MATHEMATICAL ANALYSIS OF ROLL WAVES IN HERSCHEL BULKLEY FLUIDS

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Abstract. *The purpose of this work is to study the roll waves generation in hyperconcentrated fluids flowing in a sloping canal. Whith the expectation of predicting a mathematical model generating roll waves, this article presents a mathematical model based on Navier-Stokes equations integrated in vertical, including the Herschel-Bulkley rheological model in the tension tensor. An analysis of linear stability is made and a analytical theory of permanent roll waves is imployed to determine under what flow conditions roll waves can exist. Moreover, a roll wave equation is established and a numerical analysis verifies the generation of such instabilities.*

Keywords*: Roll waves, Herschel-Bulkley, Hiperconcentrated fluids.*

1. INTRODUCTION

The flows that occur on inclined channel with a resistance due to friction, can develop instabilities in hydraulic jump form or bore waves. Those instabilities can appear in both Newtonian fluids (clear water) and non Newtonian fluids (hyperconcentrated fluids). Such disturbances, with constant wavelength, are traveling waves, particularly called roll waves. Some what rare in natural flows, those waves appear more frequently in artificial canals and spillways of dams.

In 1925, Jeffreys was the first to establish a criterion about the Roll Waves formation from a linear stability analysis. He inferred that the uniform flow made it self unstable if the Froude number was superior to 2. Dressler (1949) carried out an analysis based on the Saint Venant formulation without diffusion terms, combined with bore wave equation. However, his syntetically and correct analysis does not permit to establish the length of these waves.

A first theoretical attempt consisted in an approximation of shallow water equations with roll waves diffusion, allowing us to define stability and production standards (Maciel et al., 1997).

As regards the roll waves formation in hyperconcentrated fluids, several studies were done, although what appears more often in the literature is the study of roll waves generated in Bingham's fluids (Liu and Mei, 1994), (Maciel, 1998), (Noble, 2004).

Roll waves formation was undertaken by (Ng and Mei, 1994), starting from a rheological proposal of fluids with pseudoplastic behavior (power law), carry out an analiytical investigation by seeking roll waves solutions characterized as periodic shocks connected by smoothly increasing depth profiles. Pascal (2005) investigate the generation and structure of roll waves developing on the surface of a power-law fluid layer flowing down a porous incline.

In the present paper we consider a non Newtonian fluid flowing down an incline, using the Herschel-Bulkley rheological model. In parallel, an experimental study has been performed by team, where mixtures of water+clay and water+fine sand+clay are prepared and rheometry tests are realized. Based on (Coussot, 1992) studies, (Piau, 1996), (Huang and Garcia, 1998), (Lledo, 2003) and (Kiryu, 2003), it's proves that those fluids rheology (considering the sedimentation/ressuspension phenomena) could be describe through the nonlinear rheological model like Herschel-Bulkley, in simple laminar shear and steady regime. In the sequence those mixtures flow on a long platform with 10m length, producing, in some situations, roll waves.

2. GOVERNING EQUATIONS

Considering the two-dimensional laminar flow and using the Herschel-Bulkley rheological model, a coordinate system (*x*,*z*) is defined as the *x*-axis downslope along and the *z*-axis upward normal to the plane bed. The equations of motion for the layer are obtained from the Navier-Stokes equations, including the Herschel-Bulkley rheological model in the tension tensor, given by:

$$\tau = \tau_c + K_n \left(\frac{\partial u}{\partial z} \right)^n \tag{2.1}$$

where τ_c is the yield stress, n is the flow index and K_n is the dynamic viscosity.

The longitudinal and vertical velocity components are denote by (u, w), the pressure by P, and the total flow depth normal to be bed by h. The characteristic flow length and the flow depth changes relatively slowly in the longitudinal direction. Then the flow is governed by equations

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad . \tag{2.2}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g \sin\theta + \frac{\partial}{\partial z}\left[\tau_c + K_n\left(\frac{\partial u}{\partial z}\right)^n\right] \tag{2.3}$$

$$\frac{\partial P}{\partial z} = -\rho g \cos\theta \tag{2.4}$$

where ρ is the fluid density and g is the gravitational acceleration.

The stress condition at the surface is given by

$$P,\ \tau = 0 \qquad \text{at} \qquad z = h \tag{2.5}$$

The Kinematic condition at the surface is given by

$$w = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} \qquad \text{at} \qquad z = h \tag{2.6}$$

While at the bottom we have

$$u = w = 0 \qquad \text{at} \qquad z = 0 \tag{2.7}$$

Integrating Eq. (2.4) and using the boundary condition (2.5) we find the expression for the pressure to be:

$$P = -\rho g \cos\theta (z - h) \tag{2.8}$$

Inserting (2.8) into (2.3), the momentum equation reads

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = g \sin\theta - g \cos\theta \frac{\partial h}{\partial x} + \frac{K_n}{\rho}\frac{\partial}{\partial z}\left[\tau_c + K_n\left(\frac{\partial u}{\partial z}\right)^n\right] \tag{2.9}$$

The limiting velocity profile for a steady uniform flow is obtainable from equation:

$$\rho g \sin\theta (h - z) = \tau_c + K_n\left(\frac{\partial u}{\partial z}\right)^n \tag{2.10}$$

The flow can be divided into a plug layer having velocity $u = u(z_0)$ on top of a shear layer in which u increases from zero to $u(z_0)$.

$$u = u(z_0) \qquad\qquad z_0 \le z \le h \tag{2.11}$$

$$u = u(z_0)\left[1 - \left(\frac{z}{z_0}\right)^{(n+1)/n}\right] \qquad\qquad 0 \le z \le z_0 \tag{2.12}$$

where

$$u(z_0) = \frac{n}{n+1}\left(\frac{\rho g z_0^{\,n+1}\sin\theta}{K_n}\right)^{1/n} \qquad (2.13)$$

Considering following the dimensionless variables,

$$Z = \frac{z}{z_0},\ U = \frac{u(z)}{u(z_0)} \qquad (2.14)$$

The velocity profile is verified through numerical resolution.

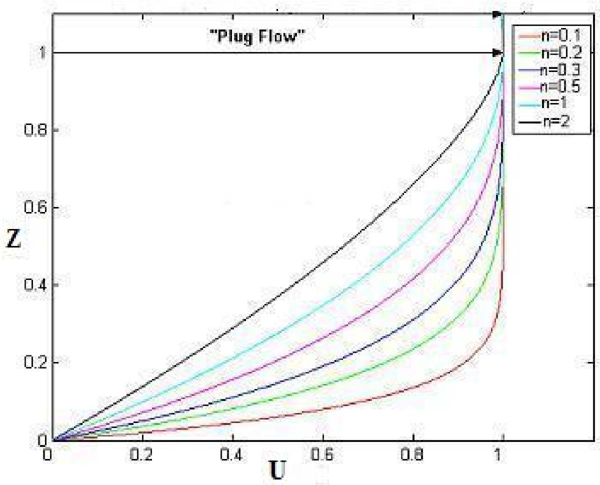

Figure 1.Velocity profiles for different values of *n*

Profiles of velocity in fig.1, show a plug layer on top of a shear layer, as shown by (Huang, 1998).
The depth-averaged velocity is given by

$$\bar{u} = \int_0^h u\,dz = \frac{n}{n+1}\left(\frac{\rho g\sin\theta}{K_n}\right)^{\frac{1}{n}}\left(\frac{h\rho g\sin\theta-\tau_c}{\rho g\sin\theta}\right)^{\frac{n+1}{n}}\left[1-\frac{n}{2n+1}\left(\frac{h\rho g\sin\theta-\tau_c}{h\rho g\sin\theta}\right)\right] \qquad (2.15)$$

Integrating (2.2) and (2.8) with respect to 0 to *h*, using the Leibniz and applying the boundary conditions (2.6) and (2.7), we get

$$\frac{\partial\bar{u}h}{\partial x}+\frac{\partial h}{\partial t}=0 \qquad (2.16)$$

$$\frac{\partial\bar{u}h}{\partial t}+\frac{\partial\alpha\bar{u}^2h}{\partial x}+\frac{1}{2}g\cos\theta\frac{\partial h^2}{\partial x} = g.hsen\theta-\frac{\tau_c}{\rho}-\frac{K_n}{\rho}\left[\frac{\bar{u}(\rho g sen\theta)^2(n+1)(2n+1).h}{(h\rho g\sin\theta-\tau_c)\left((n+1)n\rho g\sin\theta-n^2\tau_c\right)}\right]^n \qquad (2.17)$$

The velocity distribution coefficient is given by

$$\alpha = \frac{1}{\bar{u}^2h}\int_0^h u^2dz = \frac{(2n+1)}{(3n+2)}\left[\frac{\left(2(2n+1)^2h\rho g\sin\theta+\tau_c(4n+3)n\right)}{\left((n+1)^2h\rho g\sin\theta+2n\tau_c(n+1)+n^2\tau_c^{\,2}/h\rho g\sin\theta\right)}\right] \qquad (2.18)$$

in which

h : flow depth;
$\bar{u}$: average vertical velocity;

τ_c : yield stress;
θ : canal declivity;
ρ : fluid density;
K_n : consistency index ;
n : flow index.

Substituting (2.16) into (2.1) and setting $z = 0$, the bottom stress follows

$$\tau_p = \tau_c + K_n \left[\frac{\bar{u}(\rho g \sin\theta)^2 (n+1)(2n+1)h}{(h\rho g \sin\theta - \tau_c)\left(n(n+1)h\rho g \sin\theta + n^2 \tau_c\right)} \right]^n \tag{2.19}$$

The equations (2.17) and (2.18) constitute the governing equations for laminar flows following the Herschel-Bulkley model.

3. DIMENSIONLESS VARIABLES

To investigate the relative magnitude of the terms in these equations, dimensionless variables are introduced by using a number of scales as follows:

$$x = l_0 x^*,\ (h,z) = h_0\left(h^*, z^*\right),\ t = \frac{l_0}{u_0} t^*,\ \bar{u} = \bar{u}_0 \bar{u}^* \tag{3.1}$$

The index $\left({}_0\right)$ represents the uniform draining conditions and asterisk $\left({}^*\right)$ the dimensionless variables, in which

h_0 the length scale in *z* and l_0 is the length scale in *x* given by

$$l_0 = \frac{\bar{u}_0^{\,2}}{g \sin\theta} \tag{3.2}$$

$$\bar{u}_0 = \frac{n}{n+1}\left(\frac{\rho g \sin\theta}{\mu_n}\right)^{\frac{1}{n}} \left(\frac{h_0 \rho g \sin\theta - \tau_c}{\rho g \sin\theta}\right)^{\frac{n+1}{n}} \left[1 - \frac{n}{2n+1}\left(\frac{h_0 \rho g \sin\theta - \tau_c}{h_0 \rho g \sin\theta}\right)\right] \text{ (Velocity scale)} \tag{3.3}$$

$$Fr = \frac{\bar{u}_0^{\,2}}{(g h_0)} \quad \text{(Froude number)} \tag{3.4}$$

We define a characteristic bottom stress in terms of h_0 and $\bar{u}_0$, it follows:

$$\tau_p = \tau_c + K_n \left[\frac{\bar{u}_0(\rho g \sin\theta)^2 (n+1)(2n+1)h_0}{(h_0 \rho g \sin\theta - \tau_c)\left(n(n+1)h_0 \rho g \sin\theta + n^2 \tau_c\right)} \right]^n \tag{3.5}$$

Introducing these dimensionless variables in equations (2.17)-(2.18) and omitting the asterisks, the system is obtained, after some mathematical developments.

$$\frac{\partial h}{\partial t} + \frac{\partial (h\bar{u})}{\partial x} = 0 \tag{3.6}$$

$$\frac{\partial \bar{u}}{\partial t} + \alpha \bar{u}\frac{\partial \bar{u}}{\partial x} - (\alpha - 1)\frac{\bar{u}}{h}\frac{\partial h}{\partial t} + \beta \frac{\partial h}{\partial x} = h - C^* - \left(1 - C^*\right)\left\{\left[\frac{\bar{u}\left(1 - C^*\right)}{h - C^*}\right]\left[\frac{\left((n+1) + nC^*\right)h}{(n+1)h + nC^*}\right]\right\}^n \tag{3.7}$$

where

$$\beta = \frac{gh_0 \cos\theta}{\bar{u}_0^{\ 2}} \tag{3.8}$$

$$C^* = \frac{\tau_c}{\rho g h_0 \sin\theta} \tag{3.9}$$

$$\alpha = \frac{(2n+1)}{(3n+2)}\left[\frac{\left(2(n+1)^2 h + C^*(4n+3)n\right)}{\left((n+1)^2 h + 2(n+1)nC^* + \frac{n^2}{h}\left(C^*\right)^2\right)}\right] \tag{3.10}$$

4. LINEAR STABILITY ANALYSIS

To establish a stability analysis a small disturbance will be added to the equations (3.5)and (3.6) as follows:

$$h = 1 + H(x,t) \tag{4.1}$$

$$u = 1 + V(x,t) \tag{4.2}$$

Considering $H, V << 1$, through of a process of linearizing of the system, the linearized equations can be combined to produce the following equation in H :

$$\frac{\partial^2 H}{\partial t^2} + (\alpha - \beta)\frac{\partial^2 H}{\partial x^2} + 2\alpha\frac{\partial H}{\partial x \partial t} + (2n+1)\frac{\partial H}{\partial x} + n\frac{\partial H}{\partial t} = 0 \tag{4.3}$$

Considering a normal mode for the disturbances

$$H = \hat{H}e^{i(kx - \omega t)} \tag{4.4}$$

in which *k* is the wavenumber, and $\omega = \omega_r + i\omega_i$ is complex.. For the dispersion equation we obtain:

$$\omega^2 - (2\alpha k - ni)\omega + (\alpha - \beta)k^2 - (2n+1)ki = 0 \tag{4.5}$$

Solving this equation we have:

$$\omega = \frac{1}{2}\left[2\alpha k - ni \pm \sqrt{a + bi}\right] \tag{4.6}$$

where

$$a = 4\left(\alpha^2 - a + \beta\right)k^2 - n^2 \tag{4.7}$$

$$b = 4(2n + 1 - \alpha n)ki \tag{4.8}$$

Considering $0 < n \le 1$, $0 < \alpha \le 1.2$, we have $b > 0$ for $k \ne 0$, we obtain:

$$I(\omega) = \frac{1}{2}\left[-n \pm \sqrt{\frac{1}{2}\left(\sqrt{a^2 + b^2} - a\right)}\right] \tag{4.9}$$

$$\Re(\omega) = \frac{1}{2}\left[2\alpha k \pm \sqrt{\frac{1}{2}\left(\sqrt{a^2 + b^2} + a\right)}\right] \tag{4.10}$$

If $I(\omega) < 0$, therefore $I(\omega)^-$ it tends to decay and stability. Then $I(\omega) > 0$ if and only if

$$\alpha^2 - \alpha + \beta < \frac{(2n+1-n\alpha)^2}{n^2} \tag{4.11}$$

Substituting (3.9) into (4.11), the flow will be unstable if

$$\beta < \frac{2n+1}{n^2}\left\{(2n+1) - \frac{\left[2(n+1)^2 + (4n+3)nC^*\right]n}{\left[(n+1)^2 + 2n(n+1)C^* + n^2 C^{*2}\right]}\right\}^n = \beta_n \tag{4.12}$$

The figure 2 shows the growth rate of disturbance as a function of k and α for $n = 0.2$ and $n = 0.4$, respectively, for some values of β.

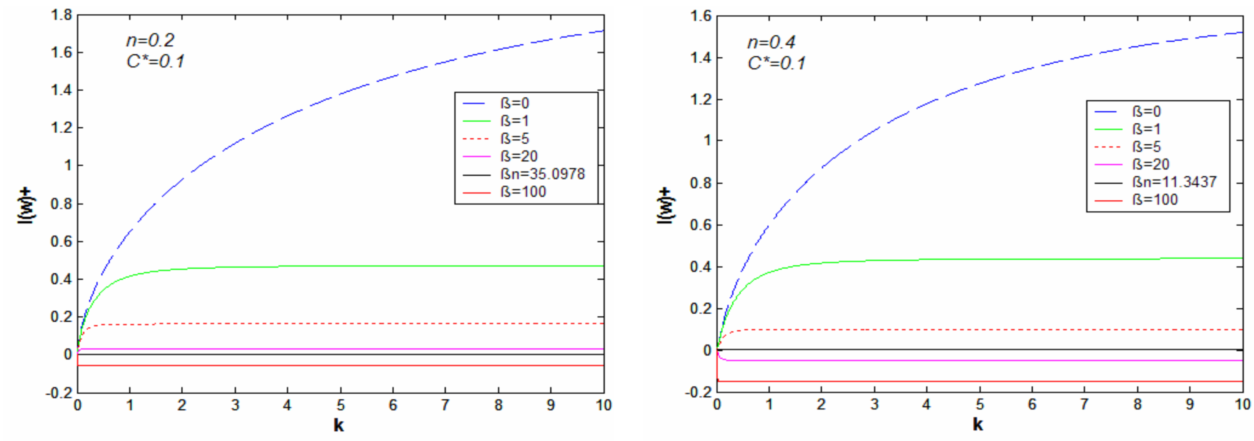

Figure 2: Growth rate of disturbance.

The figure 3 shows the phase velocity of disturbance as a function of k and α for $n = 0.2$ and $n = 0.4$, respectively, for some values of β.

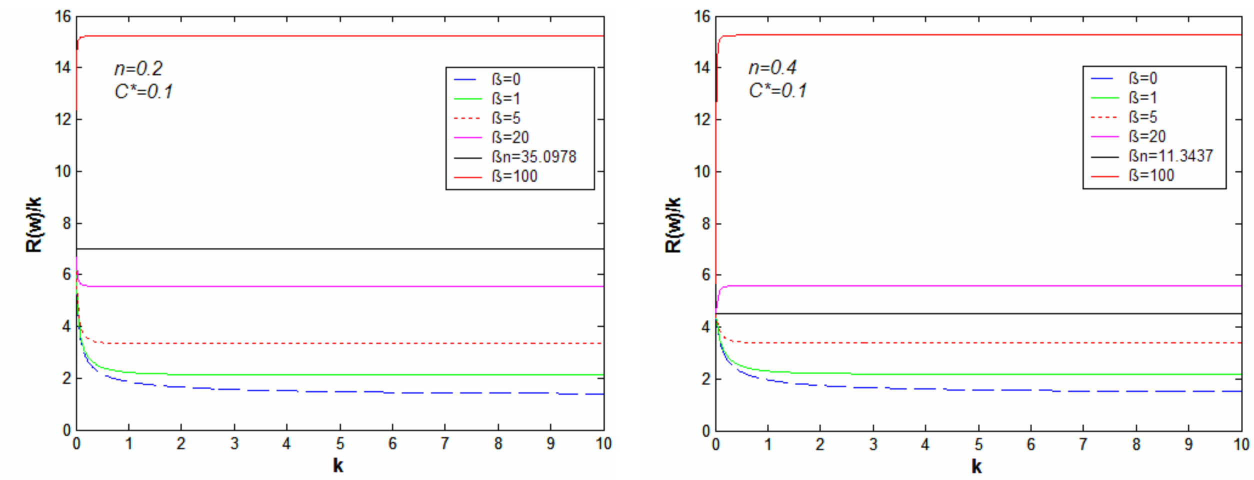

Figure 3.Phase velocity of disturbance.

The threshold of instability β_n depends only on n. The neutral stability curves are given the two lines $k = 0$ and $\beta = \beta_n$. If the yield stress will be null, the determined results has compared very well with that found in (Ng and Mei, 1994), when it used the rheological model of the type to power law. For Newtonian fluid the instability criterion becomes $\beta < 3$ and the phase velocity is *3*. This has been obtained by (Prokopiou et al, 1991) and (Ng and Mei 1994).

5. EQUATION OF THE ROLL WAVE

In order to analyses the almost permanent study of the system, we make a variable change : $z = x - Ut$, in which U is the uniform propagation velocity of a roll wave. With that, and applying a variable change (3.5) and (3.6) equations, it is obtained:

Mass Conservation

$$h(\bar{u} - U) = 1 - U \tag{5.1}$$

Momentum equation

$$-U\frac{\partial u}{\partial z} + \alpha\bar{u}\frac{\partial h}{\partial z} + (\alpha-1)U\frac{\bar{u}}{h}\frac{\partial h}{\partial z} + \beta\frac{\partial h}{\partial z} = 1 - \frac{\tau_c}{h\rho g sen\theta} -$$

$$-\frac{1}{h}\left(1 - \frac{\tau_c}{h_0\rho g sen\theta}\right)\left[\left(\frac{\bar{u}\left(h_0\rho g sen\theta - \tau_c\right)}{h_0\rho g sen\theta - \tau_c}\right)\left(\frac{\left((n+1)h_0\rho g sen\theta + n\tau_c\right)h}{(n+1)h_0 h\rho g sen\theta + n\tau_c}\right)\right] \tag{5.2}$$

Replacing (4.13) in (4.14) and cutting the variable *u*, we have a first order differencial equation in the variable *h*:

$$\frac{\partial h}{\partial z} = \frac{h - C^* - \left(1 - C^*\right)\left[\left(\frac{\left(1 + U(h-1)\right)\left(1 - C^*\right)}{\left(h - C^*\right)}\right)\left(\frac{(n+1) + nC^*}{(n+1)h + nC^*}\right)\right]^n}{\left((\alpha-1)U^2 - \frac{\alpha\left(1-U\right)^2}{h^2} + \beta h\right)} = \frac{F(h)}{G(h)} \tag{5.3}$$

therefore

$$h(z) = \int \frac{F(h)}{G(h)}dz \tag{5.5}$$

When the yield stress will be null, that is, for a rheological proposal of fluids with pseudoplastic behavior (power law), the obtained equations are in accord with the equations found by (Ng and Mei 1994).

5.1 Shock conditions

The wavelength of the roll wave it can be defined of the following form

$$\lambda = \int_{h_1}^{h_2} \frac{G(h)}{F(h)}dh \tag{5.1.1}$$

where h_1 is the depth before of the shock, h_2 is the depth after of the shock and $\langle h \rangle$ the average depth of the roll wave profile given by

$$\langle h \rangle = \frac{1}{\lambda}\int_0^{\lambda} h\,dz = \frac{1}{\lambda}\int_{h_1}^{h_2} \frac{h\,G(h)}{F(h)}dz \tag{5.1.2}$$

The shock conditions are derived from the conservation laws of mass and momentum in (3.5) and (3.6)

$$U\left[\,h\,\right]_1^2 = \left[\,\bar{u}\,h\,\right]_1^2 \tag{5.1.3}$$

$$U\left[\bar{u}\,h\right]_1^2 = \left[\alpha\,\bar{u}^2 h + \frac{1}{2}\beta\,h^2\right]_1^2$$
(5.1.4)

where $[h]_1^2 = h_2 - h_1$. Using the eq. (5.1) and eliminating $\bar{u}$ from (5.1.2), we have a relation between the two depths h_1 and h_2:

$$\alpha(1-U)^2 - h_1 h_2 U^2(\alpha - 1) - \frac{1}{2}\beta h_1^2 h_2 - \frac{1}{2}\beta h_1 h_2^2 = 0 \quad (5.1.5)$$

Solving for h_2, with $\beta > 0$, follows:

$$h_2 = \left\{\left[\frac{h_1}{2} + (\alpha - 1)\frac{U^2}{\beta}\right]^2 + \frac{2\alpha(1-U)^2}{h_1\beta}\right\}^{\frac{1}{2}} - \frac{h_1}{2} - (\alpha - 1)\frac{U^2}{\beta} \quad (5.1.6)$$

For $\beta = 0$:

$$h_2 = \frac{(1-U)^2\alpha}{(\alpha - 1)U^2 h_1} \quad (5.1.7)$$

6. NUMERICAL RESULTS

The resolution of the eq. (5.3) it was obtained through a developed of a rountine calculation, using the computational packet Python, showing the profile of the roll wave. Figure 5, shows the profile of the roll waves for $n = 0.4$, $\beta = 1$ and $U = 2.18$, varying the value of C^*.

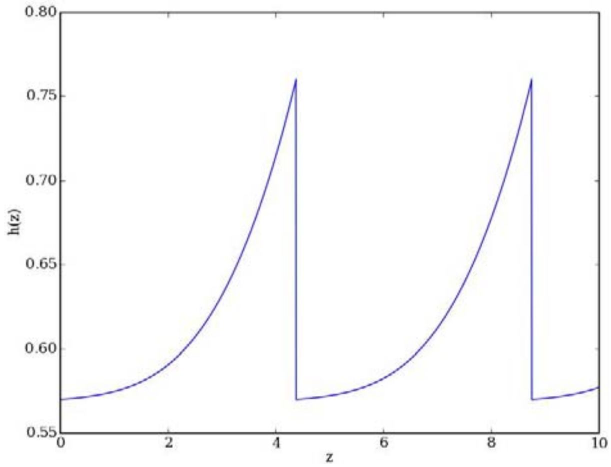

Figure5.a. Numerical results for $n = 0.4$, $\beta = 1$, $U = 2.18$ and $C^* = 0.1$.

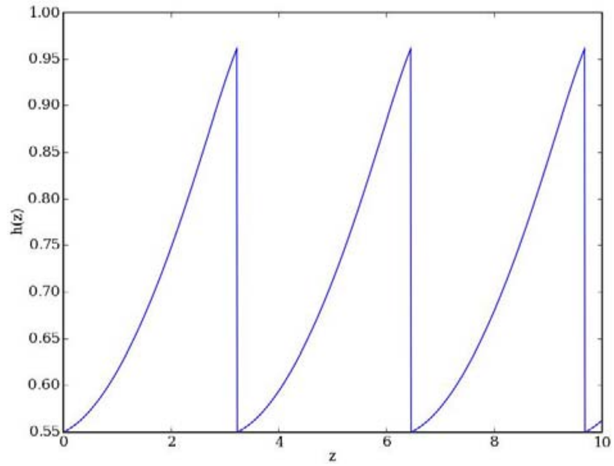

Figure5.a. Numerical results for $n = 0.4$, $\beta = 1$, $U = 2.18$ and $C^* = 0.4$.

Figure 5. Profile of the roll waves varying the parameter C^*.

Based in fig. 5, we can observe that an increase of parameter C^*, cause an increase in the amplitude and a decrease in the amplitude and decrease in the wavelength generated.

The figure 6, shows the profile of the roll waves for $n = 0.4$, $C^* = 0.1$ and $\beta = 1$, varying the value of propagation velocity (U).

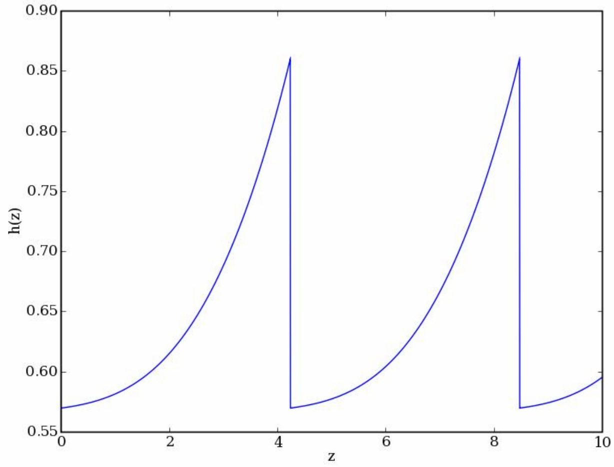

Figure 6.a. Numerical results for $n = 0.4$, $\beta = 1$, $C^* = 0.1$ and $U = 2.17$.

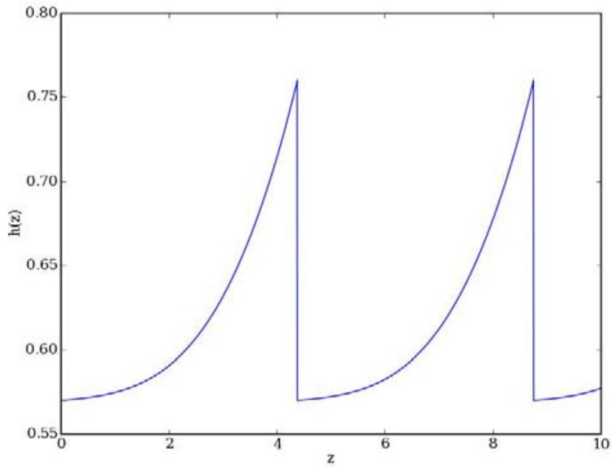

Figure 6.a. Numerical results for $n = 0.4$, $\beta = 1$, $C^* = 0.1$ and $U = 2.18$.

Figure 6. Profile of the roll waves varying the propagation velocity (U)

Fixing the values of β, n, C^* and increasing the propagation velocity (U), we observe a decrease of the amplitude and small variation of the wavelength.

The fig. 6, shows the profile of the roll waves for $n = 0.4$, $C^* = 0.1$ and $U = 3.38$, varying the value of parameter β.

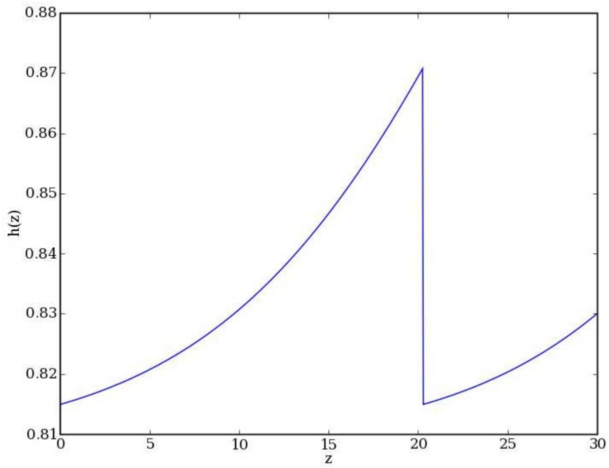

Figure 7.a. Numerical results for $n = 0.4$, $C^* = 0.1$, $U = 3.38$ and $\beta = 1$.

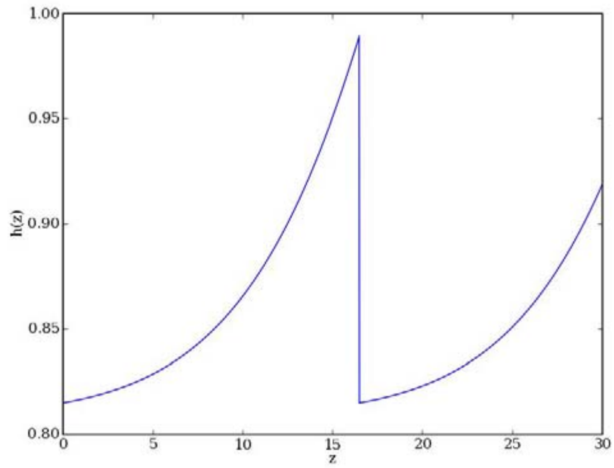

Figure 7.b. Numerical results for $n = 0.4$, $C^* = 0.1$, $U = 3.38$ and $\beta = 5$.

Figure 7. Profile of the roll waves varying the parameter β.

It is observed that increasing the value of parameter β, occurs a significant decrease in the wavelength and a increase in the amplitude of the wave.

7. CONCLUSIONS

It was presented in this article a mathematical model to roll waves generation in hyperconcentrated fluids flowing in a sloping canal, including the Herschel-Bulkley rheological model.

An analysis of linear stability was made, showing the conditions of stability of the system. Through the numerical results, we can observe the appearance of roll waves stabilized for $\beta < \beta_n$.

Using an analytical permanent roll waves theory we determined flow conditions under which roll waves solutions are possible. The method determines a mathematical model generating roll waves and e a numerical analysis shows the evolution of such instabilities. These roll waves patterns are in agreement with those predicted by (Ng and Mei, 1994) study.

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9. RESPONSIBILITY NOTICE

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