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ANALYTICAL MODELS FOR VIBRATION OF BEAM REINFORCED PLATES

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Abstract. *This work is aimed at developing a beam reinforced plate model for structures used in oil platforms. The beam is modeled using plate theory and is placed in the union of two horizontal plates. The model takes in account the in-plane and the out-of-plane waves propagating at both plates and on the union beam. The results are then compared with Finite Element Model, through the use of a commercial software, showing an excellent level of accordance. The advantage of analytical models is that computational costs are significantly reduced when compared to Finite Element Model.*

Keywords*: plates, vibration, in-plane waves, out-of-plane waves*

1. INTRODUCTION.

Offshore oil platforms are assembled basically with beam reinforced plates. This type of structure is characterized by a high density of machines and equipments assembled directly through the main structure. Vibrations generated by this sources are propagated through the main structure and can reach the lodging area, where the noise level should be controlled.

For beam reinforced plates, the traditional existing models are developed with plates coupled with a beam model, as example Euler or Timoshenko beam models. These models do not take into account the wedge and web resonances.

Both the power flow transmitted through the structure and the vibratory energy could be calculated using the correct value of the average spatial mean squared velocity, $<\overline{v^2}>$, of each structure. One way of calculating this parameter is using the Finite Element Method (FEM). This method, however, is used only for low modal densities or low/medium frequencies, otherwise the processing time demanded would be extremely high. This comes from the fact that for high frequencies or high modal density the wavelength are smaller, so it is required a large number of elements to correct represent the system, making difficult the application for higher structures.

The deterministic methods permit the analysis up to higher frequencies with low processing time, and without restrictions with components dimensions.

This work consists in determining the response of a system composed as three plates connected in a “T” configuration. The plate on the union represents the beam modeled using theory of plates, where it is possible to simulate the beam in a more real way, considering the out-of-plane and the in-plane waves. The results are compared with the one obtained using a Finite Element Model commercial software.

2. FUNDAMENTAL ASPECTS.

2.1. OUT-OF-PLANE WAVES IN THIN PLATES

This model deals with bending waves propagating in thin plates, being appropriate to thickness up to one decimal of the wavelength. For example, to a 10 mm plate thickness, the frequency range of analysis can be extended to 5 000 Hz using this model.

For a thin plate, with thickness h and finite dimensions *a* and *b*, as showed on Figure 1, with a differential element with volume $h\,dx\,dy$, the shear forces, twist moment and bending moment are supposed to act.

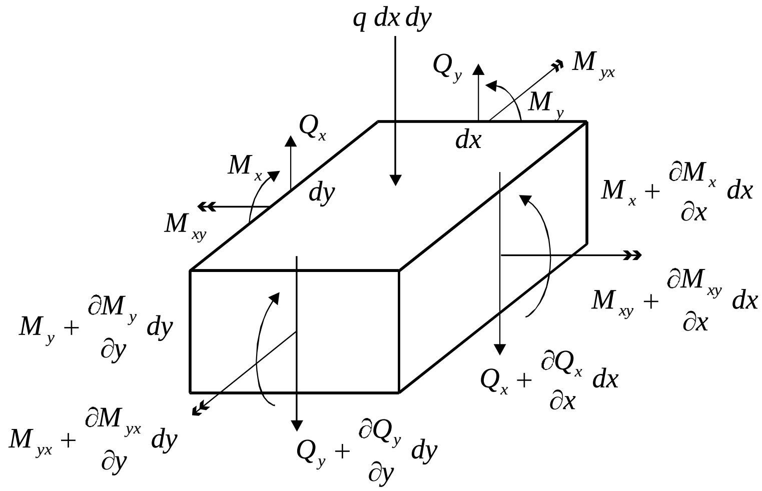

Figure 1. Thin plate element submitted to forces and moments.

The governing equation for a thin plate submitted to a distributed load $q(x,y,t)$ is (Graff, 1991):

$$D\nabla^4 w(x,y,t) + \rho h \frac{\partial^2 w(x,y,t)}{\partial t^2} = q(x,y,t) \tag{1}$$

where the term $w(x,y,t)$ is the medium plane deflection

This equation is generally applied to plate thickness up to ten times the wavelength and the solution describes the behavior of the out-of-plane waves.

2.2. IN-PLANE WAVES IN THIN PLATES

For a thin plate element, submitted to in-plane forces, as showed on Figure 2, the in plane efforts N_x and N_y by length unity, arising from normal stress σ_x and σ_y and the effort N_{xy} generated by τ_{xy}, are given by (Flugg, 1966).

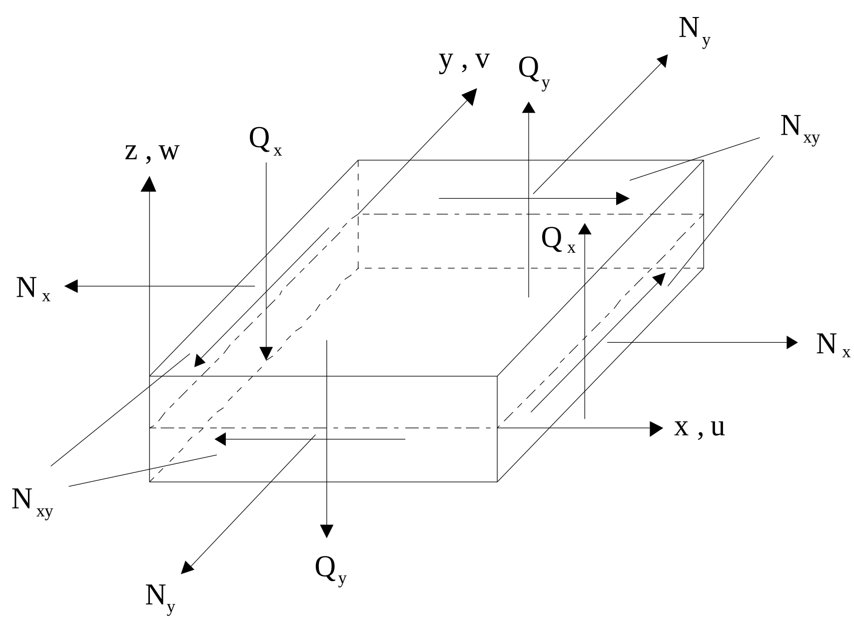

Figure 2. Thin plate submitted to in-plane loads.

$$N_x = \frac{Eh}{1-\nu^2}\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right) \tag{2}$$

$$N_y = \frac{Eh}{1-\nu^2}\left(\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\right) \tag{3}$$

$$N_{xy} = \frac{Eh}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{4}$$

The resultant equation of motion for x and y directions are given by (Hwang and Pi, 1972):

$$\frac{\partial^2 u}{\partial x^2} + \frac{(1-\nu)}{2}\frac{\partial^2 u}{\partial y^2} - \frac{(1-\nu^2)\rho}{E}\frac{\partial^2 u}{\partial t^2} + \frac{(1+\nu)}{2}\frac{\partial^2 v}{\partial x\,\partial y} = 0 \tag{5}$$

$$\frac{(1+\nu)}{2}\frac{\partial^2 u}{\partial x\,\partial y} + \frac{(1-\nu)}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - \frac{(1-\nu^2)\rho}{E}\frac{\partial^2 v}{\partial t^2} = 0 \tag{6}$$

2.3. BENDING WAVES SOLUTION

For a thin plate, with a zero distributed load $q(x, y, t)$, Equation (1) can be rewritten again as:

$$D\nabla^4 w(x, y, t) + \rho h \frac{\partial^2 w(x, y, t)}{\partial t^2} = 0 \tag{7}$$

The general solution for a bi-supported plate that satisfies the differential equation (10) can be expressed in the following way:

$$w(x, y) = \sum_{n=1}^{\infty} \psi_n(x)\,\mathrm{sen}(k_y\, y)\, e^{j\omega t} \tag{8}$$

The base functions $\psi_n(x)$ are given by (Bonifácio, 1998):

$$\psi_n(x) = A_1 e^{s_1 x} + A_2 e^{s_2 x} + A_3 e^{s_3 x} + A_4 e^{s_4 x} \tag{9}$$

where $s_1 = \sqrt{k_y^2 + k^2}$, $s_2 = \sqrt{k_y^2 - k^2}$, $s_3 = -\sqrt{k_y^2 + k^2} = -s_1$, $s_4 = -\sqrt{k_y^2 - k^2} = -s_2$ and $k = \sqrt{\frac{\rho\omega^2}{D}}$

Then, the transversal displacement in the plate are written as:

$$w(x, y) = \sum_{n=1}^{\infty} \{A_1 e^{s_1 x} + A_2 e^{s_2 x} + A_3 e^{-s_1 x} + A_4 e^{-s_2 x}\}\,\mathrm{sen}(k_y\, y)\, e^{j\omega t} \tag{10}$$

where the constants A_1, A_2, A_3 and A_4 are calculated through the boundary conditions applied on the plates edges.

2.4. IN-PLANE WAVES SOLUTION

The configuration of in-plane forces are showed in Figure 2. The Equations (8) and (9) are the differential Equations for in-plane displacements and are coupled. Those equations should be uncoupled and solved separately. This is solved writing the displacements in the plane, called u and v , in terms of potential functions for longitudinal deformation and shear rotation, represented by ϕ_d and ψ_s , respectively, as (Mccolumn, 1988):

$$u(x, y) = \frac{\partial \phi_d}{\partial x} + \frac{\partial \psi_s}{\partial y} \tag{11}$$

$$v(x,y) = \frac{\partial \phi_d}{\partial y} - \frac{\partial \psi_s}{\partial x} \tag{12}$$

and the potential functions are given by:

$$\phi_d = E(x)\sin\frac{n\pi y}{b} \tag{13}$$

$$\psi_s = T(x)\cos\frac{n\pi y}{b} \tag{14}$$

Using Equations (16) and (17), the uncoupled differential equations are written in the following way:

$$\nabla^2\left(\nabla^2 + k_l^{\,2}\right)\phi_d = 0 \tag{15}$$

$$\nabla^2\left(\nabla^2 + k_s^{\,2}\right)\psi_s = 0 \tag{16}$$

where the wavenumbers are given by $k_l = \omega/c_l$ and $k_s = \omega/c_s$, and c_l and c_s are the longitudinal and transversal shear waves velocities, respectively.

The terms $E(x)$ and $T(x)$ are written as the sum of two propagating waves in opposite, one in the positive axis direction and the other in the negative axis direction:

$$E(x) = A_1 e^{k_1 x} + B_1 e^{-k_1 x} \tag{17}$$

$$T(x) = A_2 e^{k_2 x} + B_2 e^{-k_2 x} \tag{18}$$

where $k_1^2 = k_y^2 - k_l^2$, $k_2^2 = k_y^2 - k_s^2$ and $k_y = n\pi/b$.

After substitution of Equations (15) and (16) in (13) and (14), the uncoupled equations that describes the in-plane movement are obtained:

$$u(x,y) = \sum_{n=1}^{\infty}\{k_1 A_1 e^{k_1 x} - k_1 B_1 e^{-k_1 x} - k_y A_2 e^{k_2 x} - k_y B_2 e^{-k_2 x}\}\,\mathrm{sen}\left(k_y y\right)e^{j\omega t} \tag{19}$$

$$v(x,y) = \sum_{n=1}^{\infty}\{k_y A_1 e^{k_1 x} + k_y B_1 e^{-k_1 x} - k_2 A_2 e^{k_2 x} + k_2 B_2 e^{-k_2 x}\}\cos\left(k_y y\right)e^{j\omega t} \tag{20}$$

where A_1, B_1, A_2 and B_2 are the constants to be determined.

The boundary conditions are applied through Equations (19) and (20) for the displacements and (2), (3) and (4) for the efforts applied on the edges.

3. PROPOSED ALGEBRAIC FORMULATION

3.1. CONFLATION OF THREE PLATES INTO A “T” SHAPE

Considering three coupled plates jointed in a T configuration, as shown on Figure 3, plate 1 can be submitted to four different load distributed at the edge, $M_x(y)$, $N_x(y)$, $N_{xy}(y)$ and $Q_x(y)$.

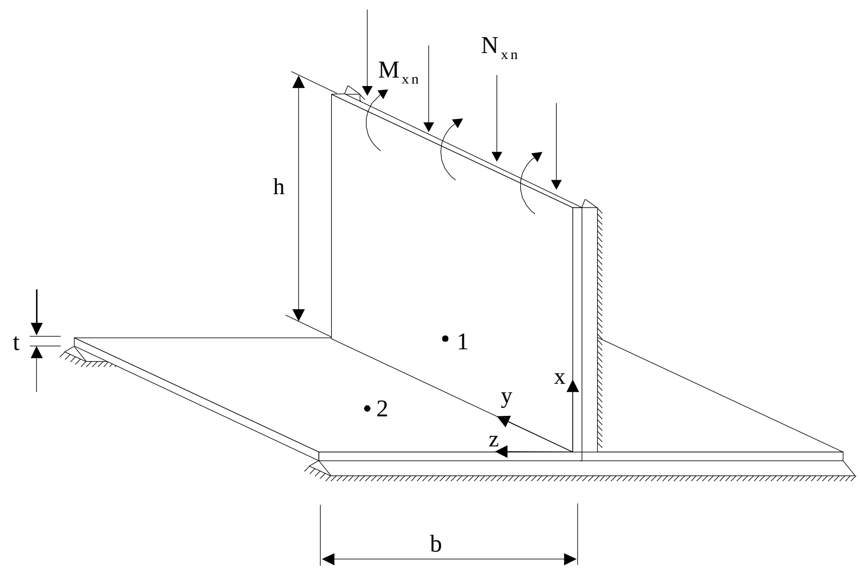

Figure 3 – Three coupled plates in a T configuration.

The functions describing the in-plane response, u and v, and the transversal displacement, called w, at the plates 1, 2, or 3 are given by:

$$u_1(x,y)=\sum_{n=1}^{\infty}\{k_1\,A e^{k_1 x}-k_1\,B e^{-k_1 x}-k_y\,C e^{k_2 x}-k_y\,D e^{-k_2 x}\}\,sen(k_y\,y)e^{j\omega t} \quad (21)$$

$$v_1(x,y)=\sum_{n=1}^{\infty}\{k_y\,A e^{k_1 x}+k_y\,B e^{-k_1 x}-k_2\,C e^{k_2 x}+k_2\,D e^{-k_2 x}\}\cos(k_y\,y)e^{j\omega t} \quad (22)$$

$$w_1(x,y)=\sum_{n=1}^{\infty}\{A_1\,e^{s_1 x}+A_2\,e^{s_2 x}+A_3\,e^{-s_1 x}+A_4\,e^{-s_2 x}\}\,sen(k_y\,y)e^{j\omega t} \quad (23)$$

$$u_2(z,y)=\sum_{n=1}^{\infty}\{k_1\,F e^{k_1 z}-k_1\,G e^{-k_1 z}-k_y\,H e^{k_2 z}-k_y\,I e^{-k_2 z}\}\,sen(k_y\,y)e^{j\omega t} \quad (24)$$

$$v_2(z,y)=\sum_{n=1}^{\infty}\{k_y\,F e^{k_1 z}+k_y\,G e^{-k_1 z}-k_2\,H e^{k_2 z}+k_2\,I e^{-k_2 z}\}\cos(k_y\,y)e^{j\omega t} \quad (25)$$

$$w_2(z,y)=\sum_{n=1}^{\infty}\{A_5\,e^{s_1 z}+A_6\,e^{s_2 z}+A_7\,e^{-s_1 z}+A_8\,e^{-s_2 z}\}\,sen(k_y\,y)e^{j\omega t} \quad (26)$$

$$u_3(x,y)=\sum_{n=1}^{\infty}\{k_1\,J e^{k_1 z}-k_1\,K e^{-k_1 z}-k_y\,L e^{k_2 z}-k_y\,M e^{-k_2 z}\}\,sen(k_y\,y)e^{j\omega t} \quad (27)$$

$$v_3(x,y)=\sum_{n=1}^{\infty}\{k_y\,J e^{k_1 z}+k_y\,K e^{-k_1 z}-k_2\,L e^{k_2 z}+k_2\,M e^{-k_2 z}\}\cos(k_y\,y)e^{j\omega t} \quad (28)$$

$$w_3(x,y)=\sum_{n=1}^{\infty}\{A_9\,e^{s_1 z}+A_{10}\,e^{s_2 z}+A_{11}\,e^{-s_1 z}+A_{12}\,e^{-s_2 z}\}sen\left(k_y\,y\right)e^{j\omega t} \tag{29}$$

Figure 4 shows the representation of the loads acting on each plate and the continuity condition that must be satisfied. In these cases, there are 24 equations and 24 constants to be determined for each frequency and for each load component.

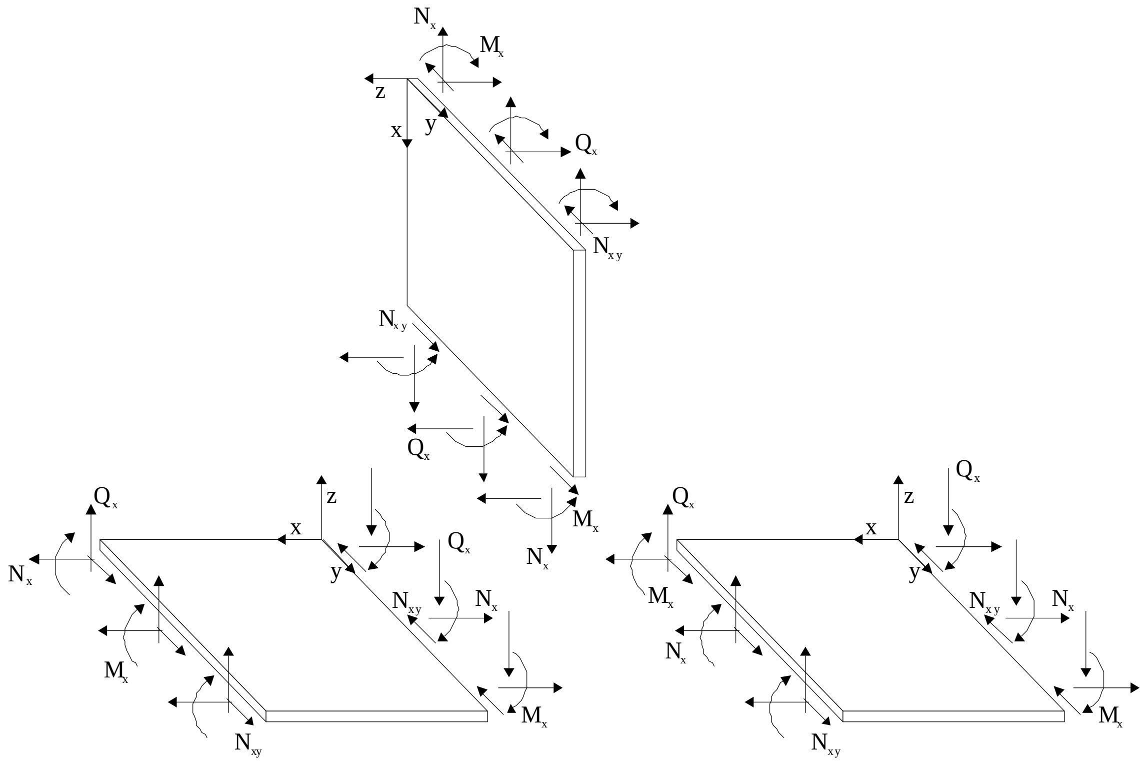

Figure 4 – Loads acting on plates 1, 2 and 3.

The distributed loads acting on plate 1 are written for each component as:

$$\mathrm{N}_{\mathrm{x\,n}}=\frac{2}{\mathrm{b}}\int_0^{\mathrm{b}}\mathrm{N}_{\mathrm{x}}(\mathrm{y})\,\mathrm{sen}\left(\frac{\mathrm{n}\,\pi}{\mathrm{b}}\mathrm{y}\right)\mathrm{dy} \tag{33}$$

$$\mathrm{M}_{\mathrm{x\,n}}=\frac{2}{\mathrm{b}}\int_0^{\mathrm{b}}\mathrm{M}_{\mathrm{x}}(\mathrm{y})\,\mathrm{sen}\left(\frac{\mathrm{n}\,\pi}{\mathrm{b}}\mathrm{y}\right)\mathrm{dy} \tag{34}$$

$$\mathrm{N}_{\mathrm{xy\,n}}=\frac{2}{\mathrm{b}}\int_0^{\mathrm{b}}\mathrm{N}_{\mathrm{xy}}(\mathrm{y})\,\mathrm{sen}\left(\frac{\mathrm{n}\,\pi}{\mathrm{b}}\mathrm{y}\right)\mathrm{dy} \tag{35}$$

$$\mathrm{Q}_{\mathrm{x\,n}}=\frac{2}{\mathrm{b}}\int_0^{\mathrm{b}}\mathrm{Q}_{\mathrm{x}}(\mathrm{y})\,\mathrm{sen}\left(\frac{\mathrm{n}\,\pi}{\mathrm{b}}\mathrm{y}\right)\mathrm{dy} \tag{36}$$

The continuity conditions at the plate union and the internal force and moment equilibrium conditions are written on the appendix.

The result is a system of 24 equations which solution gives the 24 constants (A_1 to A_8 and A to I), for each frequency. After the constants are calculated, it is possible to calculate the response at any point on the plate, using equations 21 to 29. The terms at matrix T represent the equations coefficients and vector P is the load vector.

$$\begin{bmatrix} T_{11} & T_{12} & \cdots & \cdots & T_{1\,24} \\ T_{21} & \ddots & T_{23} & T_{24} & \vdots \\ \vdots & T_{32} & \ddots & T_{34} & \vdots \\ \vdots & T_{42} & T_{43} & \ddots & \vdots \\ T_{24\,1} & \cdots & \cdots & \cdots & T_{24\,24} \end{bmatrix} \begin{Bmatrix} A \\ \vdots \\ M \\ A_1 \\ \vdots \\ A_{12} \end{Bmatrix} = \begin{Bmatrix} P_1 \\ \vdots \\ P_{12} \\ P_{13} \\ \vdots \\ P_{24} \end{Bmatrix} \quad (37)$$

4. RESULTS AND DISCUSSION

4.1 VALIDATION

In the following example, there is an analytical model of a three-coupled inverted T plate. The plates are 5 mm thick, 1 m wide and 0.5 m long. The material is steel, with Young Modulus $E = 210 \cdot 10^9\ N/m^2$, material density $\rho = 7850\,kg/m^3$ and Poison coefficient $\nu = 0.3$. The structural damping is $\eta = 0.01$.

The following distributed load is considered for the edge:

$$M_x = M_n \cdot \text{sen}\left(\frac{n\pi}{L} y\right) \quad (38)$$

where n is assumed to vary from 1 to 2. In this way, the load is given by the addition of two sinusoidal components. The amplitudes of the loads (moments) are $M_1 = M_2 = 1$. The response of the plate is calculated at two points, as shown on Figure 5 (point 1 - x = 0.2, y = 0.15 and z = 0 and point 2 – x = 0, y = 0.15 and z = 0.325). Figures 6 and 7 show the comparison between the response obtained with the analytical model, calculated with the software Mathematica, and a Finite Element Model up to 1000 Hz. The Finite Element Method was applied using the software ANSYS 5.3, educational version. The full method was used for the harmonic analysis, since it is more accurate than the modal superposition, however, the processing time is greater. The plate element initially used was SHELL 63, from the ANSYS library, and has four nodes and six degrees of freedom per node. Further analysis was then carried out using the element SHELL 93, with eight nodes and six degrees of freedom per node. The results obtained were very close to those for SHELL 63. The SHELL 63 element was used, since this model requires less processing time.

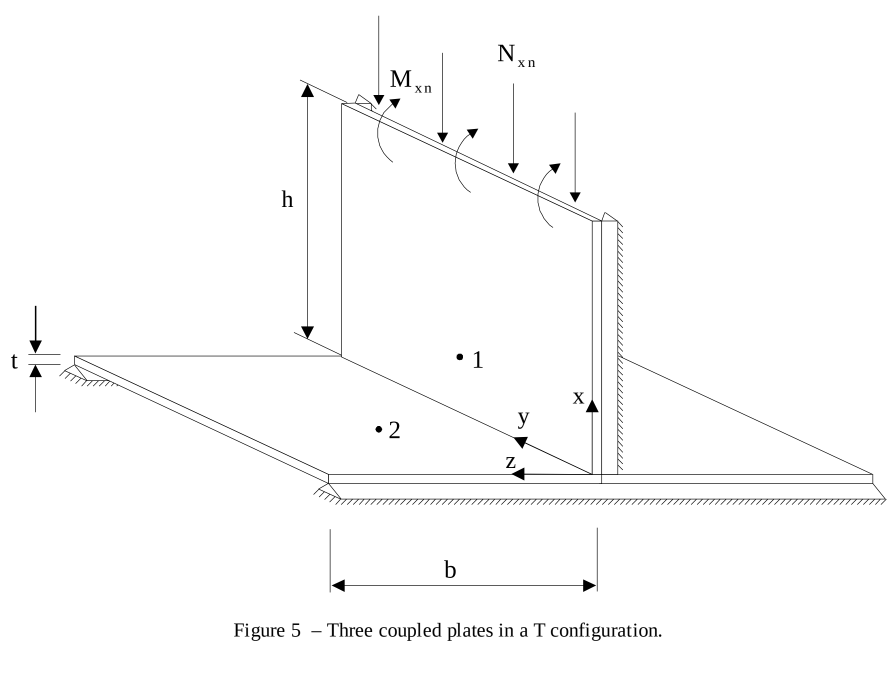

Figure 5 – Three coupled plates in a T configuration.

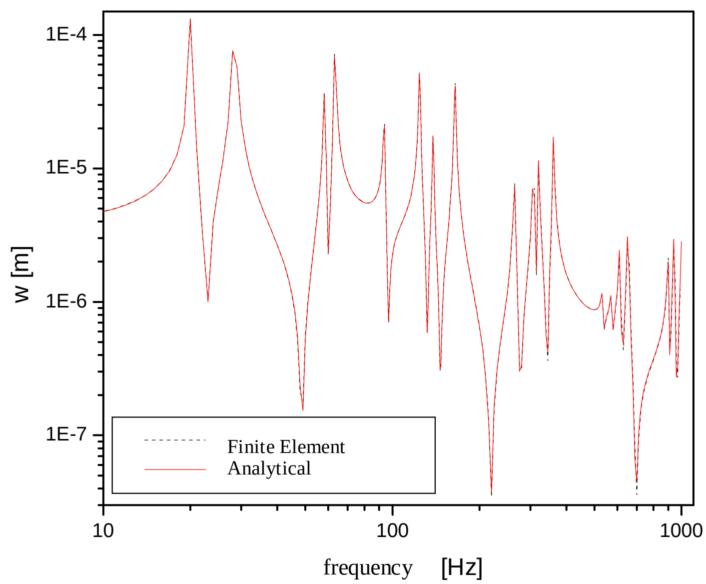

Figure 6 - Transversal displacement W on plate 1. Comparison between the analytical model and the Finite Element Model.

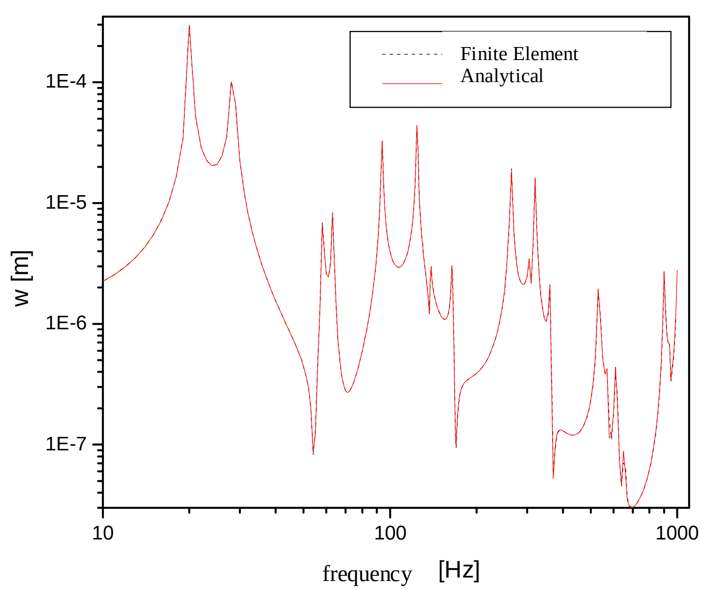

Figure 7 - Transversal displacement W on plate 2. Comparison between the analytical model and the Finite Element Model.

5. CONCLUDING REMARKS

The “T” plate model represents with very good accuracy the dynamic behaviour of beam reinforced plates, as used in offshore platforms and considering the own modes of the beams.

The results show a very good agreement between the analytical solution and the Finite Element Method. It can therefore be concluded that the solution for the connected plates, works well for all the frequencies spectrum selected.

High frequency analysis for plates reinforced beams can be easily achieved, in a really small processing cost. The response obtained can be used for verifying the importance of in-plane and out-of-plane waves, depending on the frequency range. This step will be verified in further analysis.

The next step is the application of this model to calculate the coupling loss factor between plates reinforced by beams to be used in a Statistical Energy Analysis (SEA) model.

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APPENDIX

$u_1(x = Lx_1) = -w_2(z = 0)$; $v_1(x = Lx_1) = v_2(z = 0)$; $w_1(x = Lx_1) = u_2(z = 0)$;

$\theta_1(x = Lx_1) = \theta_2(z = 0)$; $u_1(x = Lx_1) = -w_3(z = 0)$; $v_1(x = Lx_1) = v_3(z = Lz_3)$;

$w_1(x = Lx_1) = u_3(z = Lz_3)$; $\theta_1(x = Lx_1) = \theta_3(z = Lz_3)$; $Q_{1x}(x = Lx_1) - N_{2x}(z = 0) + N_{3x}(z = Lz_3) = 0$;

$N_{1x}(x = Lx_1) + Q_{2x}(z = 0) - Q_{3x}(z = Lz_3) = 0$; $N_{1xy}(x = Lx_1) - N_{2xy}(z = 0) + N_{3xy}(z = Lz_3) = 0$

$M_{1x}(x = Lx_1) - M_{2x}(z = 0) + M_{3x}(z = Lz_3) = 0$ $N_{1x}(x = 0) = N_n$ $N_{1xy}(x = 0) = S_n$;

$M_{1x}(x = 0) = M_n$; $Q_{1x}(x = 0) = Q_n$; $N_{2x}(z = L_z) = 0$; $N_{2xy}(z = L_z) = 0$;

$M_{2x}(z = L_z) = 0$; $Q_{2x}(z = L_z) = 0$; $N_{3x}(z = 0) = 0$; $N_{3xy}(z = 0) = 0$;

$M_{3x}(z = 0) = 0$; $Q_{3x}(z = 0) = 0$