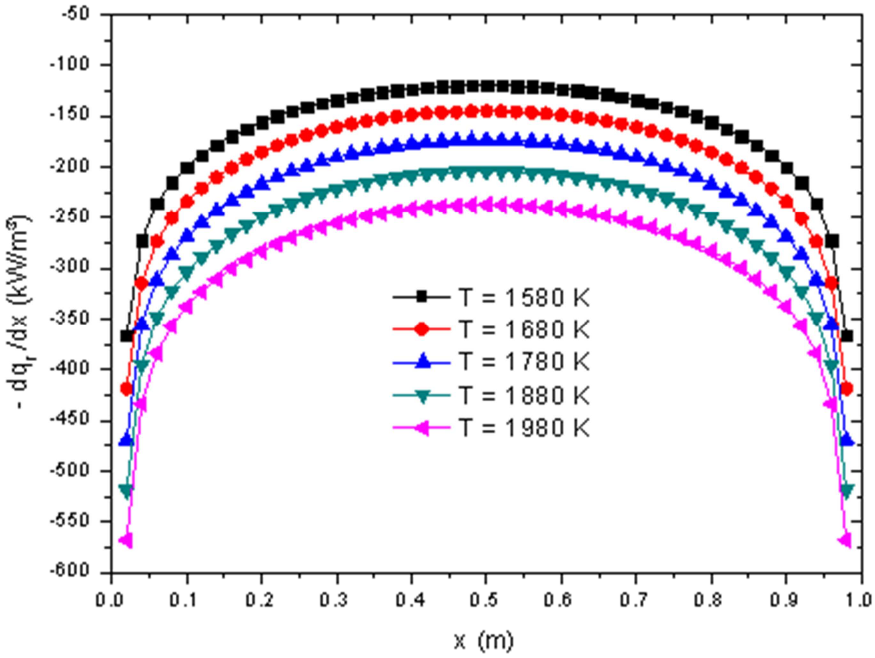

Figure 5. Radiative heat source for 20% H_2O and 10% CO_2 to different temperature between black infinite walls at 300 K.

4. CONCLUSIONS

This paper presented an analytical solution to the RTE equation in one-dimensional medium without scattering. It was derived explicit expressions to represent the heat flux and the radiative heat source. The spectral integration was build up with only three gray gases, like in the conventional Weighted-Sum-of-Gray-Gases (WSGG) model.

The solution was compared with results obtained through Monte Carlo method coupled with the absorption-line blackbody distribution function (MC-ALBDF). The results provided satisfactory agreement for isothermal and homogeneous medium.

The solution of the RTE equation embodies spectral and spatial integrations. If this last integration may be solved analytically, the error will be mainly due the spectral model. As the models are either quite complex or have significant restrictions, analytical solution provides a good tool to compare them.

5. ACKNOWLEDGEMENTS

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