NEW SYMMETRIES FOR THE HYDROMAGNETIC FALKNER-SKAN EQUATIONS

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Abstract. *A systematic search for the Lie point symmetries admitted by the steady hydromagnetic two-dimensional incompressible viscous flow boundary layer equation and associated boundary conditions is performed. Unlike previous works, the specific forms of the external velocity and transverse magnetic fields are not postulated from the very beginning. In this way a whole new class of similarity reductions for the problem is derived, for applied fields with an exponential nature. The corresponding hydromagnetic Falkner-Skan equation is numerically solved for different velocity profiles at the wall, considering stretching, expansion, injection or suction.*
Keywords: *hydromagnetic Falkner-Skan equations; steady magnetohydrodynamic boundary layer flows; Lie symmetry methods; similarity solutions.*

1. INTRODUCTION

The analysis of hydromagnetic boundary layer flows is of general physical interest, because the liquid metals in nature and industry are electrically conducting. In particular, magnetohydrodynamic (MHD) models are relevant in applications like in magnetic propulsion, power generators, accelerators, pumps, and droplet and electrostatic filters (Sutton and Sherman, 1965). Also the effect of electromagnetic forces in fluid flows is important in areas like nuclear fusion, chemical engineering, plasma physics, medicine and high-speed printing (Kumari and Nath, 1999). Recent works on hydromagnetic systems include the similarity analysis of non-Newtonian MHD flows (Afify, 2009), studies on incompressible electrically conducting fluids near the stagnation point on a stretching sheet (Ishak *et al.,* 2009), the MHD stagnation-point flow of a power-law fluid towards a stretching surface (Paullet and Weidman, 2010), MHD viscous flow due to a shrinking sheet (Noor, Kechil and Hashim, 2010) and transient heat transfer to hydromagnetic channel flow with radioactive heat and convective cooling (Makinde and Chinyoka, 2010). In this context the derivation of exact solutions for MHD model equations is a relevant task. In particular exact solutions can be used to check the accuracy of numerical codes.

In (non-hydromagnetic) boundary layer theory, a special role is played by the Falkner-Skan similarity solution (Falkner and Skan, 1931), where a power-law form is assumed for the velocity profile at infinity. Starting from the steady two-dimensional incompressible equations for a neutral fluid, an appropriate self-similar stream function is supposed so as to reduce the problem to a third-order nonlinear ordinary differential equation, with great simplification in comparison to the original spatio-temporal problem. In the flat plate particular case, the associated Falkner-Skan equation reduces to the celebrated Blasius equation (Blasius, 1908). The Falkner-Skan solution is useful not only from the mathematical point of view but also because it illustrates both favorable and contrary pressure gradients, as well as because it corresponds to wedge or stagnation point flows for certain parameter values. Moreover, the Falkner-Skan equation provides one of the few instances where the detailed mathematical analysis of boundary layer flows is possible (Padé, 2003). For instance, under specific cases like for converging channels (Magyari, 2009) or wall stretching (Fang and Zhang, 2008) even exact solutions to the Falkner-Skan equation are available. Also strange invariant sets (Swinnerton-Dyer and Sparrow, 1995) and multiple solution branches (Zaturska and Banks, 2001) have been recently reported.

In the case of hydromagnetic boundary layer flows, similarity solutions *assuming* a power-law external velocity and magnetic fields have been worked out (Abbasbandy and Hayat, 2009), (Cobble, 1980),), (Ishak *et al.,* 2009), (Kumari and Nath, 1999). However, it is well known that the use of Lie group techniques (Bluman and Kumei, 1989) provides a systematic way to derive similarity reductions for partial differential equations. For this reason, the present work is dedicated to the search for the Lie point symmetries admitted by the steady, two-dimensional hydromagnetic boundary layer equations under a transverse applied magnetic field, whose functional form *is not chosen* from the very beginning. In the same way, instead of postulating a power-law dependence, the external velocity field is left free in the analysis, as well as the boundary conditions at the wall. In other words, both stretching as well as suction/injection through the (permeable) wall are possible. However, as will be seen below, the functional dependencies of the external fields and boundary conditions should fit specific requirements so that similarity reductions exist. For neutral boundary layers the systematic search for Lie point symmetries has been already made (Burde, 1996), as well as further symmetry reductions by means of the Clarkson-Kruskal method (Ludlow, Clarkson and Bassom, 2000).

This work is organized as follows. In Section II the Lie point symmetries admitted by the equation for the stream function and the associated boundary conditions are identified. The external magnetic and velocity fields as well as the velocity profile at the wall are not prescribed *ab initio*, but treated as functions to be determined. In this way, a new exponential class amenable to point symmetry methods is found. The corresponding similarity reduction is discussed in Section III. Section IV is dedicated to the numerical simulations of the resulting hydromagnetic Falkner-Skan equation. In Section V we have the conclusions.

2. LIE POINT SYMMETRIES AND ADMISSIBLE EXTERNAL MAGNETIC AND VELOCITY FIELDS

In terms of the stream function $\psi = \psi(x, y)$, the incompressible two-dimensional steady-state hydromagnetic boundary layer model equations (Sutton and Sherman, 1965) reduces to

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = U^e U^e_x + \psi_{yyy} - B^2(\psi_y - U^e) , \tag{1}$$

where $U^e = U^e(x), B = B(x)$ and a rescaling was applied to render all quantities dimensionless. The induced magnetic fields as well as the electric field due to polarization are negligible, and external electric fields are not included. In Eq. (1), subscripts denote partial derivatives. Using a Cartesian coordinate system, the fluid velocity field is $\vec{u} = \psi_y \hat{x} - \psi_x \hat{y}$, the external velocity field far outside the boundary layer is $\vec{U}^e = U^e(x)\hat{x}$ and the transverse applied magnetic field is $\vec{B} = B(x)\hat{y}.$ To allow for maximal generality, the boundary conditions at the wall are taken as $\psi_x = -V^w(x), \psi_y = U^w(x)$ at $y=0$, with functions V^w, U^w to be specified later and interpreted as stretching and suction (or injection) velocities respectively. Finally, we have $\psi_y \to U^e(x)$ as $y \to \infty.$

Instead of directly assuming power-law forms, we keep the external velocity and magnetic fields unspecified and search for the Lie point symmetries admitted by Eq. (1) and the boundary conditions. The method for determining geometric symmetries is well-known (Bluman and Kumei, 1989) and can be performed with some of the many available symbolic packages. Proceeding in this manner, the generator of symmetries turns out to be

$$G = (k_1 x + k_2)\partial_x + [(k_1 - k_3)y + f(x)]\partial_y + (k_3\psi + k_4)\partial_\psi , \tag{2}$$

where the k_i are arbitrary numerical constants and $f(x)$ is an arbitrary function of the indicated argument. Moreover, invariance of Eq. (1) imply that the two following equations for B, U^e should be satisfied,

$$(k_1 x + k_2)(B^2)_x + 2(k_1 - k_3)B^2 = 0 , \tag{3}$$

$$(k_1 x + k_2)(U^e U^e{}_{xx} + (U^e_x)^2 + B^2 U^e_x) + (3k_1 - 4k_3)U^e U^e_x + (k_1 - 2k_3)B^2 U^e = 0 , \tag{4}$$

In addition, the boundary conditions should be also invariant under the transformation group. This requirement gives the further constraints

$$(k_1 x + k_2)U^{e,w}_x = (2k_3 - k_1)U^{e,w} , \tag{5}$$

$$(k_1 x + k_2)V^w_x = (k_3 - k_1)V^w . \tag{6}$$

The k_1 and k_3 parameters in the symmetry generator *G* are associated to the rescalings $(x, y, \psi) \to (\alpha x, \alpha y, \psi)$ and $(x, y, \psi) \to (x, \alpha y, \psi/\alpha)$ respectively, where α is a constant. The parameter k_2 is associated to parallel to the boundary layer uniform translations, $x \to x + x_0, x_0 =$ constant. Finally, the k_4 parameter and the function $f(x)$ in *G* reflect the fact that if $\psi(x, y)$ is a solution to Eq. (1), then $\psi(x, y + y_0(x)) + \psi_0$ is a solution too, where $y_0(x)$ represents an arbitrary position-dependent *y*-displacement and ψ_0 is an additive constant. The transformations due to k_4 and $f(x)$ are not relevant and will be dropped in the following. However, the seemingly superfluous *x*-

displacements linked to k_2 will be shown to be very useful, because the external magnetic field breaks down the *x*-translational invariance.

Differentiation of Eq. (5) and substitution shows that Eq. (4) is identically satisfied. Hence we are left with the decoupled linear first-order equations (3), (5) and (6) for the admissible velocities and applied magnetic fields. According to the parameters, different classes of solutions can be found. To avoid triviality, the zero magnetic field as well as the $k_1 = k_2 = k_3 = 0$ case will be excluded.

A detailed analysis shows that the solution of the system composed by Eqs. (3), (5) and (6) with $k_1 \neq 0$ reproduces the traditional hydromagnetic Falkner-Skan similarity reduction, which is fairly well described in the literature (Abbasbandy and Hayat, 2009), (Cobble, 1980),), (Ishak *et al.,* 2009), (Kumari and Nath, 1999). Hence in the following we study the other possibility, analyzing the consequences of setting $k_1 = 0$ everywhere. A complete account including the $k_1 \neq 0$ case will be reported elsewhere.

Suppose $k_1 = 0$ in Eq. (3) which determines the external magnetic fields. To avoid trivial results set $k_2 \neq 0.$ Without loss of generality it is then convenient to set $k_2 = 1,$ which gives the exponential form

$$B = B_0 \exp(k_3 x) \tag{7}$$

for the admissible magnetic fields class, where $B_0 \neq 0$ is a constant. The corresponding velocities solving Eqs. (5-6) are

$$U^{e,w} = U_0^{e,w} \exp(2k_3 x) , \quad V^w = V_0^w \exp(k_3 x) , \tag{8}$$

where $U_0^{e,w}$ and V_0^w are numerical constants. In particular, a positive (negative) V_0^w is associated to injection (suction) through the permeable wall. As far as we know, this exponential class for which symmetry is admitted is new in the context of MHD boundary layers.

3. SIMILARITY REDUCTIONS

Working with the differential invariants of the symmetry generator in Eq. (2) it is possible to infer similarity solutions. For $k_1 = 0, k_2 = 1$ with the applied magnetic field as in Eq. (7) and velocities as in Eq. (8), define

$$\psi = \frac{U_0^e}{B_0} \exp(k_3 x) \varphi(q) , \quad q = B_0 \exp(k_3 x) y , \tag{9}$$

assuming $U_0^e \neq 0.$ Using Eq. (9) in the stream function equation (1) the result is the third-order ordinary differential equation

$$\varphi_{qqq} + k \varphi \varphi_{qq} + 2k(1 - \varphi_q^2) + 1 - \varphi_q = 0, \quad k = \frac{k_3 U_0^e}{B_0^2} \quad , \tag{10}$$

with the boundary conditions

$$\varphi_q(0) = \frac{U_0^w}{U_0^e} , \quad \varphi_q(\infty) = 1 , \quad k\varphi(0) = -\frac{V_0^w}{B_0} . \tag{11}$$

Eq. (10) can be defined as the hydromagnetic Falkner-Skan equation of type II, to emphasize the difference in comparison with the usual Falkner-Skan equation arising from a power-law velocity field. In the next Section the numerical analysis of the new exponential class of similarity solutions is performed.

4. NUMERICAL SIMULATIONS

To get insight on the hydromagnetic Falkner-Skan solutions of type II, consider the asymptotic ($q \to \infty$) form of the associated stream function in Eq. (9), which gives This corresponds to both exponentially expanding ($k_3 < 0$) as well as exponentially converging ($k_3 > 0$) flows, as seen in the contour plots in Figs. 1 and 2. These flows are well suited for e.g. sharp expanding and convergent flows, more abrupt than the power-law cases described by the usual hydromagnetic Falkner-Skan solutions. However, Figs. 1 and 2 are just extrapolations, since strictly they apply only for $q \to \infty$.

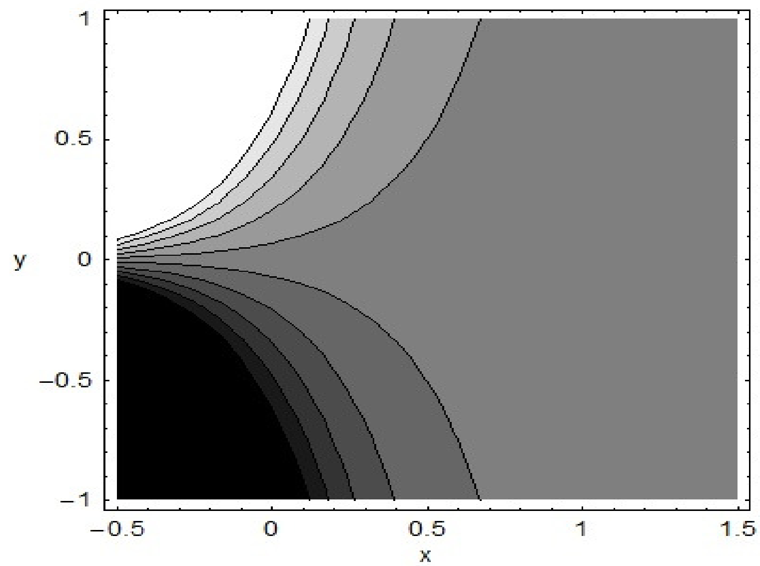

Figure 1. Contour plot of the asymptotic stream function in Eq. (9) for hydromagnetic Falkner-Skan solutions of type II and expanding flows. Here $U_0^e = 1, k_3 = -2.$

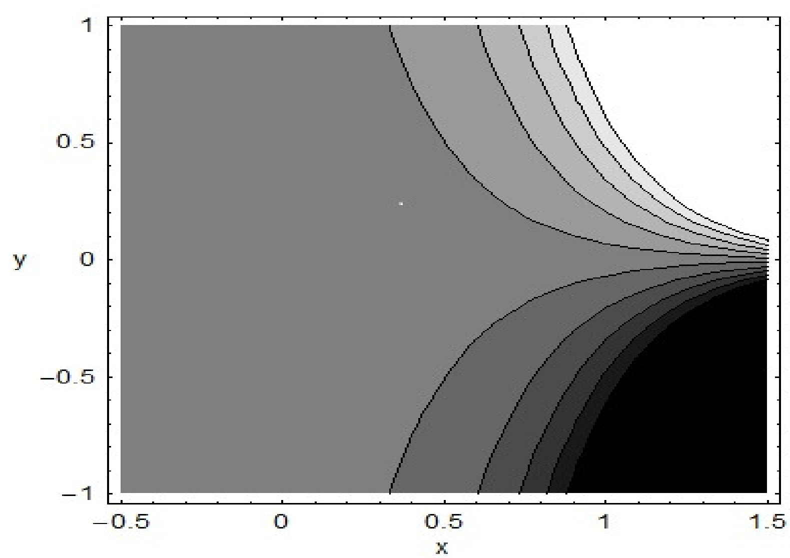

Figure 2. Contour plot of the asymptotic stream function in Eq. (9) for hydromagnetic Falkner-Skan solutions of type II and converging flows. Here $U_0^e = 1, k_3 = 2.$

More pertinent graphics can be found simulating Eq. (10) numerically and inverting the similarity transformation (9). Typical results are shown in Figs. 3 and 4, for $=U_0^e = B_0 = 1,$ initial conditions $\varphi(0) = 0.5, \varphi_q(0) = 0$ and different values of *k*.

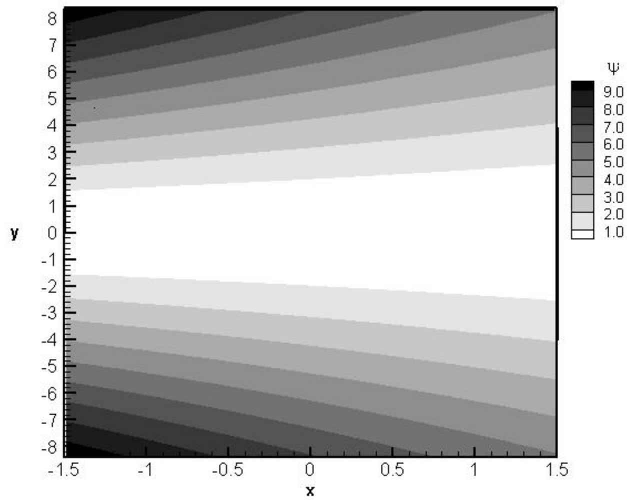

Figure 3. $\psi(x, y)$ for *k = - 0.1,* $U_0^e = B_0 = 1$ and initial conditions $\varphi(0) = 0.5, \varphi_q(0) = 0$.

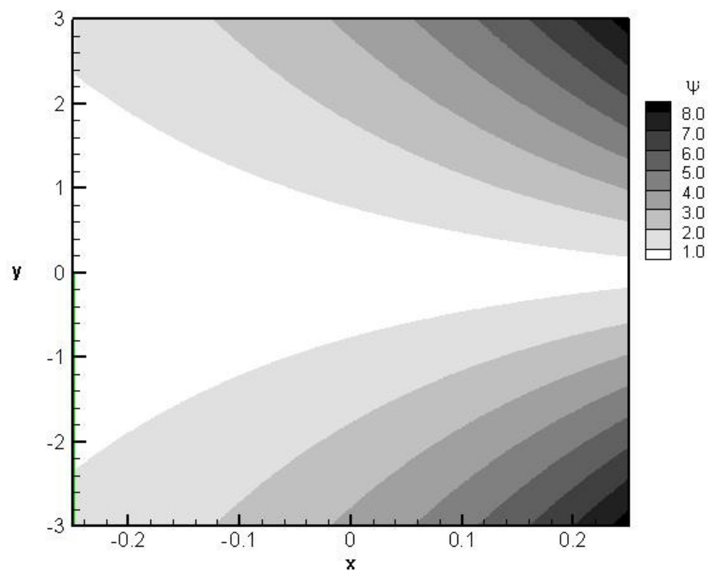

Figure 4. $\psi(x, y)$ for *k = 2.0,* $U_0^e = B_0 = 1$ and initial conditions $\varphi(0) = 0.5, \varphi_q(0) = 0$.

We have considered the numerical simulation of the hydromagnetic Falkner-Skan equation of type II, Eq. (10) under the boundary conditions (11). To deal with two-point boundary value problems there are several approaches, among which: shooting method, setting $\varphi_{qq}(0)$ by trial and error so as to complain with $\varphi_{qq}(\infty) = 0$ up to some prescribed accuracy; use of a trial function profile adapted to the boundary conditions (Kumari and Nath, 1999); Hankel-Padé method (Abbasbandy and Hayat, 2009); Crocco transformation (Chiam, 1999); conformal mapping method (Boisseau, Forgács and Giacomini, 2007). In this work the shooting method is applied since it has been proven to be enough for our purposes. Typical results are shown in Figs. 5, 6 and 7, under variation of $\varphi(0), k$ and $\varphi_q(0)$ respectively.

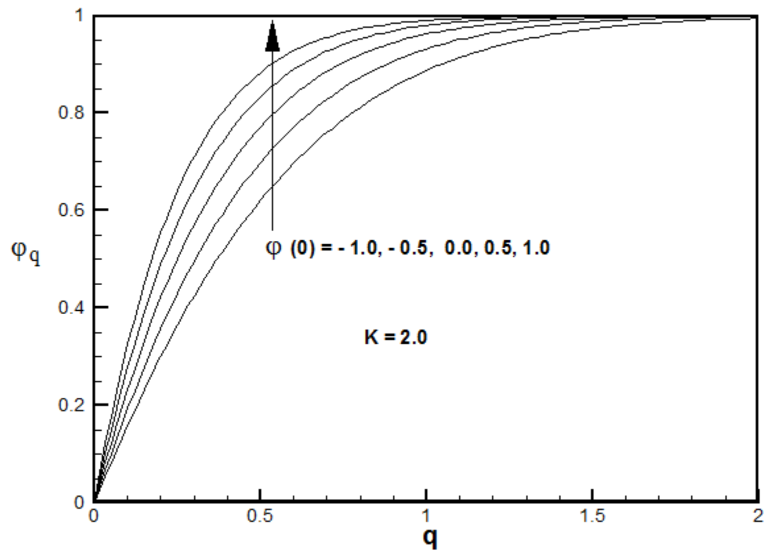

Figure 5. $\varphi_q(q)$ for $k = 2.0, \varphi_q(0) = 0$ and several values of $\varphi(0)$ as indicated.

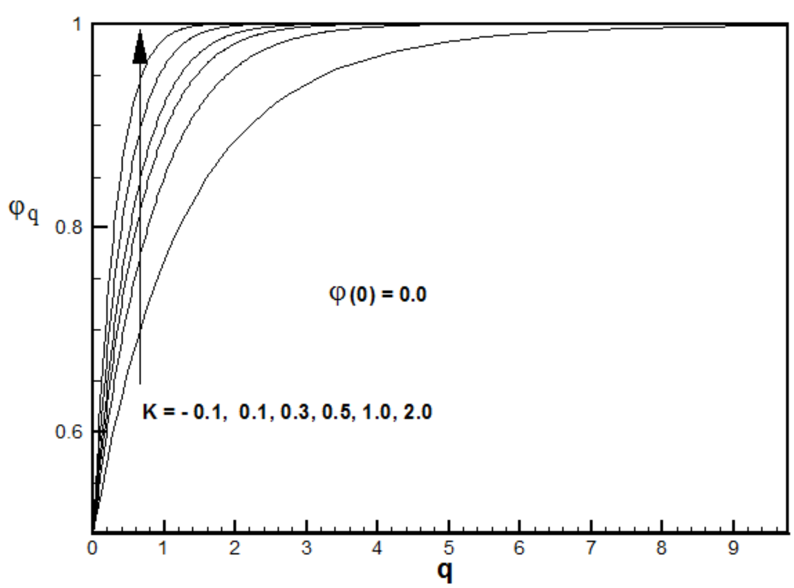

Figure 6. $\varphi_q(q)$ for $\varphi(0) = \varphi_q(0) = 0$ and several values of k as indicated.

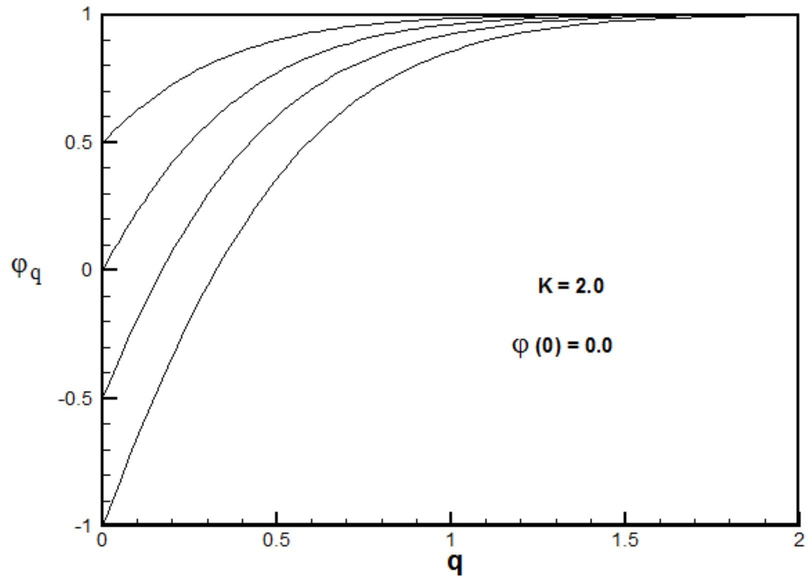

Figure 7. $\varphi_q(q)$ for $\varphi(0) = 0, k = 2$ and several values of $\varphi_q(0)$ as apparent.

5. CONCLUSIONS

In this work the relevance of exponential-type similarity solutions for the two-dimensional steady hydromagnetic boundary layer equation has been highlighted. Physically the new class of solutions applies to more abrupt expanding or converging viscous MHD flows, in comparison to the well known power-law forms. No *ad hoc* assumptions have been adopted, nor for the external velocity and magnetic fields, nor for the boundary conditions at the wall. Hence injection or suction flows were also admitted. The numerical simulation for the new hydromagnetic Falkner-Skan solutions was performed. As apparent in all simulations, the asymptotic result $\varphi_q(\infty) \to 1$ is attained. Further developments, to be shown in a separate work, comprise a series solution valid for the strong magnetic field cases.

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