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A NUMERICAL APPROACH ON DETERMINIG THE TORSIONAL STIFFNESS OF COMPLEX CROSS SECTIONS

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Abstract. *The purpose of this work is to present a solution of the torsion problem with complex cross section beams by the finite element method. The elements used are high precision Hermite type, which can express null transverse shear stress at the boundary of the cross section and further impose the continuity of the transverse shear stress at the interface between two isotropic materials. We propose an analytical/numerical comparison, numerical/numerical comparison and in the numerical/experimental comparison we take beams used currently in bycicle wheels applications.*

Keywords: *Torsion, Stress function, Cross section,Isotropic multi-phases, Finite element.*

1. INTRODUCTION

To model structures using beam elements we need to know some characteristcs of the cross section, such as: principal axes, the cross section inerties, the shear coefficient, the shear center and torsional stiffness. These parameters are easily obtained in some cases for isotropic beam withs: circular and rectangular cross sections, hollow cross sections with thin walls.

The homogenized characteristics of beams are obtained by submitting the section in bending moments, torsional moments and shear loads. The determination of the equivalent stiffness in bending have no particular problem. However, in the case of sections made of composite materials and/or with a complex shape, the determination of the transverse shear stress is complex and needs a particular development. The analytical solution of the torsion problem for prismatic beams, including beams with composite phases was proposed by Muskhelishvilli (1953) . Nevertheless, the solution proposed is difficult to apply on sections of complex shape. The solution of this problem was proposed by Shaw (1944), Southwell, (1946), Allen, (1955) , using a relaxation method. Ely, *et al.* (1960) extended this method in order to apply it to complex sections with composite phases. One of the earliest work permiting to resolve the torsion problem with any cross section by a finite element formulation was published by Zienkiewicz *et al.* (1965).

The purpose of this work is to resolve the torsion problem of beams with isotropic phases using finite elements with semi-C^1 continuity. These elements permit the continuity of the transverse shear stress at the interface between the phases, and a better precision of the transverse shear stress distribuition on the cross section of the beam. In the validation of the formulation we propose comparisons between analytical solutions and between different numerical methods. In a numerical/experimental comparison, we use bycicle wheels beams with industrial application.

2. THEORY

Consider a transverse cross section of a beam with isotropic phases working in torsion, submitted to a pure moment M_x, Fig. 1.

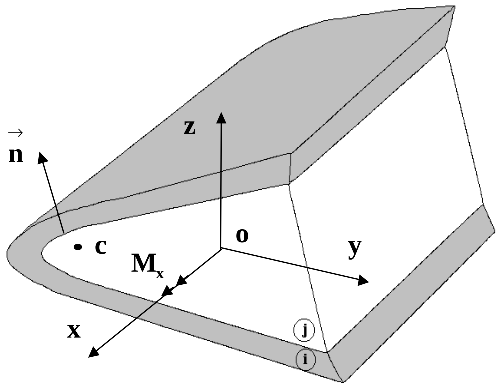


Figure 1 - Beam with isotropes phases in torsion

The displacement field relative to a plane stress state is defined as:

$$u_x = \frac{d\theta_x}{dx}\,\varphi(y,z) \tag{1}$$

$$u_y = -(z - z_c)\,\theta_x \tag{2}$$

$$u_z = (y - y_c)\,\theta_x \tag{3}$$

with: θ_x = rotation of the section around the x axe,

$\varphi(y,z)$ = warping function for torsion,

(y_c, z_c) = position of the shear centre.

The origin of the coordinate axes is defined by the position of the elastic position center and the principals axes. With Eq. (1), (2) and (3), the only stress components acting on the section are:

$$\tau_{xy} = G_i\,\frac{d\theta_x}{dx}\left(\frac{\partial\varphi}{\partial y} - (z - z_c)\right) \tag{4}$$

$$\tau_{xz} = G_i\,\frac{d\theta_x}{dx}\left(\frac{\partial\varphi}{\partial z} + (y - y_c)\right) \tag{5}$$

The equilibrium relation on the transverse cross section is written as:

$$\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \qquad (6)$$

To verify Eq. (6), the Eq. (4) and Eq.(5) are differently formulated:

$$\tau_{xy} = G_i \frac{d\theta_x}{dx} \frac{\partial \phi}{\partial z} \qquad (7)$$

$$\tau_{xz} = -G_i \frac{d\theta_x}{dx} \frac{\partial \phi}{\partial y} \qquad (8)$$

with: $\phi(y,z)$ = stress function

The new function $\phi(y,z)$, must satisfy the differential equation:

$$\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = -2 \qquad (9)$$

The stress function $\phi(y,z)$ is the solution of Eq. (9), when satisfing the boundary conditions:

$$\phi = 0 \qquad (10)$$

and,

$$\frac{\partial \phi}{\partial n} = 0 \qquad (11)$$

The perfect assembly between the *i* and *j* phases add some boundary conditions such as, the continuity of the stress function, the continuity of the strains on the longitudinal plane at the interface and the continuity of the shear stress at the interface, so:

$$\phi_i = \phi_j \qquad (12)$$

$$\left\{ \frac{\partial \phi}{\partial z} n_z + \frac{\partial \phi}{\partial y} n_y \right\}_i = \left\{ \frac{\partial \phi}{\partial z} n_z + \frac{\partial \phi}{\partial y} n_y \right\}_j \qquad (13)$$

and,

$$G_i \left\{ \frac{\partial \phi}{\partial z} n_y - \frac{\partial \phi}{\partial y} n_z \right\}_i = G_j \left\{ \frac{\partial \phi}{\partial z} n_y - \frac{\partial \phi}{\partial y} n_z \right\}_j \qquad (14)$$

The torsional moment may be expressed as:

$$M_x = \int_D \left(y\, \tau_{xz} - z\, \tau_{xy} \right) dS \qquad (15)$$

The equivalent torsional stiffness of the beam can be deduced in Eq. (16) as:

$$\langle GJ\rangle = -\int_D G_i \left\{\frac{\partial\phi}{\partial y} y + \frac{\partial\phi}{\partial z} z\right\} dS \qquad (16)$$

3. THE FINITE ELEMENT FORMULATION

The application of the weighted residual method proposed by Zienkiewicz *et al.* (1965) and Dhatt *et al.* (1984), permits to approach the solution of Eq. (9) by writing the orthogonality conditions of the error in relation with the ponderation functions depending on the element chosen. These elements shown in the Fig. 2, are of high precision Hermite type, Dhatt *et al.* (1984), and have 3 nodals variables per nodes.

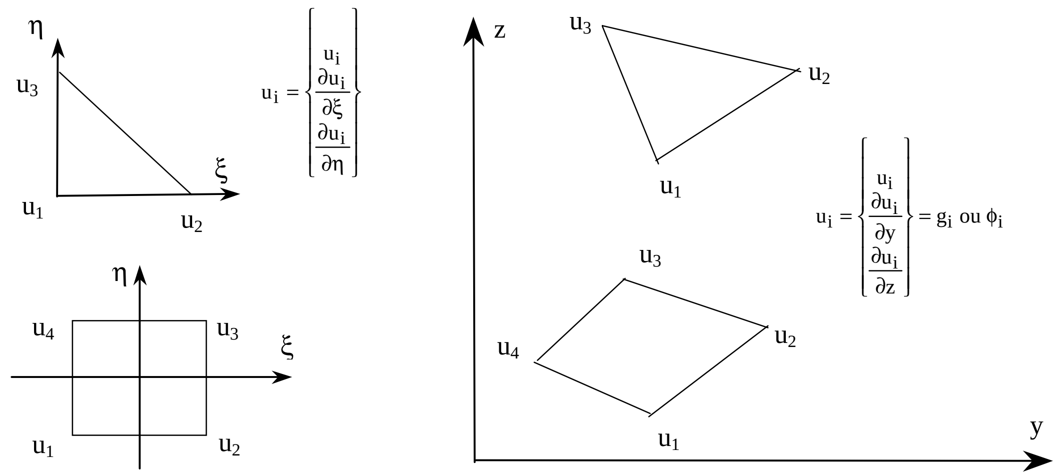


Figure 2 - Semi-C[1] finite elements

The geometric approximation is linear or quadratic and the second one allow a better definition of the geometry in the case of complex boundaries.

By considering the integral form in torsion given in Eq. (9) and by taking into account Eq. (11) we get:

$$W(\phi) = -\int_D \left\langle \frac{\partial\psi}{\partial n} \right\rangle \left\{\frac{\partial\phi}{\partial n}\right\} dS + 2\int_D \langle\psi\rangle \, dS = 0 \qquad (17)$$

with: ψ = ponderaction function

The nodal aproximation is given:

$$\phi = \langle N(1) \quad N(2) \quad \cdots \quad N(n)\rangle \begin{Bmatrix} \phi_1 \\ \partial\phi_1/\partial z \\ \vdots \\ \partial\phi_n/\partial y \end{Bmatrix} \quad e \quad \psi = \delta\phi \qquad (18)$$

Thus, the linear equation system to solve is expressed by:

$$\int_D \left\langle \frac{\partial N_i}{\partial n} \right\rangle \left\{ \frac{\partial N_j}{\partial n} \right\} \{\phi\} dS = 2 \int_D \langle N_i \rangle \, dS \tag{19}$$

This system equation is integrated in the reference space (ξ,η):

$$\int_{-1}^{1}\int_{-1}^{1} [T]^t \left[B_\xi\right]^t [Q]^t [Q]\left[B_\xi\right][T] |J| \, d\xi \, d\eta \, \{\phi\} = 2 \int_{-1}^{1}\int_{-1}^{1} [T]^t \cdot \langle N_i \rangle |J| \, d\xi \, d\eta \tag{20}$$

with: $|J|$ = determinant of the jacobian matriz

The matrices $[B_\xi]$, [Q] and [T] are given in Dhatt *et al.* (1984). The boundary conditions at the interfaces are verified when the assembly of the global matrix. The nodal variables corresponding to the material *j* are expressed as a function of the nodal variables corresponding to the material *i* using Eq. (12), Eq. (13) and Eq. (14).

4. COMPARISONS

4.1 Comparisons between analytical solutions on notched cross sections

This following type of sections were studied analytically by Cicala (1935). The comparisons between Cicala (1935) and the finite elements solution are shown in table 1.

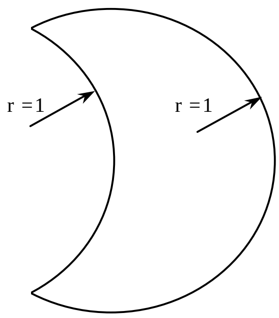


Figure 3 – Notched cross section n1

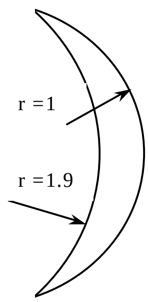


Figure 4 – Notched cross section n2

Table 1 - Torsional stiffness <GJ> for notched cross section

	<GJ>	
	Cicala (1935)	Present work
Notched Cross Section n1	0.3776	0.3693
Notched Cross Section n2	1.855e-4	2.029e-4

4.2 Comparisons between different numerical methods on multi-phase section

Bi-phase square section

The following example presents a square cross section having one interface between two isotropic materials. The finite element solution is compared with an exact solution, Timoshenko (1961), a solution by series, Muskhelishvilli (1953) and a solution by the finite difference method, Ely (1960). The results are presented in Table 2.

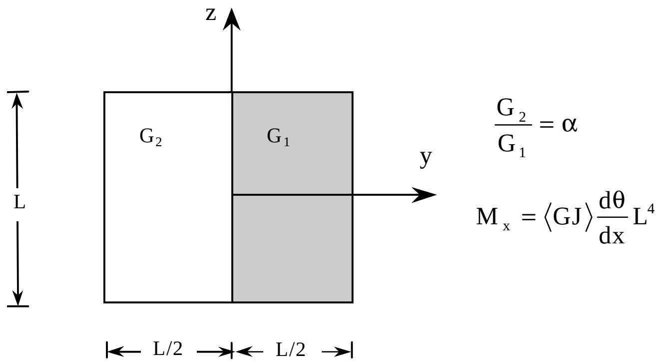


Figure 3 – Bi-phase square section

Table 2 – Torsional Stiffness <GJ> on bi-phase square section

	<GJ>			
α	Timoshenko(1961)	Muskhelishvilli (1953)	Ely (1960)	Present work
1	0.1406	0.1407	0.1388	0.1406
2	-	0.1972	0.1941	0.2023
3	-	0.2399	0.2358	0.2554

Square cross section with circular insertion

The finite element solution of this problem is compared with the one obtained by finite the difference method Ely (1960). The torsional stiffness <GJ> obtained by Ely (1960) is 0.2119 whereas the one obtained in this present work is 0.1911.

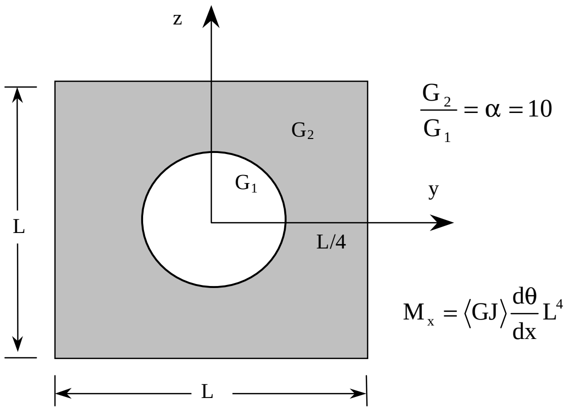
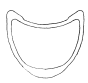
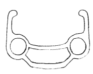
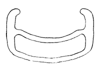
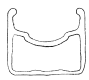


Figure 5 – Square cross section with circular insertion

4.3 Numerical/experimental comparison

The correlaction of the numerical and experimentals results is made using bycicle wheel beams (hollow complex sections). The experimental device developed to measure the torsional stiffness is presented in Pereira (1996). The properties of the material are: E = 6.95e10 N/m^2, G = 2.61e10 N/m^2 e ν = 0.33. The results are shown in Table 3.

Table 3 - Numerical/experimental comparison of the torsional stiffness

perfil	J_{exp} (mm^4)	J_{num} (mm^4)	Δ (%)
b1 	3425.6	3466.4	1.19
b2 	917.4	869.2	-5.25
b3 	1368.5	1226.1	-10.41
b4 	3752.8	3293.0	-12.25

The numerical solution of the torsion problem of hollow beams using the stress function is not simple. The elimination of the inner part disturbs the stress distribution of the cross section. This problem can be easily observed in the case of complex sections, where it is possible to verify the isovalues of the stress function for the beam filled with a virtual material (G_v) change in a regular way than the hollow beam, Pereira (1996).

The Fig. 9 shows the convergence of the torsional stiffness as function of the real material (G_r) and the virtual material (G_v). The absence of the virtual material leads to values 80 % lower than the experimental values.

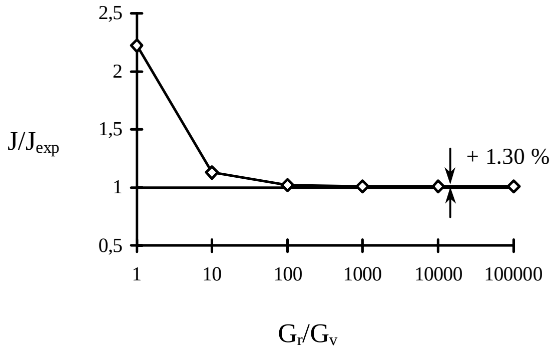


Figure 9 - Evolution of J as function of G_r/G_v - beam b1

5. COMMENTS AND CONCLUSIONS

It was noted in the analytical/numerical comparison of the notched cross section n2, where the stress concentration is high, that the numerical torsional stiffness is higher. In the square cross section with circular insertion, it was observed with a finer mesh that the result it was not so better.

The results obtained with the bycicle wheels beams are satisfactory if we consider the dispersion of ±3 % over the materials properties and of ± 20% over the geometry. The precision on the geometry is highly bounded by the fabrication process.

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