Functional behaviour of scaling velocities for turbulent separating flows

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Abstract. *The present work investigates the asymptotic structure of the turbulent boundary layer near a separation point. Using some very detailed numerical flow simulations with turbulent low Reynolds number models of the eddy viscosity and Reynolds stress types, a proposed functional relation for the prediction of the relevant velocity scales is tested. The numerical simulations are thoroughly validated through experimental data obtained with the help of a LDV system. In addition, the singular behaviour of the near separation solution is compared with the theory of other authors.*

Keywords: *turbulence, boundary layer, separation.*

1. Introduction

Near to a separation point, in a flow with a prescribed adverse pressure gradient, the asymptotic structure of the turbulent boundary layer is known to exhibit characteristics that are completely strange to its classical two-layered diagram. A point of major relevance is the total extinction of the logarithmic velocity profile, which must be replaced by a square root law. As expected, this change in the functional representation of the velocity profile is provoked by physical effects that also lead to a change in the representative scaling parameters of the problem.

The flow behaviour just described thus establishes important guidelines to which any rational theory should conform. Far away from the separation point, a classical structure based on asymptotically distinguished regions determined by Prandtl's law of the wall and by Coles' law of the wake must be obtained. In this case, the wall shear stress plays an important role in the definition of the local scaling parameters. Near to the separation point, however, the local scaling parameters are known to depend on the local pressure gradient so that the flow structure must be dominated by a Stratford type of solution

The purpose of this work is to show how a systematic application of Kaplun limits to the Reynolds averaged Navier-Stokes equations naturally uncovers the asymptotic structure of separating turbulent flows, explaining in simple terms how the two above asymptotic structures can be reconciled into a single theoretical framework. Cruz and Silva Freire (1998, 2002) have shown that appropriate limit processes can be used to split the flow domain into regions of "principal limits" and regions of "intermediate limits", yielding correct asymptotic structures that are valid far away as well as close to a separation point. Principal limits are crucial in high-Re theory for they can be shown to be uniform in overlapping intervals. In addition, associated equations obtained through passage of principal limit processes are expected to be satisfied by the corresponding limits of the exact solution. This underlying notion was introduced by Kaplun (1967) with the express purpose of relating the domain of validity of solutions with the formal domain of validity of equations. In fact, the basis for Kaplun's arguments is an unprovable principle. However, in his theory some formal properties of differential equations can be precisely defined to make plausible a discussion on such sophisticated concepts as limit processes, formal validity of equations, principal equations, intermediate equations, asymptotic expansions, domain of validity of such expansions, overlap and matching. The use of heuristic arguments is not peculiar in perturbation methods. The implication is that results must often be subjected to *a posteriori* verification.

The present work will, in particular, show how all results of Cruz and Silva Freire (1998, 2002) can be given justification through the experiments of Loureiro et al. (2007) for the flow over a steep hill. Because laboratory measurements are necessarily confined to a sparse set of measuring stations, data interpolation may lead to a spurious behaviour of the flow properties. To overcome this difficulty, numerical simulations of the flow under interest were carried out using the SST-model (Shear Stress Transport) (Menter, 1994). The sets of results furnished by the numerical simulations and validated by the experiments were then used to validate the theoretical results.

The theory of Kaplun cannot in principle explain the changes in scaling parameters; hence, it must be supplemented by a relation that can explain in physical terms the correct limiting behaviour of the flow relevant scales. Thus, a new algebraic equation needs to be advanced to specify the relevant scales of the flow. Here, we follow the procedure of Cruz and Silva Freire (1998, 2002) and specify two algebraic equations to describe the velocity and the length reference scales. These equations use some order of magnitude arguments to establish a balance between pressure and internal friction forces in the inner regions of the flow. Solutions of these equations are then used to determine the local reference scales.

2. Preamble: Relevant Scales for Separating Flows

Some remarks concerning the changes in scaling laws will now be made.

The two-layered model established by Prandtl (1925) and v. Kármán (1930) considers that across the wall layer the total shear stress deviates just slightly from the wall shear stress. Therefore, in the turbulence dominated flow region we may write

$$\partial_y \tau_t = \partial_y(-\rho \overline{u'v'}) = 0. \tag{1}$$

A simple integration of the above equation gives $ord(u') = ord(v') = ord(u_\tau)$, where we have clearly considered the velocity fluctuations to be of the same order and $u_\tau = \sqrt{\tau_w/\rho}$.

The analysis may proceed by taking as a closure assumption the mixing-length theory. Further equation integration yields the classical law of the wall for a smooth surface

$$u^+ = \varkappa^{-1} \ln y^+ + A, \qquad u^+ = u/u_\tau, \qquad y^+ = y/(\nu/u_\tau), \tag{2}$$

with $\varkappa = 0.4$, $A = 5.0$.

The essential description of the physics of a turbulent flow near to a separation point has been given by Stratford (1959). In an earlier work, Goldstein (1948) tackled the separation phenomenon for a laminar flow.

The action of an arbitrary pressure rise in the inner layer distorts the velocity profile implying that the gradient of shear stress must now be balanced by the pressure gradient. Therefore, Eq. (1) has to be specialized to

$$\partial_y \tau_t = \partial_x p. \tag{3}$$

Two successive integrations of Eq. (3) together with the mixing length hypothesis and the fact that at a separation point $\tau_w = 0$, give

$$u^+ = (2\,\varkappa^{-1})\, y^{+^{1/2}}, \tag{4}$$

with

$$u^+ = u/u_{p\nu}, \qquad y^+ = y/(\nu/u_{p\nu}), \qquad u_{p\nu} = ((\nu/\rho)\partial_x p)^{1/3}. \tag{5}$$

Thus, at a separation point, $ord(u') = ord(v') = ord(u_{p\nu})$.

The relevant velocities and length scales for flows away and close to a separation point are then (u_τ, ν/u_τ) and ($u_{p\nu}$, $\nu/u_{p\nu}$) respectively.

The noticeable result is that both relevant velocity scales – u_τ and $u_{p\nu}$ – are contained in

$$-\overline{u'v'} - (\rho^{-1}\tau_w) - (\rho^{-1}\partial_x p)\, y = 0. \tag{6}$$

In the limiting cases $\tau_w >> (y/\rho)(\partial_x p)$ and $\tau_w << (y/\rho)(\partial_x p)$, the scaling velocity tends to u_τ and $((\nu/\rho)\partial_x p)^{1/3}$ respectively, where $(\partial_x p)$ is to be considered at the wall.

To propose a characteristic velocity that is valid for the whole domain, Cruz and Silva Freire (1998, 2002) suggested to reduce Eq. (6) to an algebraic equation by considering $ord(u') = ord(v') = ord(u_R)$ and $ord(y) = ord(\nu/u_R)$. Thus, the reference velocity, u_R, is to be determined from the highest real root of Eq. (7):

$$u_R^3 - (\rho^{-1}\tau_w)\, u_R - (\rho^{-1}\nu)\partial_x p = 0, \tag{7}$$

that is,

$$u_R = \sqrt[3]{2}\, (\tau_w/\rho)\Delta_{u_R}^{-1} + (3\,\sqrt[3]{2})^{-1}\, \Delta_{u_R}, \tag{8}$$

with

$$\Delta_{u_R} = \left(3^3\, ((\nu/\rho)(\partial_x p)_w) + \sqrt{-2^2 3^3\, (\tau_w/\rho)^3 + 3^6\, ((\nu/\rho)(\partial_x p)_w)^2}\right)^{1/3}. \tag{9}$$

Note that in the very near wall region, where the viscous effects are dominant, the local governing equation is

$$\nu \partial_{yy} u = \rho^{-1} \partial_x p. \tag{10}$$

Two successive integrations of Eq. (10) and the fact that $\tau_w = 0$, give

$$u^+ = (1/2)\, y^{+^2}, \tag{11}$$

the solution of Goldstein (1948).

3. The Method of Kaplun, Limits of Equations, Principal limits

Let us consider the problem of an incompressible turbulent flow over a smooth surface in a prescribed pressure distribution. The time-averaged equations of motion – the continuity equation and the Reynolds equation – can be cast as

$$\partial_i u_i = 0, \tag{12}$$

$$u_j\,\partial_j u_i = -\partial_i p - \epsilon^2 \partial_j \left(\overline{u_j' u_i'}\right) + R^{-1}\partial^2 u_i, \tag{13}$$

where the notation is classical. Thus, in a two-dimensional flow, $(x_1, x_2) = (x, y)$ stands for a Cartesian co-ordinate system, $(u_1, u_2) = (u, v)$ for the velocities, p for pressure and $R\ (= u_e l/\nu)$ for the Reynolds number. The dashes are used to indicate a fluctuating quantity. In the fluctuation term, an overbar is used to indicate a time-average.

All mean variables are referred to the free-stream mean velocity, u_e, and to the characteristic length $l = (\rho u_e^2/(\partial_x p)_w)(w$ = wall condition). The velocity fluctuations, on the other hand, are referred to the characteristic velocity u_R defined by Eq. (7) so that $\epsilon = u_R/u_e$.

The purpose of perturbation methods is to find approximate solutions to Eqs. (12) and (13) that are valid when one or more of the variables or parameters in the problem are small or large. Provided the small parameters are taken to be ϵ and R^{-1} the classical results of perturbation theory can be used.

The present account on perturbation methods is based on the results of Kaplun (1967), Lagerstrom and Casten (1972) and Lagerstrom (1988). In the following, we use the topology on the collection of order classes as introduced by Meyer (1967). For positive, continuous functions of a single variable ϵ defined on $(0, 1]$, let $ord\,\eta$ denote the class of equivalence introduced in Meyer (1967).

The essential idea of the single limit process η-limit is to study the limit as $\epsilon \to 0$ not for fixed x near a singularity point x_d, but for x tending to x_d in a definite relationship to ϵ specified by a function $\eta(\epsilon)$. Taking without any loss of generality $x_d = 0$, we define

$$x_\eta = x/\eta(\epsilon), \quad G(x_\eta;\epsilon) = F(x;\epsilon), \tag{14}$$

with $\eta(\epsilon)$ a function defined in Ξ (= space of all positive continuous functions on (0,1]).

Definition (Kaplun limit)(Meyer, 1967). If the function $G(x_\eta; +0) = \lim G(x_\eta;\epsilon)$, $\epsilon \to 0$, exists uniformly on $\{x_\eta/|x_\eta| > 0\}$; then we define $lim_\eta\ F(x;\epsilon) = G(x_\eta; +0)$.

Thus, if $\eta \to 0$ as $\epsilon \to 0$, then, in the limit process, $x \to 0$ also with the same rate of η, so that x/η tends to a non-zero limit value.

The investigate the asymptotic structure of the turbulent boundary layer we consider

$$u(x,y) = u_1(x,y) + \epsilon u_2(x,y), \qquad v(x,y) = \eta v_1(x,y), \qquad p(x,y) = p_1(x,y), \tag{15}$$

and the following transformation

$$\hat{y} = y_\eta = y/\eta(\epsilon), \quad \hat{u}_i(x, y_\eta) = u_i(x,y)., \tag{16}$$

with $\eta(\epsilon)$ defined on Ξ.

Upon substitution of Eqs. (15) and (16) into Eqs.(12) to (13) and depending on the order class of η we then find the following formal limits:
continuity equation:

$$\mathrm{ord}(\hat{v}_i(x,y_\eta)) = \mathrm{ord}(\eta\hat{u}_i(x,y_\eta)). \tag{17}$$

x-momentum equation:

$$\mathrm{ord}\,\eta = \mathrm{ord}\,1: \qquad \hat{u}_1\,\partial_x\hat{u}_1 + \hat{v}_1\,\partial_{y_\eta}\hat{u}_1 + \partial_x\hat{p}_1 = 0 \tag{18}$$

$$\mathrm{ord}\,\epsilon^2 < \mathrm{ord}\,\eta < \mathrm{ord}\,1: \qquad \hat{u}_1\,\partial_x\hat{u}_1 + \hat{v}_1\,\partial_{y_\eta}\hat{u}_1 + \partial_x\hat{p}_1 = 0 \tag{19}$$

$$\mathrm{ord}\,\epsilon^2 = \mathrm{ord}\,\eta: \qquad \hat{u}_1\,\partial_x\hat{u}_1 + \hat{v}_1\,\partial_{y_\eta}\hat{u}_1 + \partial_x\hat{p}_1 = -\partial_{y_\eta}\overline{u_1' v_1'} \tag{20}$$

$$\mathrm{ord}\,(1/\epsilon^2 R) < \mathrm{ord}\,\eta < \mathrm{ord}\,\epsilon^2 = \mathrm{ord}\,\eta: \qquad \partial_{y_\eta}\overline{u_1' v_1'} = 0 \tag{21}$$

$$\mathrm{ord}\,(1/\epsilon^2 R) = \mathrm{ord}\,\eta: \qquad -\partial_{y_\eta}\overline{u_1' v_1'} + \partial^2_{y_\eta}\hat{u}_1 = 0 \tag{22}$$

$$\mathrm{ord}\,\eta < \mathrm{ord}\,(1/\epsilon^2 R): \qquad \partial^2_{y_\eta}\hat{u}_1 = 0. \tag{23}$$

y-momentum equation:

$$\text{ord}\,\eta = \text{ord}\,1: \qquad \hat{u}_1\,\partial_x\hat{v}_1 + \hat{v}_1\,\partial_{y_\eta}\hat{v}_1 + \partial_{y_\eta}\hat{p}_1 = 0 \tag{24}$$

$$\text{ord}\,\eta < \text{ord}\,1: \qquad \partial_{y_\eta}\hat{p}_1 = 0. \tag{25}$$

The set of Eqs. (17) to (25) induces to every order of η a correspondence, $lim_\eta \to$ associated equation, on that subset of Ξ for which the associated equation exists.

Definition. The formal local domain of an associated equation E is the set of orders η such that the η-limit process applied to the original equation yields E.

Given any two associated equations E_1 and E_2, Kaplun defines

$$\mathbb{R}(x_\eta;\epsilon) = E_1(x_\eta;\epsilon) - E_2(x_\eta;\epsilon), \tag{26}$$

where ϵ denotes a small parameter.

Definition (of equivalence in the limit) (Kaplun, 1967). Two equations E_1 and E_2 are said to be *equivalent in the limit* for a given limit-process, lim_η, and to a given order, $\delta(\epsilon)$, if

$$\mathbb{R}(x_\eta;\epsilon)/\delta(\epsilon) \to 0,\ as\ \epsilon \to 0,\ x_\eta \text{ fixed.} \tag{27}$$

The following definitions are now possible.

Definition (of formal domain of validity). The formal domain of validity to order δ of an equation E of formal local domain D is the set $D_e = D \cup D_i's$, where $D_i's$ are the formal local domains of all equations E_i' such that E and E_i' are equivalent in D_i' to order δ.

Definition (of principal equation). An equation E of formal local domain D, is said to be principal to order δ if:
i) one can find another equation E', of formal local domain D', such that E and E' are equivalent in D' to order δ;
ii) E is not equivalent to order δ to any other equation in D.

An equation which is not principal is said to be intermediate.

The intermediate equation, Eq. (23), together with the boundary condition $\hat{u}_1(x,0) = 0$, imply that the near wall solution is $\hat{u}_1(x,y_\eta) = y_\eta$. This solution has to be contained by the principal solution furnished by Eq. (22). The outer flow equations, on the other hand, imply that $\hat{u}_1(x,y_\eta) = u_e(x,y)$. Thus, we appear to be faced by a dilemma for the inner solution is unbounded in the limit $y_\eta \to \infty$ and hence no matching can be achieved with the bounded outer solution. In fact, the matching process that involves the inner and outer solutions is to be performed in a region dominated by Eq. (21). As it turns out, Eq. (21) yields a solution with a limiting logarithmic behaviour that bridges an inner solution of order ϵ to the outer solution of order unity through the relationship $\epsilon = \text{ord}(\ln^{-1} R)$. This problem has been investigated by many authors (see, e.g., Izakson (1937), Millikan (1939), Yajnik (1970), Tennekes (1973), Afzal (1976)) and is sometimes called a 'generation gap' (Mellor, 1972). An important additional implication is the deduction of an algebraic relationship that can be used for the prediction of the local skin-friction.

Since the leading order solution in the inner regions of the flow is ord(ϵ), it follows that $(1/\epsilon^2 R)$ has to be replaced by $(1/\epsilon R)$ in Eqs. (21) to (23) so that the inner region principal equation is given by

$$\text{ord}\,(1/\epsilon R) = \text{ord}\,\eta: \qquad -\partial_{y_\eta}\overline{u_1'v_1'} + \partial^2_{y_\eta}\hat{u}_1 = 0. \tag{28}$$

Therefore, the principal equations to the turbulent boundary layer problem are Eqs. (20), (28) and (24). The relevant scales ϵ^2 and $1/\epsilon R$ coincide with the scales proposed by Sychev and Sychev (1987) for the description of their two internal layers.

To relate the formal properties of equations described above to the actual problem of determining the uniform domain of validity of solutions, Kaplun(1967) advanced two assertions, the Axiom of Existence and the Ansatz about domains of validity. These assertions constitute primitive and unverifiable assumptions of perturbation theory.

Axiom (of existence) (Kaplun, 1967). If equations E and E' are equivalent in the limit to the order δ for a certain region, then given a solution S of E which lies in the region of equivalence of E and E', there exists a solution S' of E' such that as $\epsilon \to 0$, $|S - S'|/\delta \to 0$, in the region of equivalence of E and E'.

To the axiom of existence there corresponds an Ansatz; namely that there exists a solution S of E which lies in the region of equivalence of E and E'. More explicitly, we write.

Ansatz (about domains of validity) (Kaplun, 1967). An equation with a given formal domain of validity D has a solution whose actual domain of validity corresponds to D.

The overlap domain of Eqs. (20) and (28) can now be determined through $\mathbb{R}(x_\eta;\epsilon)$ by taking $\delta(\epsilon) = \epsilon^\alpha$. Then upon substituting E_1 by Eq. (20), E_2 by Eq. (28) and passing the limit as ϵ tends to zero, one finds

$$D_{overlap} = \left\{\text{ord}(\epsilon^{1+\alpha}R)^{-1} < \text{ord}\eta < \text{ord}(\epsilon^{2+\alpha})\right\}. \tag{29}$$

The two principal equations then provide approximate solutions that are accurate to order($\epsilon^{\alpha_{max}}$) where

$$\alpha_{max} = -(1/2)\left((\ln R/\ln\epsilon) + 3\right). \tag{30}$$

As the flow approaches a separation point, however, we have already seen that the structure depicted by Eqs. (17) and (25) breaks down. To account for the flow behaviour, we must consider Kaplun limits in x-direction.

Let us define

$$\hat{x} = x_\Delta = x/\Delta(\epsilon), \qquad \hat{y} = y_\eta = y/\eta(\epsilon), \quad \hat{u}_i(x_\Delta, y_\eta) = u_i(x,y), \tag{31}$$

with $\Delta(\epsilon)$ and $\eta(\epsilon)$ defined on Ξ.

The idea is to approach the separation point by taking simultaneously the η- and Δ-limits at a fixed rate $\zeta = \Delta/\eta$ = ord (1). Note that under this condition, Eq. (6) (or even Eq. (7)) implies that ord (ϵ^2) = ord ($1/\epsilon R$).

The resulting flow structure is given by
continuity equation:

$$\text{ord}(\hat{v}_i(x, y_\eta)) = \text{ord}(\hat{u}_i(x, y_\eta)). \tag{32}$$

x-momentum equation:

$$\text{ord}\,\Delta = \text{ord}\,1: \qquad \hat{u}_1\,\partial_{x_\Delta}\hat{u}_1 + \hat{v}_1\,\partial_{y_\eta}\hat{u}_1 + \partial_{x_\Delta}\hat{p}_1 = 0 \tag{33}$$

$$\text{ord}\,\epsilon^2 < \text{ord}\,\Delta < \text{ord}\,1: \qquad \hat{u}_1\,\partial_{x_\Delta}\hat{u}_1 + \hat{v}_1\,\partial_{y_\eta}\hat{u}_1 + \partial_{x_\Delta}\hat{p}_1 = 0 \tag{34}$$

$$\text{ord}\,\epsilon^2 = \text{ord}\,\Delta: \qquad \hat{u}_1\,\partial_{x_\Delta}\hat{u}_1 + \hat{v}_1\,\partial_{y_\eta}\hat{u}_1 + \partial_{x_\Delta}\hat{p}_1 = -\partial_{x_\Delta}\overline{u_1'^2} - \partial_{y_\eta}\overline{u_1'v_1'} + \partial^2_{x_\Delta}\hat{u}_1 + \partial^2_{y_\eta}\hat{u}_1 \tag{35}$$

$$\text{ord}\,\Delta < \text{ord}\,\epsilon^2: \qquad \partial^2_{x_\Delta}\hat{u}_1 + \partial^2_{y_\eta}\hat{u}_1 = 0. \tag{36}$$

y-momentum equation:

$$\text{ord}\,\Delta = \text{ord}\,1: \qquad \hat{u}_1\,\partial_{x_\Delta}\hat{v}_1 + \hat{v}_1\,\partial_{y_\eta}\hat{v}_1 + \partial_{y_\eta}\hat{p}_1 = 0 \tag{37}$$

$$\text{ord}\,\epsilon^2 < \text{ord}\,\Delta < \text{ord}\,1: \qquad \hat{u}_1\,\partial_{x_\Delta}\hat{v}_1 + \hat{v}_1\,\partial_{y_\eta}\hat{v}_1 + \partial_{y_\eta}\hat{p}_1 = 0 \tag{38}$$

$$\text{ord}\,\epsilon^2 = \text{ord}\,\Delta: \qquad \hat{u}_1\,\partial_{x_\Delta}\hat{v}_1 + \hat{v}_1\,\partial_{y_\eta}\hat{v}_1 + \partial_{y_\eta}\hat{p}_1 = -\partial_{x_\Delta}\overline{u_1'v_1'} - \partial_{y_\eta}\overline{u_1'^2} + \partial^2_{x_\Delta}\hat{v}_1 + \partial^2_{y_\eta}\hat{v}_1 \tag{39}$$

$$\text{ord}\,\Delta < \text{ord}\,\epsilon^2: \qquad \partial^2_{x_\Delta}\hat{v}_1 + \partial^2_{y_\eta}\hat{v}_1 = 0. \tag{40}$$

The principal equations are Eqs. (35) and (39). They show that near to a separation point the two principal equations, Eqs. (20) and (28), merge giving rise to a new structure dominated basically by two regions: a wake region (ord(η), ord(Δ)> ϵ^2) and a viscous region (ord(η), ord(Δ) < ϵ^2). The system of Eqs (33) to (40) indicates that the pressure gradient effects become leading order effects for orders higher than ord(ϵ^2) = ord(Δ). Thus, at about ord(x) = ord(Δ) = ord(ϵ^2), that is ord($(\nu/u_{p\nu})/l$) = ord($(u_{p\nu}/u_e)^2$), we should have ord(u_τ) = ord($u_{p\nu}$).

4. Near Wall Solution

The proposition of a near wall solution that is valid for regions of attached flow as well as regions of separated flow can now be advanced provided Eqs. (3) is taken as the basis for our discussion, that is, we consider the dominant equations in region $(\epsilon R)^{-1} << \eta << \epsilon^2$ and $\Delta \to 0$. In fact, two straight integrations of Eq. (3) together with the mixing-length hypothesis lead to

$$u^+ = \sigma 2\varkappa^{-1}\sqrt{u_\tau^{+\,2} + u_{p\nu}^{+\,3} L_c^+ y^+} + \sigma u_\tau^+ \varkappa^{-1}\ln\left(y^+\right), \tag{41}$$

where $u^+ = u/u_R$, $u_\tau^+ = u_\tau/u_R$, $u_{p\nu}^+ = u_{p\nu}/u_R$, $y^+ = y/L_c$ and $L_c^+ = L_c u_R/\nu$ and $\sigma = \tau_w/|\tau_w|$ is used to indicate the flow direction.

Many other different treatments of the lower boundary condition can be appreciated in literature. Loureiro et al. (2007b), for example, have investigated the numerical prediction of flows over two-dimensional, smooth, steep hills according to the above formulation and the formulations of Mellor (1966) and of Nakayama and Koyama (1984). The standard κ-ϵ model was then used to close the averaged Navier-Stokes equations. The results are shown to vary greatly.

Table 1. Properties of undisturbed profile (x/H = -12.5) and of hill.

Boundary layer thickness	δ	100 mm
External velocity	u_δ	0.0482 m s^{-1}
Friction velocity	u_τ	0.0028 m s^{-1}
Roughness length	z_0	0.27 mm
Longitudinal velocity fluctuations at (z/δ=0.05)	$\sqrt{\overline{u'^2}}/u_\tau$	2.50
Transversal velocity fluctuations at (z/δ=0.08)	$\sqrt{\overline{w'^2}}/u_\tau$	0.83
Hill height	H	60 mm
Hill length	L	600 mm
Maximum slope	θ_{max}	18.6^0

5. Results: Experimental and Numerical Validation

The structure of a separating turbulent boundary layer, as described above, will be tested against the data of Loureiro et al. (2007a). This work gives an account of the problem of a separating flow which furnish reasonably detailed near wall measurements. In particular, reliable wall shear stress measurements are highly coveted for they permit all hypotheses concerning the relevant scales of a separating flow to be tested. The data of Loureiro et al. (2007a) in particular furnish wall shear stress data even in the region of reverse flow. Details of the experimental conditions can be obtained from the original sources. The general flow conditions, however, are presented in Table 1. Here, we just mention that the experiments of Loureiro et al. (2007a) were conducted in a water channel. Loureiro et al. (2007a) used laser anemometers for velocity measurements.

The experimental data of Loureiro et al. (2007a) must be supplemented by data obtained through numerical simulations of the same flow geometries. The reason here is to obtain data that for one reason or another have not been reported by the original work but that are crucial for theory evaluation. A critical example is the pressure distribution at the wall, which has not been given by Loureiro et al (2007a). A secondary objective of the numerical simulations is to have data with a sufficiently fine domain discretization so as to allow for accurate data interpolation.

The simulations for both flow geometries were conducted with the code ANSYS CFX, release 5.7. The code solves the Reynolds averaged Navier-Stokes equations (RANS) through a finite-volume formulation coupled with a scheme for the treatment of the convective and diffusive terms simultaneously. Turbulence closure was achieved by choosing the (κ-ω)-SST formulation of Menter (1994). In fact, six different types of turbulence modeling were applied to the hill problem (Loureiro et al., 2007c). For the prediction of mean velocity profiles and wall shear stress, the SST-model proved to be the best choice.

The κ-ω based SST model is supposed to give highly accurate predictions of the onset and the extent of separation from a smooth surface under an adverse pressure gradient. Standard formulations based on the κ-ϵ or κ-ω models do not account properly for the transport of the turbulent shear stress. This results in an over prediction of the eddy viscosity. Menter (1994) proposed to rectify this problem by imposing a limiter to the eddy viscosity. The limiter contains a blending function whose arguments depend on κ, ω, S = (local strain rate), ν, y and some constants.

The near wall treatment is provided by a method that automatically shifts from a wall function approach to a low-R formulation as mesh size is refined. By resorting to the analytical solutions of the κ-ω formulation in the logarithmic and viscous regions of the flow, a blending procedure can be implemented to specify an algebraic equation for ω that can be used throughout the inner regions of the flow. This equation is then solved together with the fluxes for the momentum and κ (which is artificially kept to zero) equations. Wall functions are replaced by a low Reynolds number formulation once a near wall grid resolution of at least $y^+ < 2$ is achieved.

The coordinate systems used for the computations were the same as defined by the experiments. They are shown in Fig. 1. Typical numerical simulation details were: 1) computational domains: 22.5x4 H (=hill height); 2) mesh sizes: 110,376 elements; 3) running time: 1h 06 min. Boundary conditions were taken directly from the experimental data, including the mean and fluctuating quantities. The computations were performed on a Pentium 4, 3.0 GHz, with 1 Gb DDR400 RAM operating in dual channel mode.

The computed general flow patterns for both flow geometries are shown in Fig. 2. A detailed comparison between the measured and the computed regions of separated flow is given in Table 2.

The pressure and shear stress profiles at the wall are presented in Fig. 3. These two figures are crucial for an evaluation of the characteristic flow velocities u_τ, $u_{p\nu}$ and u_R.

The behaviour of the characteristic velocities u_τ, $u_{p\nu}$ and u_R is shown in Fig. 4. A negative value of $u_{p\nu}$ indicates this parameter has been evaluated in a region of favorable pressure gradient; a negative u_τ indicates a region of adverse

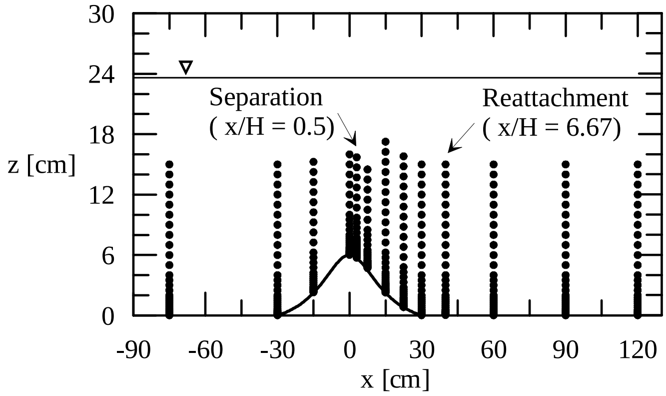

Figure 1. Co-ordinate systems and measuring stations.

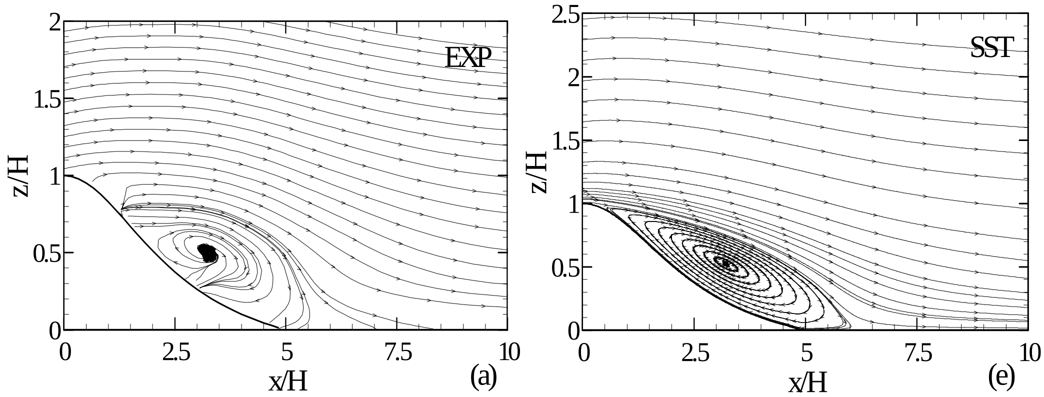

Figure 2. Flow streamlines. a: experiments; b: computations.

Table 2. Length of separated flow according to predictions. H = hill height.

Work	Sep. Point (x/H)	Reatt. Point (x/H)	Length (x/H)
Experiments	0.50	6.67	6.17
Numerical Simulation	0.53	5.53	5.00

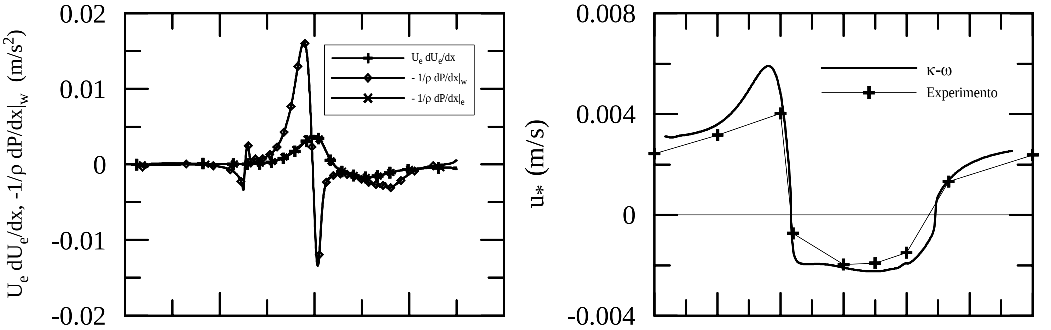

Figure 3. Wall pressure and wall shear stress.

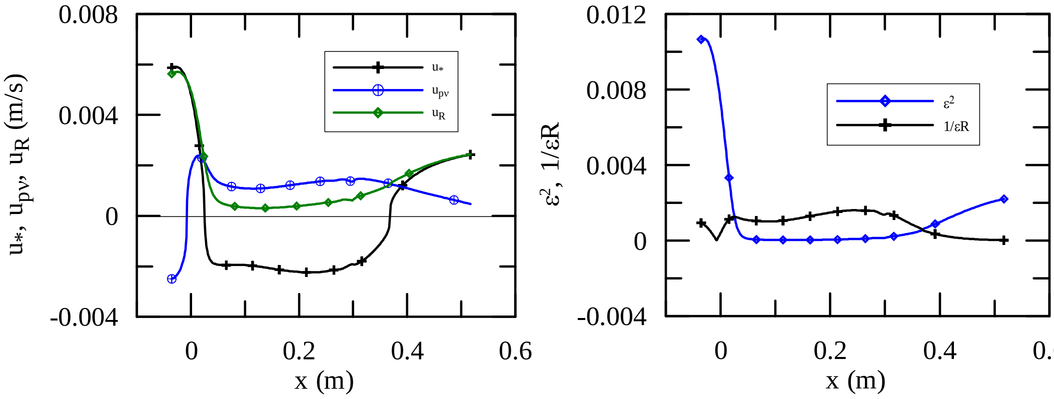

Figure 4. Characteristic behaviour of u_τ, $u_{p\nu}$, u_R and the merging between the inner and outer regions (ϵ^2 and $1/\epsilon R$).

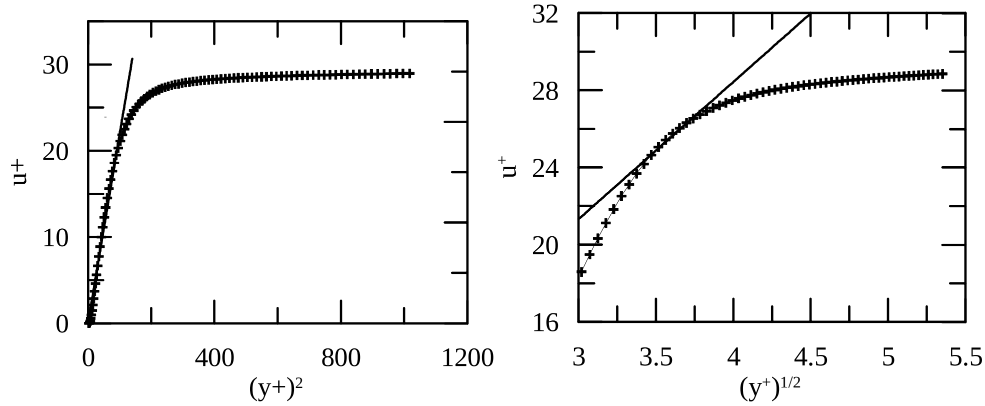

Figure 5. The local solutions of Goldstein (1948) and Stratford (1959).

pressure. The points ahead separation and behind re-attachment respectively where u_τ and $u_{p\nu}$ assume a same value can be noted. Clearly, this defines the order of the magnitude of the flow regions where the pressure gradient effects have to be considered. The numerical computations furnish $u_\tau = u_{p\nu} = 0.00225$ ms^{-1}. Since $u_e = 0.0482$ ms^{-1}, then $\epsilon^2 = 0.00218$. This value can be compared with the reference length $((\nu/u_{p\nu})/l) = (0.00048/0.36250) = 0.0016$. Thus, we have cause to say that the near separation sub-layer structure is governed by a scale of ord(ϵ^2), just as stated by Eqs. (35) and (39).

Fig. 4 also illustrates the merging between the inner viscous region and the external inertial region. The points where the two curves cross are defined by the separation and reattachment locations. Thus, at the separation point the two former distinct layers are governed by the same scales; the flow is then observed to change from a two-layered structure to a one layered structure.

At the separation point, local solutions based on Goldstein's (1948) and Stratford's (1959) approximations should hold. Fig. (5) shows the velocity profile with $u_\tau = 0$ plotted in $(y^+)^2$ and $(y^+)^{1/2}$ coordinates. The existence of local solutions that comply with the theories of Goldstein and Stratford is evident. The fit for the Goldstein solution ($u^+ = 0.22\ (y^+)^2$) was determined with 28 points located in the region $1.62 < y^+ < 10.49$. The coefficient of determination R-squared was 0.993305. Stratford' solution ($u^+ = 7.11\ (y^+)^{1/2}$) was determined with 9 points located in the region $11.56 < y^+ < 14.44$. The coefficient of determination R-squared was 0.999986.

The law of the wall formulation of Eq. (41) is compared to the numerical data in Fig. 6. The good agreement with the experimental data confirms u_R and L_c as the most appropriate scales for the problem of separating boundary layers.

6. Final remarks

The present work has shown how a straightforward application of Kaplun limits to the problem of a separating boundary layer is capable of explaining in simple terms how the two-layered structure for the canonical turbulent boundary layer reduces to a single-layered structure near to a separation point. The work has shown how the characteristic velocities behave in the different regions of the flow, yielding local solutions that corroborate the results implied by the analysis of Kaplun limits. The present analysis is different from those of other authors in the sense that the small parameter ϵ is governed by an algebraic equation obtained through asymptotic arguments specially devised to accommodate the various predominant physical effects.

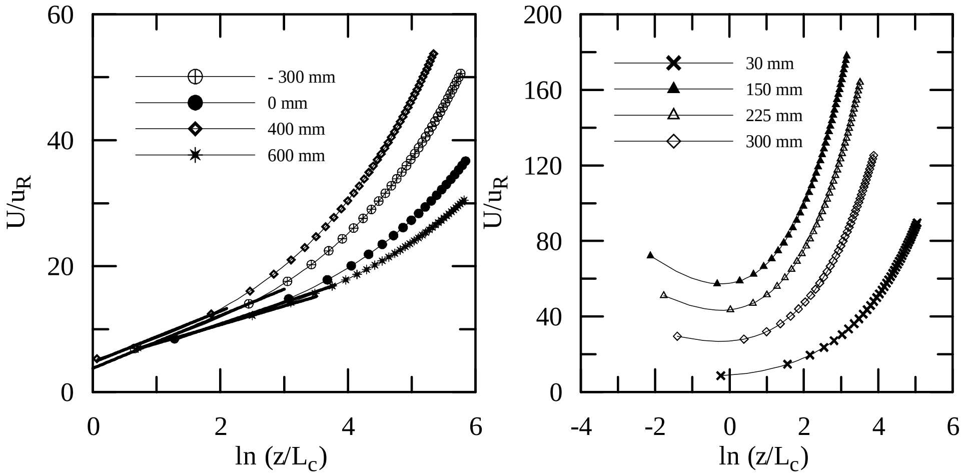

Figure 6. Law of the wall in terms of the new inner scales.

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