QUATERNION FEEDBACK REGULATOR WITH CONTROL AND SLEW RATE CONSTRAINTS

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Abstract : *This paper addresses the problem of reorienting a rigid spacecraft from arbitrary initial condition to a prescribed final condition under control and slew rate constraint in a circular orbit. The rest-to-rest reorientation of the spacecraft is accomplished using three reaction wheels with independent three axis control. The control law proposed is based on a non linear quaternion feedback theory applied to a rigid body in translation and rotation movement.*

1. Introduction

The attitude control of a spacecraft has been subject of much consideration in the recent years. In this case the precision points control; rapid multi target acquisition and tracking capabilities are indispensable requirements. Numerous studies have been conducted for different spacecraft configuration as 1) complete rigid 2) a combination of rigid and flexible parts, or gyrostat-type systems. In this paper quaternions feedback control logic with control and slew rate constraints are considered. A brief overview of the three axes attitude orientation is shown in a way to establish a parallel with the results presented on this paper. In Feiyue and Bainum (1988) an optimal control theory was applied to the slewing motion of a general rigid spacecraft. Jan and Chiou (2003) developed an algorithm of minimum time in order to provide a robust tracking control. Wie and Weiss (1989) presented a linear quaternions feedback regulator with open-loop decoupling control torque for gyroscope forces to ensure eigenaxis rotations. Wie and Jianbo (1995) proposed a globally stable quaternions feedback control law.

2. Eigenaxis Rotation Methods

In this section three main methods will be showed; Euler, Direction Cosine and Quaternion method, which are used to describe the rotation motion of a rigid body around its body-fixed axes with origin at its center of mass. Using one of the methods showed, the rigid body attitude can be changed from any given orientation to any other orientation by rotating the body around an axis called Euler axis or eigenaxis. A relationship between the rigid body rotations about the orbital axes is derived and it will be the base to establish a set of kinematics differential equation of motion.

2.1 Euler Method

The Euler attitude angles represent the most common method to describe the spatial orientation of a rigid body (Wie and Jianbo, 1995). The rigid body axis is inicially aligned with the inertial reference axis; the Euler angles are specified by three successive rotations to bring the inertial reference coordinates into alignment with the rigid body axis; twelve different rotations are possible. In this work the following sequence was used: the first rotation about the z-axis (Ψ), then about the y-axis (θ) and finally rotation about the x-axis (ϕ).

The Euler angles have the following ranges:

$$-\pi \le \phi \le +\pi, \quad \frac{\pi}{2} \le \theta \le +\frac{\pi}{2}, \quad -\pi \le \Psi \le +\pi$$

There is a direct relation between the Euler attitude angles and the angular velocity of the rigid body axis. From this relationship the kinematics differential equation can be derived:

$$\dot{\phi} = p + nSec[\theta]Sin[\psi] + rCos[\phi]Tan[\theta] + qSin[\phi]Tan[\theta]$$
$$\dot{\theta} = qCos[\phi] + nCos[\psi] - rSin[\phi]$$
$$\dot{\psi} = rCos[\phi]Sec[\theta] + qSec[\theta]Sin[\phi] + nSin[\psi]Tan[\theta]$$

The above equation can be rewritten as,

$$p = \dot{\phi} - \dot{\psi}Sin[\theta] - nCos[\theta]Sin[\psi]$$
$$q = \dot{\theta}Cos[\phi] + \dot{\psi}Cos[\theta]Sin[\phi] - n(Cos[\phi]Cos[\psi] + Sin[\theta]Sin[\phi]Sin[\psi])$$
$$r = \dot{\psi}Cos[\theta]Cos[\phi] - \dot{\theta}Sin[\phi] - n(-Cos[\psi]Sin[\phi] + Cos[\phi]Sin[\theta]Sin[\psi])$$

where,

(p, q, r) are the rigid body axis components of the angular velocity of the body with respect to inertial axes, and (ψ, ϕ, θ) are the Euler angles of the body axis with respect to the orbital fixed reference frame that rotates with the orbital angular velocity *n*.

2.2 Direction Cosines Method

The direction cosines method is useful to represent the transformation between two different coordinates. Using the direction cosines, the body axis coordinates can be transformed into to the inertial reference frame and vice versa by a matrixial operation. The direction cosines matrix that represents the transformation from body axis to the inertial reference axis, according to Wertz, 1978, is expressed by:

$$\begin{bmatrix} c[\theta]c[\psi] & c[\theta]s[\psi] & -s[\theta] \\ s[\psi]s[\theta]s[\phi] - c[\phi]s[\psi] & c[\phi]c[\psi] + s[\theta]s[\phi]s[\psi] & c[\theta]s[\phi] \\ s[\phi]c[\psi]s[\theta] + s[\phi]s[\psi] & -c[\psi]s[\phi] + c[\phi]s[\theta]s[\psi] & c[\theta]c[\phi] \end{bmatrix}$$

where,

$$c[\theta] = \cos[\theta] \qquad s[\theta] = \sin[\theta]$$

2.3 Quaternion Method

Nevertheless, the Euler method to be frequently used to describe the kinematics of the rigid body and showing as the biggest advantages in the face of other methods its physical interpretation, this method has a singular point that happen when the pitch angle, θ, goes through the vertical $\pm\frac{\theta}{2}$. This problem was settled when Sir Willian Hamilton developed quaternion algebra in 1843, after long researches on the hyper complex numbers (Tasora and Righttini, 1999). Since then quaternions have been widely used in Dynamics modelling because they can easily represent rotations of reference frames in space as soon as a correspondence between them and the Euler angles. The quaternions are four-dimensional hypercomplex vector, with one real and three imaginary components:

$$q = a\hat{i} + b\hat{j} + c\hat{k} + d\ ,\ q \in \{\Re^1, \Re^3\},$$

were,

$i^2 = j^2 = k^2 = -1,\ ij = k,\ ji = -k$, with cyclic permutation $i \to j \to k \to i$

In the rigid body orientation the first three components indicates the direction of the Euler axis. The scalar part of the quaternions, the fourth component, is related to the rotation angle about the Euler axis. In this work the quaternion will be written in the vectorial form:

$$\hat{q} = \{q_1, q_2, q_3, q_4\} = q[s, v]$$

Using the notation referred above some property of the quaternions shall be showed:

The Euclidian Norm of the quaternion q is defined as $\|q\| = (q_1^2 + q_2^2 + q_3^2 + q_4^2)^{\frac{1}{2}}$, q=1.

The product between two quaternions are given by the following equation

$$q_1 q_2 = (s_1 s_2 - v_1 v_2,\ s_1 v_2 + s_2 v_1 + v_1 \times v_2)$$

The four elements of quaternions are defined as

$$q_i = c_i\, Cos(\frac{\vartheta}{2}), \qquad i = 1,2,3$$

$$q_4 = Cos(\frac{\vartheta}{2})$$

Where ϑ, is the magnitude of the Euler axis rotation, and (c_1, c_2, c_3) are the direction cosines of the Euler axis relative to reference frames.

The direction cosine matrix parametrized in Section 2.2 can also be parametrized in terms of quaternions, as follows:

$$C(q, q_4) = \begin{bmatrix} 1 - 2(q_2^2 + q_3^2) & 2(q_1 q_2 + q_3 q_4) & 2(q_1 q_3 - q_2 q_4) \\ 2(q_2 q_1 - q_3 q_4) & 1 - 2(q_1^2 + q_3^2) & 2(q_2 q_3 + q_1 q_4) \\ 2(q_3 q_1 + q_2 q_4) & 2(q_3 q_2 - q_1 q_4) & 1 - 2(q_1^2 + q_2^2) \end{bmatrix}$$

At this point we are ready to develop the equation of motion for a rigid body satellite that moves in a circular orbit under the gravitational force. The set of kinematics and dynamics equations will be deduced in the next section.

3. Euler's Equation of Motion

The satellite will be considered in a circular orbital with an angular velocity in the orbital reference frame, $\hat{\omega}_{orbital} = -n\hat{b}_2$, and an angular velocity of rotation related to the inertial reference frame given by $\hat{\omega}_{body} = p\hat{n}_1 + q\hat{n}_2 + r\hat{n}_3$. According to Wie and Jianbo (1995), the vector body angular rate referring to orbit coordinates is given by:

$$\hat{\omega}_{body}^{orbital} = \hat{\omega}_{body} + C(q, q_4)\hat{\omega}_{orbital}\,. \qquad (1)$$

Then, the kinematics equation of motion can be described as follows,

$$\dot{q} = \frac{1}{2}(q_4 \omega_{inertial} - \omega_{inertial} \times q) \qquad (2)$$

$$\dot{q}_4 = \frac{-1}{2}(\omega_{inertial} \times q) \qquad (3)$$

In order to derive the dynamic equation of motion the vector position from the Earth's center and the rigid body $R_{orbital} = -\hat{b}_3$ must be also transformed into the inertial reference frame as showed below,

$$R_{inertial} = C(q, q_4) R_{orbital} . \quad (4)$$

The gravity gradient torque $\Gamma_{gravity}$ can be expressed by the following approximation, (Wie and Jianbo, 1995):

$$\Gamma_{gravity} = 3n^2.(R_{inertial} \times (J.R_{inertial})) \quad (5)$$

were,

J is the satellite inertial matrix about a body-fixed reference frame at the center of mass and n is the orbital rate.

The orientation of the spacecraft is accomplished using three reaction wheels that are aligned along three body-fixed control axes. The total angular momentum of reaction wheels and the angular moment of the spacecraft are given by,

$$H_{wheels} = J_a.\dot{\omega}_{body} + \hat{u} \quad (6)$$
$$H_{craft} = J.\dot{\omega}_{body} \quad (7)$$
$$H_{total} = H_{wheels} + H_{craft} \quad (8)$$

were,

J_a is the inertial matrix of the gyrostat spacecraft about a body-fixed reference frame, which is assumed to be aligned with principal axis of the gyrostat and $\hat{u}$ is control input vector.

The rotational motions of a rigid spacecraft are described by Euler's equations:

$$H_{inertial} = \frac{dH_{total}}{dt} + \omega_{body} \times H_{total} - \Gamma_{gravity} . \quad (9)$$

Substituting Eqs. (5) and (8) into Eq. (9), we obtain the dynamics equation of motion of a satellite in a circular orbit, as follows, for roll, pitch, and yaw, respectively:

$$\dot{p} = \frac{-1}{J_1 + J_a}(-6(J_2 - J_3)n^2(-1 + 2q_1^2 + 2q_2^2)(q_2 q_3 + q_1 q_4) + u_1 + h_3 q - h_2 r - (J_2 - J_3)qr) \quad (10a)$$

$$\dot{q} = \frac{-1}{J_2 + J_a}(6(J_1 - J_3)n^2(-1 + 2q_1^2 + 2q_2^2)(q_1 q_3 - q_2 q_4) + u_2 - h_3 p + h_1 r + (J_1 - J_3)pr) \quad (10b)$$

$$\dot{r} = \frac{-1}{J_3 + J_a}(-12(J_1 - J_2)n^2(q_2 q_3 + q_1 q_4)(-q_1 q_3 + q_2 q_4) + u_3 + h_2 p - h_1 q - (J_1 - J_2)pq) \quad (10c)$$

4. Control Strategy

The control strategy used in this work is based in feedback controller with saturation control and slew rate constraints, (Wie and Jianbo, 1995). Let a spacecraft performing a reorientation maneuver. This maneuver shall be performed as soon as possible but within the saturation limits of the reaction wheels. In other words,

$|u| \leq N_{\max}$, where $N_{\max}$ is the maximum torque available to reaction wheels.

The globally stable quaternion feedback control law of Wie and Jianbo (1995) was modified to be implemented not as initially referenced but as an orbit referenced pointing control law which is showed below,

$$u = KQ_{erro} + C\omega_{body}^{orbita} - \omega_{body} \otimes (J.\omega_{body} + H_{wheels}) + \Gamma_{gravity} + d_i(t) \quad (11)$$

where,

K=kJ, C=cJ *Positive scalar diagonal matrix.*

$d_i(t)$ External disturbance

$Q_{erro}=[q_{1erro}, q_{2erro}, q_{3erro}]$ *Vector part of the error quaternion and* $\otimes$ denote the vector cross product.

The error quaternion matrix will be the difference between the commanded quaternion q_c and the current quaternion q_i .

$$\begin{bmatrix} q_{1e} \\ q_{2e} \\ q_{3e} \\ q_{4e} \end{bmatrix} = \begin{bmatrix} q_{4c} & q_{3c} & -q_{2c} & -q_{1c} \\ -q_{3c} & q_{4c} & q_{1c} & -q_{2c} \\ q_{2c} & -q_{1c} & q_{4c} & -q_{3c} \\ q_{1c} & q_{2c} & q_{3c} & q_{4c} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

To perform a reorientation eigenaxis maneuver with slew constraint a saturation controller was developed in order to provide torque saturation of reaction wheels but which still ensure the eigenaxis rotation under specific slew rate constraint. Using the Cascade-Saturation Control method described in Wie (1998), the Eq. (11) can be rewritten as follows,

$$u = Ksat(PQ_{erro}) + C\omega_{body}^{orbital} - \omega_{body} \otimes (J.\omega_{body} + H_{wheels}) + \Gamma_{gravity} + d_i(t) \quad (12)$$

where,

$P=diag(p_1,p_2,p_3)$,
C=cJ

And furthermore we chose

$k_i = \dfrac{c|q_i(0)|}{\|q(0)\|}\dot{\theta}_{\max}$, $\dot{\theta}_{\max}$ is the maximum slew rate about the eigenaxis.

$KP = kJ$

Where $k \cong k_i \, p_i$ is a positive scalar constant; then we have the following results for all $t \in [0, t^*]$:

a. The close-loop satellite attitude dynamic is obtained substituting the Eq. (12) into Eq. (9) which the result is an eigenaxis rotation about *q(0).*
b. The actual slew rate about the eigenaxis is bounded as $\|\omega_{body}(t)\| \le \dot{\theta}_{\max}$ and increases monotonically.
c. The attitude error $\|q_{erro}(t)\|$ decreases monotonically.
d. At time t^*, we have $|p_i \, q_i \, (t^*) = 1|$ for all *i*=1,2,3.

The results described above are deduced from the Lemma 7.3 showed in Wie (1998).

Using the results showed above it was developed an algorithm in order to obtain the saturation control logic.

Let the normalized reaction wheels vector torque and the error quaternion vector defined, respectively as,

$$N_{wheels}=diag(1/N_{1max},\ 1/N_{2max},\ 1/N_{3max}) \qquad (13)$$

$$erro = Sat(PQ_{erro}) = \begin{cases} PQ_{erro} & if \quad \|PQ_{erro}\| < 1 \\ \dfrac{PQ_{erro}}{\|PQ_{erro}\|} & if \quad \|PQ_{erro}\| \geq 1 \end{cases} \qquad (14)$$

Substituting the Eq. (14) into to Eqs. (12) we have,

$$u = Kerro + C\omega_{bady}^{orbital} - \omega_{body} \otimes (J.\omega_{body} + H_{wheels}) + \Gamma_{gravity} + d_i(t) \qquad (15)$$

The control input vector acting in the spacecraft, becomes,

$$u_c = \begin{cases} u & if \quad \|N_{weels}u\| \leq 1 \\ \dfrac{u}{\|N_{weels}u\|} & if \quad \|N_{weels}u\| > 1 \end{cases} \qquad (16)$$

Verify in the Eq. (16) that the control input vector is function of the saturated torque vector.

5. Stability Analysis

The closed-loop attitude dynamics of a rigid spacecraft employing the saturation control logic of Eq. (15) are then described by:

$$J\dot{\omega}_{body} = -KSat(PQ_{erro}) - C\omega_{body}^{orbital} \qquad (17)$$

where the kinematics equations are described by Eqs. (2) and (3).

The asymptotical stability of the closed-loop system can be verified using a positive definite function of the form

$$V = (\frac{1}{K})V_\omega + V_q \qquad (18)$$

where V_ω, V_q are the *quaternions Lyapunov function* and *angular velocity Lyapunov function*, respectively. The Lyapunov functions are defined as follows:

$$V_q = q_1^2 + q_2^2 + q_3^2 + (1 - q_4)^2 \qquad (19)$$

$$V_\omega = \frac{1}{2}(p^2 + q^2 + r^2) \qquad (20)$$

The time derivative of *V* along the closed-loop trajectory by Eq.(18) becomes

$$\dot{V} = -(\frac{c}{k})\omega^T \omega < 0 \qquad (21)$$

This means that the closed-loop system is asymptotically stable, and *q(t)* and $\omega(t)$ will become zero.

At $t = t^*$, $q(t^*)$ and $\omega(t^*)$ lie along the vector *q(0)*, for $t \geq t^*$, the derivative of Lyapunov functions are given by

$$\dot{V}_q = q^T \omega \tag{22}$$

$$\dot{V}_\omega = -c\omega^T \omega \ -kq^T \omega \tag{23}$$

if $\dot{V}_q < 0$, then it is possible to have $\dot{V}_\omega > 0$, it is also possible that the rate limitation be violated. This situation can be avoided by properly choosing *c,* as shown below.

If the angular velocity vector $\omega(t)$ lies along the direction of $q(0)$, *i.e.*,

$$\omega(t) = a(t)q(0) \tag{24}$$

where *a(t)* is a scalar function with *a(0)=0*, then *q(t)* of Eqs. (2) and (3) will remain along the same direction of *q(0)*; the resulting motion is an eigenaxis rotation about *q(0)*, Theorem 7.2, Wie (1998).

Since $\omega(t)$ lies along *q(0)*, this implies that *q(t)* also lies entirely along *q(0)*, so there is a scalar function *g(t)* such that

$$q(t) = g(t)q(0) \quad t > 0 \tag{25}$$

Substituting the maximum slew rate, $\dot{\theta}_{\max}$, into the Eq. (17), we obtain

$$\dot{\omega} = \frac{-c\dot{\theta}_{\max}}{\|q(0)\|} q(0) - c\omega \tag{26}$$

the solution of Eq. (26) can be written as

$$\omega(t) = -f(t)q(0) \tag{27}$$

where

$$f(t) = \frac{(1-e^{-ct})\dot{\theta}_{\max}}{\|q(0)\|} > 0 \tag{28}$$

Substituting Eqs. (25) and (27) into the quaternions kinematics differential equation, we obtain

$$\dot{g}(t) = \frac{-1}{2} q_4 f(t)\,. \tag{29}$$

Furthermore exists a time instant t^* satisfying

$$\left| g(t^*) = \frac{c\dot{\theta}_{\max}}{k\|q(0)\|} \right|. \tag{30}$$

Using the results described in Theorem 7.2 the kinematics differential equation can be rewritten as a second order equation since the maneuver is an eigenaxis rotation.

$$\ddot{\theta} + c\dot{\theta} + ks(\frac{\theta}{2}) = 0 \qquad t \geq t^* \tag{31}$$

with the following initial condition

$$\theta(t^*) = 2s^{-1}[g(t^*)] \tag{32}$$

where θ is the rotation angle about the eigenaxis. Assuming that θ is small the Eq. (31) becomes

$$\ddot{\theta} + c\dot{\theta} + k\frac{\theta}{2} = 0 \qquad t \geq t^* \tag{33}$$

Then we can properly choose *k* and *c* as follows:

$$k = 2\omega_n^2$$
$$c = 2\zeta\omega_n$$

where ζ and ω_n, are the damping ration and the natural frequency of a second-order dynamic system. It is clear that the slew rate will not exceed $\dot{\theta}_{\max}$ for all $t \in [0,\infty)$ if we do not choose a small ζ.

6. Numerical Results

In this section we presented an example with numerical simulation results in order to demonstrate the performance of the saturation control logic applied for a satellite, rigid body, in a circular orbital. This controller was developed to ensure a reorientation eigenaxis maneuver with a specific maximum wheel torque and a maximum slew rate. The initial and final satellite orientation expressed in quaternion (sequence x-y-z) are set as q(0)=[0.3919,0.2006,0.5320,0.7233]^T, $\omega(0)$=[0.05,0.03,0.05] deg/s and $q(t_f)$=[0,0,0,1]^T . The external disturbances are represented by a White Gaussian Noise *(wgn)* injected by $d_i(t) = wgn\ (N.m),\ i = 1,2,3..$

The reaction wheels configuration and all satellite parameters used in this simulation are showed in Table 1.

Table 1 – Reaction Wheels and Satellite Parameters

Satellite inertial matrix (*J*)	[1.4286 1 1.0476]Kg.m^2
Gyrostat inertial matrix (J_a)	[0.1 0.1 0.1] Kg.m^2
Maximum reaction wheels torque (N_{wheels})	[0.3 0.25 0.20] N.m
Maximum slew rate ($\dot{\theta}_{\max}$)	0.2 deg/s
Orbital Rate (*n*)	0.01 deg/s

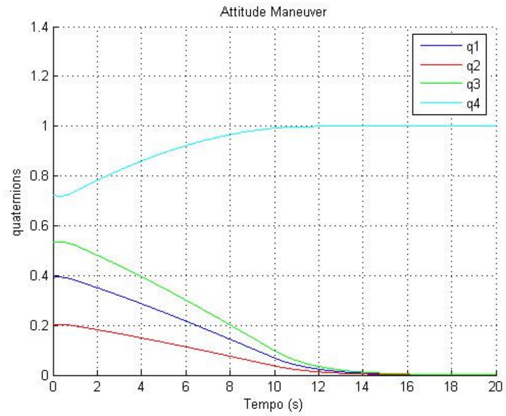

Figure 1 – Time histories of quaternions attitude tracking.

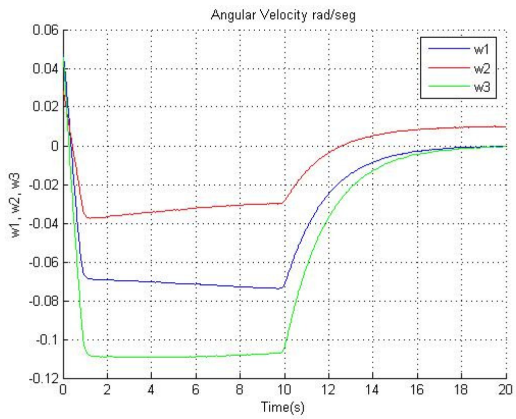

Figure 2 – Time histories of angular velocities.

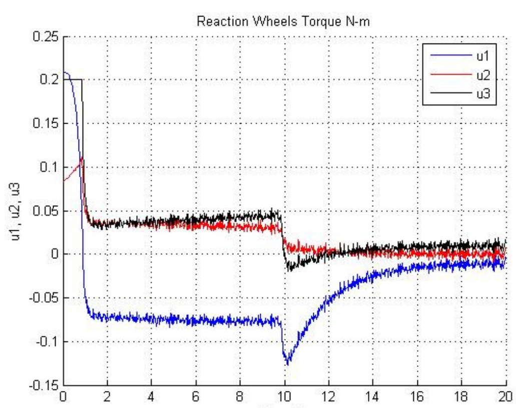

Figure 3 – Reaction wheels control torque histories.

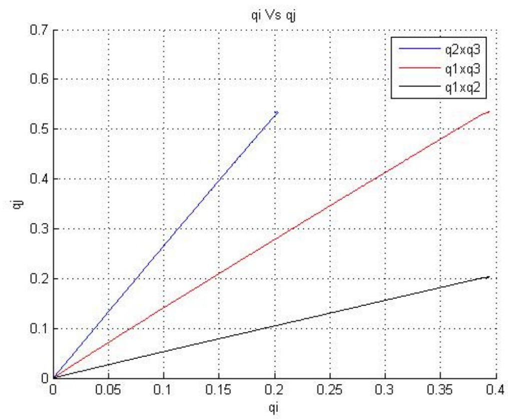

Figure 4 – Time histories of $q_i x q_j$.

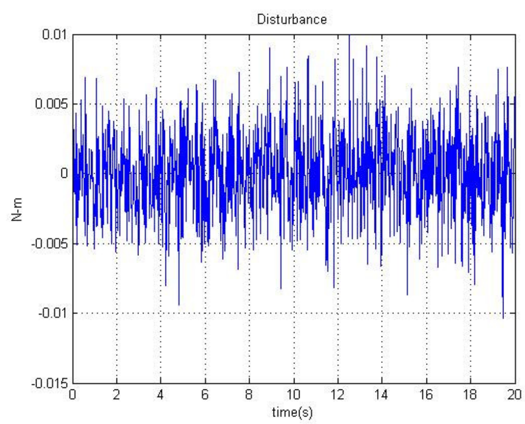

Figure 5 – Time histories of external disturbances.

Figure 1 shows that the reorientation about the eigenaxis maneuver was successfully performed. Figures 2 and 3 demonstrate the satisfaction of constraints of the control variables, where the reaction wheels control torque and the wheel speed are shown within the design limits developed in Section 4.0. It should be pointed out that the component of the wheel speed around the y axis reaches the orbital rate in steady state phase. The $q_i x q_j$ plots in Figure 4 clearly indicate the speed deviation of the instantaneous rotational axis with respect to the initial eigenaxis. The perfect eigenaxis rotation becomes a straight line in the $q_i x q_j$ plot. Figure 5 illustrates the external disturbance injected in the system in order to check the performance of the controller projected.

7. Conclusion

A useful quaternion feedback regulator with control and slew rate constraints has been developed and successfully applied to a satellite in a circular orbital. The proposal non-linear feedback control law showed effectiveness upon actuator and sensor saturation limits. Future works will extend the quaternion feedback regulator with slew rate and control constraints into a set of hybrid state equation of motion for a flexible satellite.

8. References

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