COMPLIANCE MINIMIZATION USING TOPOLOGY OPTMIZATION METHOD WITH TETRAHEDRICAL ELEMENTS (GRID #4)

Karilany Dantas Coutinho, karilany@ufrnet.br
João Carlos Arantes Costa Júnior, arantes@ufnet.br
Federal University of Rio Grande do Norte (UFRN), DEM (PPGEM), Campus Universitário, Lagoa Nova, Technological Center, CEP 59072-790, Natal-RN, Brazil.

Marcelo Krajnc Alves, krajnc@emc.ufsc.br
Federal University of Santa Catarina (UFSC), EMC (POSMEC), Campus Universitário, Trindade, Technological Center, CEP 88040-900, Florianópolis-SC, Brazil.

Abstract. *The topology optimization problem characterizes and determines the optimum distribution of material into the domain project. The objective of this work is to propose a competitive formulation for optimum structural topologies determination in 3D problems in formulation of compliance minimum problems and able to provide high-resolution layouts. The classical structural optimization problem of compliance minimization is a consistent optimum formulation problem, differently what account with the optimization problem has stress constraints. The procedure combines the Galerkin Finite Elements Method with the optimization method, searching the improve material distribution along the fixed domain of project. The finite element used for the approach is a four nodes tetrahedron with a selective integration scheme, which interpolates not only the components of the displacement field but also the relative density field. According to mentions above, the proposed consist in the formulation of a compliance topology optimization problem. The Layout Optimization Method is based on approach material with considers homogenized constitutive equation that depends only of the relative density of the material. The microstructure used in this procedure was the SIMP (Solid Isotropic Material with Penalty). The approach reduces considerably the computational cost, showing to be efficient and robust. The results provided a well-defined structural layout, with a sharpness distribution of the material and a boundary condition definition. The layout quality was proportional to the medium size of the element and a considerable reduction of the project variables was observed due to the tetrahedrical element.*

Keywords*: topology, compliance, optimization, FEM, layout.*

1. INTRODUCTION

The objective is define the excellent topology of structures, Bendsoe and Kikuchi (1988) proposed the Topology Optimization Method, based in the Theory of the Homogenization. In this paper, the problem of topology optimization is transformed in a problem of material redistribution under one preset project domain using Compliance Minimization in 3D elements. The properties effective of the composed material are according to Homogenization Theory. The concept of this method was used to decide minimization problems of the internal energy of deformation for Suzuki and Kikuchi (1991), Díaz and Bendsoe (1992), Tenek and Hagiwara (1993), Bendsoe *et. al*. (1995) and Krog and Olhoff (1999). Formularizations had been proposals for Mlejnek and Schirrmacher (1993), Yang and Chuang (1994) and Costa Jr. and Alves (2000, 2002). Costa Jr. and Alves (2000, 2002) work the density of the material is considered constant inside of the element. Moreover, the properties effective of the material are determined by a homogenized constitutive equation, that depends only on the relative density of the material and is based on the model considered by Gea (1996).

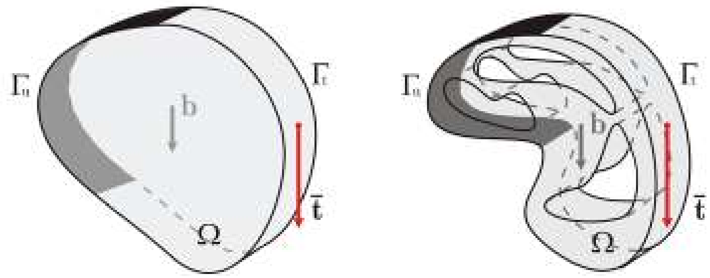
In this work will be developed a computational process competitive and efficient in the definition of the 3D topology structures under compliance minimization. On TOM (Topology Optimization Method) will be applied to the minimization criterion of the internal energy of deformation, being the constraints associates to the problem: Volume and Side. With target of to stabilize the solution and prevent problems, stability constraints are imposed. The state equations are approached by the Galerkin Finite Element Method, with four nodes tetrahedral elements that interpolate relative density fields. The material model is characterized by an artificial microstructure of type SIMP.

2. PROBLEM DEFINITION

The Fig. 1, define a topology optimization problem.

Here, we define the following sets: Ω – is the domain of the body and the boundary is denoted for $\partial\Omega = \Gamma_{\mathbf{u}} \cup \Gamma_{\mathbf{t}}$ and $\Gamma_{\mathbf{u}} \cap \Gamma_{\mathbf{t}} = \varnothing$, $\Gamma_{\mathbf{u}}$ – is the part of the boundary with prescribed displacement, i.e., $\mathbf{u} = \overline{\mathbf{u}}$; $\Gamma_{\mathbf{t}}$ – is the part of the boundary with prescribed traction, i.e., $\boldsymbol{\sigma}\mathbf{n} = \overline{\mathbf{t}}$; $\mathbf{b}$ – is the body force.

$H_o(\Omega)=\left\{\mathbf{v} \middle| \mathbf{v}\in\left[H^1(\Omega)\right]^2, \mathbf{v}=\mathbf{0} \text{ at } \mathbf{x}\in\Gamma_u\right\}$ and $H=\{\bar{\mathbf{u}}+H_0\}$ are variations of the displacement field and set of admissible displacements, respectively.



(a) Initial Condition (b) Final Condition
Figure 1: Compliance optimization problem.

2.1. Formulation of the Problem

The optimization problem has like objective the compliance minimization of 3D topology under constraint volume and side constraints.
The layout optimization problem may be formulated as follows:

$$\min_{\rho} l(\mathbf{u})=\min_{\rho}\left(\int_{\Omega}\mathbf{b}\cdot\mathbf{u}d\Omega+\int_{\Gamma_t}\bar{\mathbf{t}}\cdot\mathbf{u}d\Gamma\right) \tag{1}$$

Subject to

(i) volume constraint:

$$\int_{\Omega}\rho\, d\Omega=V_o, \tag{2}$$

(ii) side constraints

$$\rho_{\inf}-\rho\le 0 \quad \text{and} \quad \rho-\rho_{\sup}\le 0, \quad \forall\,\mathbf{x}\in\Omega, \tag{3}$$

(iii) stability constraint

$$\left(\frac{\partial\rho(\mathbf{x})}{\partial j}\right)^2-\left(c_j\right)^2\le 0;\ j=x,y \text{ and } z \quad \forall\,\mathbf{x}\in\Omega, \tag{4}$$

The constants c_x, c_y e c_z impose a superior limit to the components of relative density gradient, Petersson and Sigmund (1998) and Sigmund and Petersson (1998).
The displacement field $\mathbf{u}(\boldsymbol{\rho}(\mathbf{x}),\mathbf{x})$ is solution of:

$$\begin{aligned} &a(\mathbf{u},\mathbf{v})=l(\mathbf{v}), \quad \forall\,\mathbf{v}\in H_o \\ &\therefore div\,\boldsymbol{\sigma}+\mathbf{b}=\mathbf{0} \end{aligned}, \tag{5}$$

where $\boldsymbol{\sigma}=\mathbf{D}^H(\rho)\boldsymbol{\varepsilon}(\mathbf{u})$ and $\bar{\mathbf{u}}=\mathbf{0}$ on Γ_u, $\boldsymbol{\sigma}\mathbf{n}=\bar{\mathbf{t}}$ on Γ_t, with:

$$a(\mathbf{u},\mathbf{v})=\int_{\Omega}\mathbf{D}^H(\rho)\boldsymbol{\varepsilon}(\mathbf{u})\cdot\boldsymbol{\varepsilon}(\mathbf{v})d\Omega \tag{6}$$

The integral defined in equation (6) is approach by select integration:

$$a(\mathbf{u},\mathbf{v}) = a_{vol}(\mathbf{u},\mathbf{v}) + a_{dist}(\mathbf{u},\mathbf{v}) \tag{7}$$

been $a_{vol}(\mathbf{u},\mathbf{v})$ are volumetric terms and $a_{dist}(\mathbf{u},\mathbf{v})$, distortion terms. And

$$l(\mathbf{v}) = \int_{\Omega} \mathbf{b}\cdot\mathbf{v}\, d\Omega + \int_{\Gamma_t} \overline{\mathbf{t}}\cdot\mathbf{v}\, d\Gamma \tag{8}$$

2.2. Model Material Definition

The porous material concept employed is modeled with the so-called, proportional "fictitious material" model, also name as the solid isotropic material with penalization model (SIMP). So, a continuous variable ρ, $0 \le \rho \le 1$ is introduced. In numerical implementations, a small lower bound $\rho_{\inf}$, such that $0 < \rho_{\inf} \le \rho \le 1$, is imposed, in order to avoid a singular FEM problem, when solving for equilibrium conditions equations in the full domain Ω, see Bendsoe and Sigmund (1999). The homogenized constitutive equation of the effective material may be fully expressed in terms of the relative density of the porous material.

$$\boldsymbol{\sigma} = \left[\mathbf{D}^H(\rho)\right]\boldsymbol{\varepsilon}(\mathbf{u}) \tag{9}$$

where

$$\boldsymbol{\sigma}^T = \left\{\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}\right\} \tag{10}$$

$$\left[\mathbf{D}^H(\rho)\right] = \left[\mathbf{D}^H_{vol}\right] + \left[\mathbf{D}^H_{dist}\right] \tag{11}$$

and

$$\left[\mathbf{D}^H_{vol}\right] = \begin{bmatrix} \mathbf{D}^H_{11}(\rho) & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{\varepsilon}^{\mathrm{T}} = \left\{\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, 0, 0, 0\right\} \tag{12}$$

$$\left[\mathbf{D}^H_{dist}\right] = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{D}^H_{22}(\rho) \end{bmatrix}, \quad \boldsymbol{\varepsilon}^{\mathrm{T}} = \left\{0, 0, 0, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}\right\} \tag{13}$$

with

$$\left[\mathbf{D}^H_{11}(\rho)\right] = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 \end{bmatrix} \tag{14}$$

$$\left[\mathbf{D}^H_{22}(\rho)\right] = \frac{E}{2(1+\nu)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{15}$$

ν is Poisson's ratio.

$$E(\rho) = \rho^{\eta} E_o \tag{16}$$

E_o is the Young's modulus of solid material and η is a penalty parameter of "porous material concept".

3. DISCRETIZATION OF THE PROBLEM

3.3. Formulation of the discretized problem

In order to discretize the problem is applied the Galerkin Finite Element Method. It is employed a four nodes tetrahedron finite element that interpolates not only the displacement components but also the relative density ρ. Consequently, the discretization formulation of the problem way be stated as:

$$\min_{\rho} f\left(\mathbf{u}(\boldsymbol{\rho})\right) \tag{17}$$

subject to:

$$\int_{\Omega} \rho \, d\Omega - V_o \le 0 \tag{18}$$

$$\rho_{\inf} - \rho_i \le 0 \quad \text{and} \quad \rho_i - \rho_{\sup} \le 0\,; \quad i=1,...,n \quad (n \text{ is the number of nodes in the mesh}) \tag{19}$$

remember that $\rho_{\sup} = 1$, and

$$h_{ej}(\boldsymbol{\rho}) = \frac{\sqrt{\left(\dfrac{\partial \rho}{\partial j}\right)^2}}{c_j} - 1\,; \quad j = x, y \text{ and } z\,, \tag{20}$$

for $e = 1,\ldots,n_e$; $\boldsymbol{\rho} \in \mathbf{X}$ with $\mathbf{X} = \left\{ \boldsymbol{\rho} \in \mathbb{R}^n \,\middle|\, \rho_{\inf} \le \rho_i \le \rho_{\sup}\,, i = 1,\ldots,n \right\}$ and n_e is the number of elements in the mesh.

Augmented Lagrangian Method

Step – 1. Initial Conditions: $k = 0$, $\lambda^k = 0$, $\mu^k = \mathbf{0}$, $erro = 1.0$, ζ, ω^k and tol.

Step – 2. While $erro > tol$, to do:

(i) Solution of the minimization problem with side constraints:

$$\min \Pi(\boldsymbol{\rho}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \zeta, \boldsymbol{\omega})\,, \quad \forall\, \boldsymbol{\rho} \in \mathbf{X} \tag{21}$$

where,

$$\Pi(\rho, \lambda, \mu; \zeta, \omega) = f(\rho) + \frac{1}{\zeta}\sum_{e=1}^{n_e} \Lambda_e(g_e, \zeta\lambda_e) + \sum_{j=1}^{3}\left[\frac{1}{\omega_j}\sum_{e=1}^{n_e} \Psi_e^j\left(h_e^j, \omega_j \mu_{ej}\right)\right], \tag{22}$$

with

$$\Lambda_e(g_e, \zeta\lambda_e) = \begin{cases} g_e(g_e + \zeta\lambda_e) & , \text{ if } g_e \ge -\dfrac{\zeta\lambda_e}{2} \\ -\left(\dfrac{\zeta\lambda_e}{2}\right)^2 & , \text{ if } g_e < -\dfrac{\zeta\lambda_e}{2} \end{cases}, \tag{23}$$

and

$$\Psi_e^j\left(h_e^j, \omega_j \mu_{ej}\right) = \begin{cases} h_e^j\left(h_e^j + \omega_j \mu_{ej}\right) & , \text{ if } h_e^j \ge -\dfrac{\omega_j \mu_{ej}}{2} \\ -\left(\dfrac{\omega_j \mu_{ej}}{2}\right)^2 & , \text{ if } h_e^j < -\dfrac{\omega_j \mu_{ej}}{2} \end{cases} \; ; \; j = 1,...,3. \tag{24}$$

(ii) Update of Lagrange multipliers

$$\lambda_e^{k+1} = \max\left\{0, \lambda_e^k + \frac{2}{\zeta} g_e\left(\mathbf{x}^k\right)\right\} \tag{25}$$

and

$$\mu_{ej}^{k+1} = \max\left\{0, \mu_{ej}^{k} + \frac{2}{\omega_j} h_e\left(\mathbf{x}^k\right)\right\} \;\; ; \;\; j = 1,...,3. \tag{26}$$

(iii) Update of penalty parameters

$$\zeta^{k+1} = \begin{cases} \gamma_1 \zeta^k & \text{with } \gamma_1 \in (0,1), \;\; \text{if } \;\; \gamma_1 \zeta^k > \zeta^{crit} \\ \zeta^{crit} & \end{cases} \tag{27}$$

and

$$\omega_j^{k+1} = \begin{cases} \beta_j \omega_j^k & \text{with } \beta_j \in (0,1), \;\; \text{if } \; \beta_j \omega_j^k > \omega_j^{crit} \\ \omega_j^{crit} & \end{cases} \;\; ; \;\; j = 1,...,3. \tag{28}$$

(iv) Error

$$a = \max_e \left|\lambda_e^{k+1} - \lambda_e\right|, \; b = \max_e \left|\mu_{e1}^{k+1} - \mu_{e1}\right|, \; c = \max_e \left|\mu_{e2}^{k+1} - \mu_{e2}\right| \;\text{ and }\; d = \max_e \left|\mu_{e3}^{k+1} - \mu_{e3}\right| \tag{29}$$

so, $erro = \max\{a, b, c, d\}$.

Step – 3. End

The problem can be formulated as: $\lambda, \mu_1, \mu_2, \mu_3 \in \mathbb{R}^{n_e}$ and $\zeta, \omega_1, \omega_2, \omega_3 \in \mathbb{R}$, determinate $\boldsymbol{\rho}^* \in \mathbb{R}^n$, such that:

$$\boldsymbol{\rho}^* = \arg\min \Pi\left(\boldsymbol{\rho}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \zeta, \boldsymbol{\omega}\right) \;, \quad \forall\, \boldsymbol{\rho} \in \mathbf{X}. \tag{30}$$

4. NUMERICAL EXAMPLES

4.1. Problem #01

Consider a problem according to Fig. 2. The case consists in one block of dimensions: $a = 4.0\,m$, $b = 7.0\,m$ and $c = 5.0\,m$. On block is applied a prescribed vertical load of $\mathbf{P} = 5.0 \times 10^6\, N$. The optimum layout is submitted to volume constraint $\alpha = 0.2$. It was analyzed the structure with 13,389 elements and 3,130 nodes.

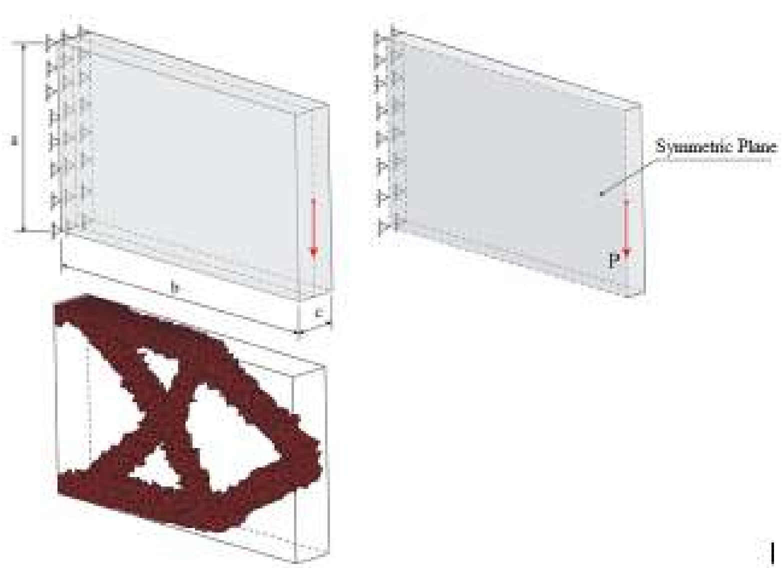


Figure 2: Problem #1 Result.

4.2. Problem #02

In this case, consider a problem according to Fig. 3. The block dimensions are: $a = 2.0\,m$, $b = 10.0\,m$ and $c = 1.0\,m$. On block is applied a prescribed vertical load of $\mathbf{P} = 5.0\times10^6\,N$ and with a volume constraint $\alpha = 0.25$. It was analyzed the ¼ of block structure with 35,555 elements and 7,289 nodes.

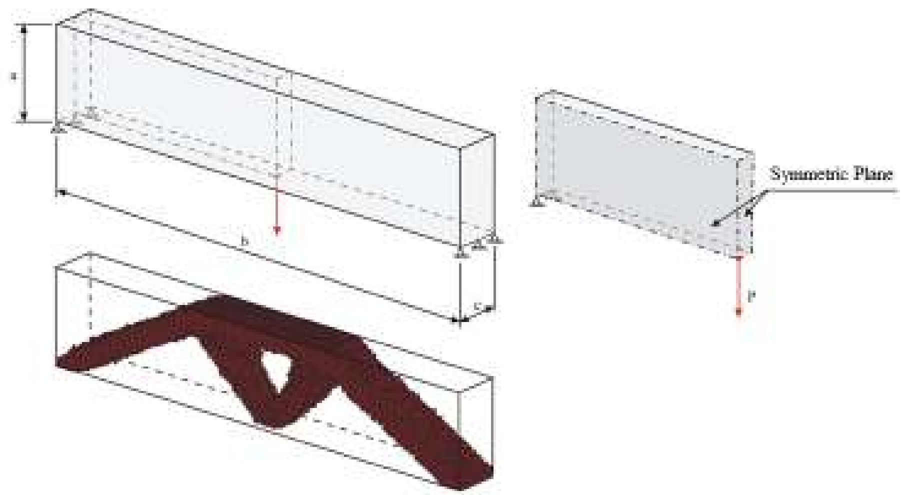


Figure 3: Problem #2 Result.

4.3. Problem #03

Consider a problem according to Fig. 4. The block dimensions are: $a = 1.0\,m$, $b = 1.0\,m$ and $c = 1.0\,m$. On block is applied a prescribed vertical load of $\mathbf{P} = 5.0\times10^6\,N$ and with a volume constraint $\alpha = 0.2$. The block presents a prescribed displacement of $[0,0,0]$ on $\overline{\Omega}_1 = \overline{\Omega}_2 = \overline{\Omega}_3 = \overline{\Omega}_4$. It was analyzed the ¼ of block structure with 16,205 elements and 3,218 nodes.

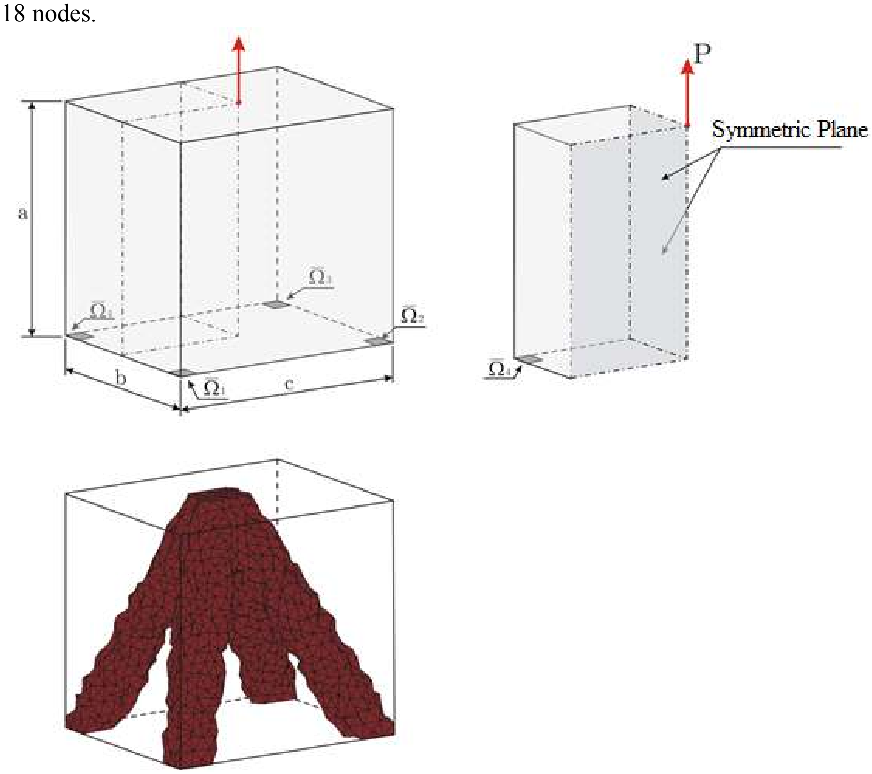


Figure 4: Problem #4 Result.

5. CONCLUSION

According to results the algorithm proposed showed to be effective and robust in the generation of excellent structural problems 3D structural topologies. The final resolution of the material contour is directly dependent of the average size of the finite element employee, representing a direct relation with the computational cost, as can be verified presents excellent clearness with clear disposal of material. In relation to a formulation for pixel this method has the disadvantage of the computational cost for the calculation of the stiffness in each element, but this disadvantage can disappear with the implementation of a *h*-adaptivity process. Note that, the compliance optimization problems without stress constraints are not sufficient to project structures, so will be necessary to process a stress analysis, and to make adjustments to the sizing. The optimum compliance should be applied to get an initial proposal of topology structure (conception model).

The strategy of the implementation of adaptivity resources, that is, the implementation of an intelligent process of refinement of the mesh with information of the topology obtained in the original mesh, is detailed in the works of Costa Jr. (2003), Costa Jr. and Alves (2003a-b). For the solution optimization problem, according to described in the main body of this paper, the Augmented Lagrangian Method is applied. The solver presented excellent performance, in a posterior version will compare with computational structure TANGO of Andreani *et al*. (2004), Andreani *et al*. (2005) and Birgin and Martinez (2002).

6. REFERENCES

Andreani, R., Birgin, E.G., Martínez, J.M. and Schuverdt, M.L., 2004, “Augmented Lagrangian Methods under the Constant Positive Linear Dependence Constraint Qualification”, Mathematical Programming.

Andreani, R., Birgin, E.G., Martínez, J.M. and Schuverdt, M.L., 2005, “On Augmented Lagrangian methods with general lower-level constraints”, Technical Report MCDO-050304, Department of Applied Mathematics, UNICAMP, Brazil.

Bendsoe, M.P., 1995, “Optimization of Structural Topology, Shape, and Material, Berlin Heidelberg: Springer-Verlag”, 271 p.

Bendsoe, M.P. and Kikuchi, N., 1988, “Generating optimal topologies in structural design using a homogenization method, Comput. Meth”. Appl. Mech. Engrg., v. 71(2), p. 197-224.

Bendsoe, M.P. and Sigmund, O., 1999, “Material interpolation schemes in topology optimization, Archive of Applied Mechanics”, v. 69, p. 635-54.

Birgin, E.G. and Martínez, J.M., 2002, “Large-scale active-set box-constrained optimization method with spectral projected gradients”, Computational Optimization and Applications 23, pp. 101-125.

Costa Jr., J.C.A., 2003, "Otimização Topológica com Refinos *H*-adaptativos", PhD Dissertation (Thesis document), UFSC, Santa Catarina, Brazil.

Costa Jr., J.C.A. and Alves M.K., 2003a, “Layout optimization with *h*-adaptivity of structures”, Int. J. Numer. Meth. Engng., v. 58(1), p. 83-102.

Costa Jr., J.C.A. and Alves M.K., 2003b, “*h*-adaptivity procedures of a stress based layout optimization”, Cilamce 2003, Ouro Preto-MG, Brasil, CDRom media.

Duysinx, P. and Sigmund, O., 1998, “New Development in handling stress constraints in optimal material distribution”, In 7th AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Design Optimization, American Institute of Aeronautics and Astronautics, Saint Louis, Missouri, EUA, paper 98/4906/1–9.

Duysinx, P. and Bendsøe, M.P., 1998, “Topology Optimization of Continuum Structures with Local Stress Constraints”, Int. J. Numer. Meth. Engng., vol. 43, pp. 1453-1478.

Kohn, R.V. and Strang, G., 1986, “Optimal design and relaxation of variational problem”, Comm. Pure. Appl. Math., 39, p. 1-25, 139-82, 353-77.

Petersson, J. and Sigmund O., 1998, “Slope constrained topology optimization”, Int. J. Numer. Meth. Engng., v. 41(8), 1417-34.

Sigmund, O. and Petersson, J., 1998, “Numerical instabilities in topology optimization: a survey on procedures dealing with checkerboards, mesh dependencies and local minima, Structural Optimization”, v. 16, p. 68-75.

7. RESPONSIBILITY NOTICE

The authors are the only responsible for the printed material included in this paper.