MIXED $H_2/H_¥$ CONTROL OF A TWO-FLOORS BUILDING MODEL USING THE LINEAR MATRIX INEQUALITY APPROACH

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Abstract. *This paper presents the mixed H_2/H_∞ control strategy formulated by means of the Linear Matrix Inequality (LMI) approach to attenuate the vibrations of a two-floor building model under seismic excitation. The structure considered is manufactured by Quanser Consulting Inc., and represents a building controlled by an active mass driver. Here, the H_∞ and H_2 strategies are combined as a mixed control problem by means of a system of LMIs. The performance of the mixed H_2/H_∞ control strategy in both the frequency and time domains are analyzed based on a numerical optimization technique, using an efficient convex optimization software. The feasibility and the effectiveness of the mixed control strategy are demonstrated by the active vibration control of the flexible structure.*

Keywords*: active vibration control, mixed H_2/H_∞ control, linear matrix inequality (LMI), two-floor building model.*

1. INTRODUCTION

In the last two decades, robust control problems have been studied effectively in many fields of control engineering. Active vibration control is one of the main topics in these works and still remains attractive for new control design schemes. Generally, time domain specifications, like H_2 control, and frequency domain specifications, like H_∞ control, are mostly considered in active vibration control problems as valuable criterions. Combining H_2 and H_∞ control objectives in a controller is one further step in robust control theory (Lu and Skelton, 2000; Du et al, 2008).

In general, convexity is an important specification and many linear control problems can be reduced to convex optimization problems which involve linear matrix inequalities (LMI). LMI has more flexibility for combining various design constraints on the closed loop system. Recently LMI-based control system analysis has become popular since it encompasses many control subjects (Boyd et al, 1994).

A mixed H_2/H_∞ control problem using convex optimization is formulated by (Khargonekar and Rotea, 1991). State feedback H_2/H_∞ design is studied using LMI approach by (Sivrioglu and Nonami, 1997). The goal of this problem is to design a state feedback controller which guarantees not only a pre-specified H_∞ disturbance attenuation level, but also the minimum H_2 performance index.

The present work is concerned with the design of robust control systems to satisfy both these sets of performance specifications for an active vibration problem in practice. For this purpose, this paper presents the mixed H_2/H_∞ control strategy formulated by means of the LMI approach to attenuate the vibrations of a flexible structure. More precisely, the mixed control problem can be formulated as a minimization problem subject to convex constraints expressed by a system of LMIs. The control design method is tested on an active mass driver (AMD) vibration control experiment. A two-story building test-bed with AMD is used to test the designed mixed H_2/H_∞ controller on a shaking table. The structure considered is manufactured by Quanser Consulting Inc., and represents a building controlled by an AMD located at the top. The performance of the mixed H_2/H_∞ control strategy in both the frequency and time domains are analyzed based on a numerical optimization technique, using an efficient convex optimization software (Gahinet et al, 1995). Experiments are conducted to evaluate the performance of the proposed mixed controller.

2. LMI FORMULATION FOR MIXED $H_2/H_¥$ CONTROL STRATEGY

Consider the linear time invariant plant described by:

$$\dot{x} = Ax + B_1 w + B_2 u \tag{1.a}$$

$$z_1 = C_1 x + D_{11} w + D_{12} u \tag{1.b}$$

$$z_2 = C_2 x + D_{21} w + D_{22} u \tag{1.c}$$

where $x \in \Re^{n}$ is the state vector, z_1, $z_2 \in \Re^{n_z}$ are the controlled output vectors, $u \in \Re^{n_u}$ is the control input, and $w \in \Re^{n_w}$ is the exogenous input.

Suppose that the control input u is linear function of the state, i.e.,

$$u = Kx \tag{2}$$

where $K \in \Re^{n_u \times n}$ is the state feedback gain.

The closed-loop system is given by

$$\dot{x} = (A + B_2 K)x + B_1 w \tag{3.a}$$

$$z_1 = (C_1 + D_{12}K)x + D_{11}w \tag{3.b}$$

$$z_2 = (C_2 + D_{22}K)x + D_{21}w \tag{3.c}$$

Letting $T_{z_1 w}$ and $T_{z_2 w}$ as the closed-loop transfer function from w to z_1 and z_2, respectively, the multiobjective H_2/H_∞ control strategy may be described as follows. Find a static state-feedback law (Eq. 2), such that $\|T_{z_2 w}\|_2$ is minimized over all state-feedback gains K such that what also minimizes the $\|T_{z_1 w}\|_\infty$.

This approach yields a convex sub-optimal control problem as shown in Fig. 1.

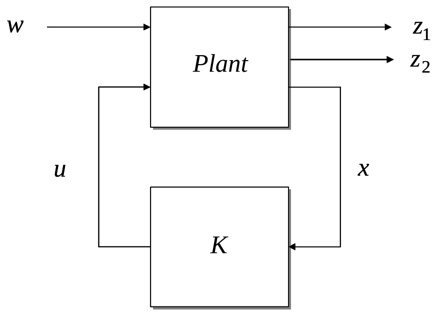


Figure 1. Block diagram of H_2/H_∞ control system with state feedback

2.1. H_2 Control Strategy

The H_2 norm of the transfer function $T_{z_2 w}$ is finite if and only if, in Eq. (3.c), $D_{21} = 0$. In this case, the H_2 norm of $T_{z_2 w}$ is given by

$$\|T_{z_2 w}\|_2^2 = \text{Trace}\left[(C_2 + D_{22}K)X_2(C_2 + D_{22}K)^T\right] \tag{4}$$

where the symmetric positive definite matrix X_2 is obtained by solving the following inequality (Boyd et al, 1994):

$$(A + B_2 K)X_2 + X_2(A + B_2 K)^T + B_1 B_1^T < 0 \tag{5}$$

By rearrangement of inequality (5), using the Schur complement (Dullerud and Paganini, 2000), and letting $Z_2 = KX_2$, the following inequality can be obtained, for $X_2 > 0$:

$$\begin{bmatrix} AX_2 + X_2 A^T + B_2 Z_2 + Z_2^T B_2^T & B_1 \\ B_1^T & -I \end{bmatrix} < 0 \tag{6}$$

The objective H_2 control can be given by minimizing the constraint (4)

$$\min_K \ \text{Trace}\left(SR^{-1}S^T\right) \tag{7}$$

or, introducing a new matrix variable M, i.e., the LMI,

$$\text{Trace}\left(M\right) < h \ , \ \begin{bmatrix} M & S \\ S^T & R \end{bmatrix} > 0 \tag{8}$$

In this way, the H_2 norm of $T_{z_2 w}$ is the minimum of Trace (M) or formally:

$$\min_{M, X_2, Z_2} \ \text{Trace } (M) \text{ subject to}$$

$$\begin{bmatrix} AX_2 + X_2 A^T + B_2 Z_2 + Z_2^T B_2^T & B_1 \\ B_1^T & -I \end{bmatrix} < 0 \tag{9}$$

$$\begin{bmatrix} M & C_2 X_2 + D_{22} Z_2 \\ X_2 C_2^T + Z_2^T D_{22}^T & X_2 \end{bmatrix} > 0 \text{, and } \ X_2 > 0$$

2.2. *H*¥ Control Strategy

For time invariant systems, the H_∞ norm the transfer function from w to z_1 is minimized when the effect of the disturbance on z_1 is diminished or that the infinity norm of the $T_{z_1 w}$ be less than g, i.e.,

$$\left\| T_{z_1 w} \right\|_\infty = \sup_w \frac{\left\| z_1 \right\|_2}{\left\| w \right\|_2} \leq g \tag{10}$$

where g which is a positive real number serves as the measure of performance.

The bounded real lemma plays a central role in obtaining the H_∞ constraint. There exists a quadratic Lyapunov function $V(x) = x^T P x$ such that for all time t,

$$\frac{d}{dt} V(x) + z_1^T z_1 - g^2 w^T w < 0 \tag{11}$$

where P is a symmetric positive definite matrix.

Substituting Eq. (3.b) into inequality (11), and assuming $D_{11} = 0$, the following inequality can be obtained

$$[(A + B_2 K)x + B_1 w]^T Px + x^T P[(A + B_2 K)x + B_1 w] + (C_1 + D_{12} K)^T x^T (C_1 + D_{12} K)x - g^2 w^T w < 0 \tag{12}$$

By rearrangement of inequality (12), yields

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} (A + B_2 K)^T P + P(A + B_2 K) + (C_1 + D_{12} K)^T (C_1 + D_{12} K) & PB_1 \\ B_1^T P & -g^2 I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < 0 \tag{13}$$

Using the Schur complement for inequality (13)

$$(A + B_2 K)^T P + P(A + B_2 K) + (C_1 + D_{12} K)^T (C_1 + D_{12} K) - PB_1 g^{-2} B_1^T P < 0 \tag{14}$$

Multiplying (14) by P^{-1} from right and left, the following inequality can be obtained

$$P^{-1}(A + B_2 K)^T + (A + B_2 K)P^{-1} + P^{-1}(C_1 + D_{12} K)^T (C_1 + D_{12} K)P^{-1} - B_1 g^{-2} B_1^T < 0 \tag{15}$$

By putting Eq. (15) in the LMI form again

$$\begin{bmatrix} P^{-1}(A + B_2 K)^T + (A + B_2 K)P^{-1} + P^{-1}(C_1 + D_{12} K)^T (C_1 + D_{12} K)P^{-1} & B_1 \\ B_1^T & -g^2 I \end{bmatrix} < 0 \tag{16}$$

Multiplying (16) by $\begin{bmatrix} g^{1/2} & 0 \\ 0 & g^{-1/2} \end{bmatrix}$ from right and left, and letting $X_\infty = gP^{-1}$, yields

$$\begin{bmatrix} X_\infty (A+B_2K)^T + (A+B_2K)X_\infty & B_1 \\ B_1^T & -gI \end{bmatrix} + \frac{1}{g} X_\infty (C_1 + D_{12}K)^T (C_1 + D_{12}K) X_\infty < 0 \tag{17}$$

Using the Schur complement again in Eq. (17) and letting $Z_\infty = KX_\infty$, the following convex problem is obtained:

$\min_{X_\infty, Z_\infty} g$ subject to

$$\begin{bmatrix} X_\infty A^T + AX_\infty + B_2 Z_\infty + Z_\infty^T B_2^T & B_1 & X_\infty C_1^T + Z_\infty^T D_{12}^T \\ B_1^T & -gI & 0 \\ C_1 X_\infty + D_{12} Z_\infty & 0 & -gI \end{bmatrix} < 0 \text{, and } X_\infty > 0 \tag{18}$$

2.3. The Mixed H_2/H_∞ Control Strategy

The mixed H_2/H_∞ control problem is to minimize the H_2 norm of $T_{z_2 w}$ over all state-feedback gains K such that what also minimizes the H_∞ norm constraint. On the other hand, the inequalities (9) and (18) are combined letting $X = X_2 = X_\infty$ (unique solution of K), and $Z = KX$.

In this way, the multiobjective H_2/H_∞ control using H_2 and H_∞ performance constraints can be given by

$\min_{M,X,Z}$ Trace (M) subject to

$$\begin{bmatrix} M & C_2 X + D_{22} Z \\ XC_2^T + Z^T D_{22}^T & X \end{bmatrix} > 0$$

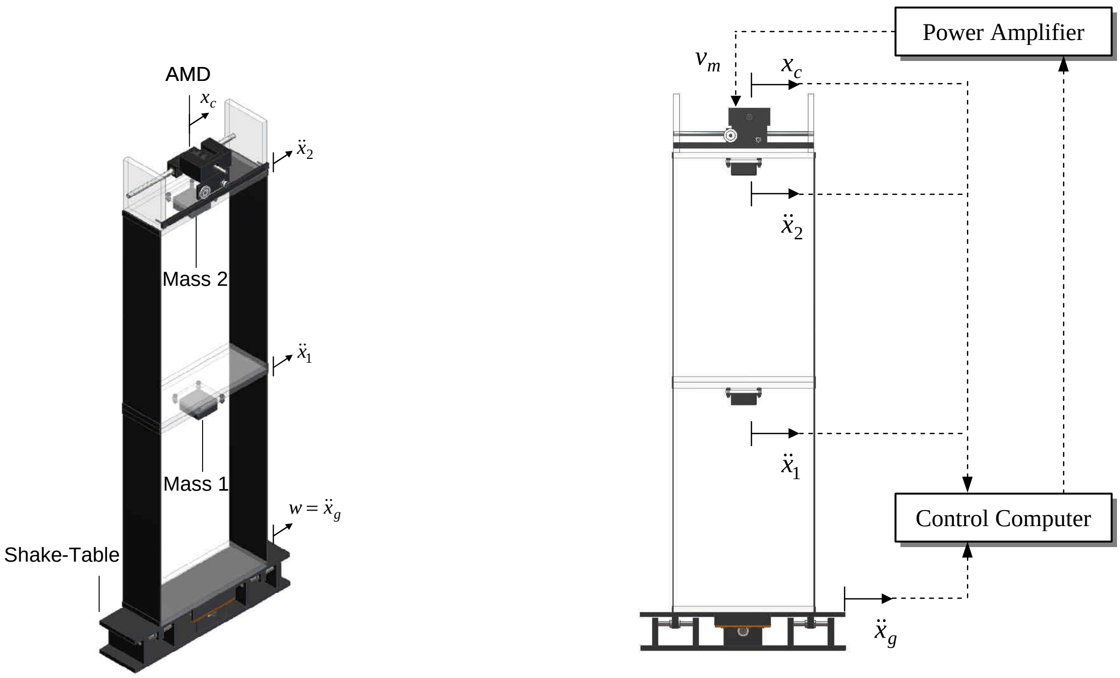
$$\begin{bmatrix} XA^T + AX + B_2 Z + Z^T B_2^T & B_1 & XC_1^T + Z^T D_{12}^T \\ B_1^T & -gI & 0 \\ C_1 X + D_{12} Z & 0 & -gI \end{bmatrix} < 0 \text{, and } X_2 > 0 . \tag{19}$$

The above inequalities are solved using the efficient convex optimization software Matlab LMI Toolbox®. After finding of a solution (M, X and Z) to this multiobjective control problem, the optimal feedback control law of control system (2) is obtained as

$$u = ZX^{-1}x \tag{20}$$

3. EXPERIMENTAL STRUCTURE

The structure specimen (see Fig. 2a), manufactured by Quanser Consulting Inc, is a two-floor building model equipped with AMD and subjected to earthquake ground acceleration ($\ddot{x}_g$) using the shake-table system. The test structure has 1125 mm in height, with each column being steel with a section of 1.75 × 108 mm. The total mass of the structure is 4.52 kg, where the first floor mass (mass 1) is 1.16 kg, the second floor mass (mass 2) is 1.38 kg. The first two modes of the structure are at 1.7 Hz and 5.1 Hz, with associated damping ratios given, respectively, by 0.042 and 0.011.



(a) (b)

Figure 2. Two-floor building model (a) and schematic of experimental setup (b)

The structure is fully instrumented to provide for a complete record of the motions undergone by the structure during testing. Each floor of the building structure is equipped witch a capacitive DC accelerometer that measures the absolute accelerations ($\ddot{x}_1$ and $\ddot{x}_2$). The Active Mass Damper (AMD) provides the control force to the structure through the control voltage (v_m). As shown in Fig. 2, it consists of a moving cart with a DC motor that drives the cart along a geared rack. Additionally, a potentiometer is attached to the motor to measure the cart position relative (x_c) to its base. The maximum stroke is ±65 mm and the total moving mass is 520 g.

Digital control is achieved by use of the MultiQ-PCI board with the QuaRC realtime controller. The controller is developed using Matlab Simulink® and executed in realtime using the QuaRC software. The Simulink code is automatically converted to C code and interfaced through the QuaRC software to run the control algorithm.

3.1. Evaluation Model

A linear time invariant state space representation of the input-output model for the structure described in the previous section has been developed. The structural dynamic system, which includes the AMD, and subjected to earthquake excitation, can be represented in state space form as

$$\dot{x} = Ax + B_1\ddot{x}_g + B_2 v_m \tag{21a}$$

$$y = Cx + D_{yw}\ddot{x}_g + D_{yu} v_m \tag{21b}$$

where $x = [x_c \quad x_1 \quad x_2 \quad \dot{x}_c \quad \dot{x}_1 \quad \dot{x}_2]^T$ is the state vector, $\ddot{x}_g$ is the ground acceleration, v_m is the control voltage, $y = [x_c \quad \ddot{x}_1 \quad \ddot{x}_2]^T$ is the vector of measured responses, A is the dynamic matrix, and the matrices C, B_1 and B_2 represent the sensors, disturbance and control input locations, respectively.

The state matrices are given as follows:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 267.89 & -23.93 & 0 & 0 \\ 0 & -431.03 & 431.03 & 0 & 0 & 0 \\ 0 & 431.03 & -767.51 & 6.55 & 0 & 0 \end{bmatrix} \tag{22a}$$

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}^T \quad (22b)$$

$$B_2 = \begin{bmatrix} 0 & 0 & 0 & 3.37 & 0 & -0.92 \end{bmatrix}^T \quad (22c)$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -431.03 & 431.03 & 0 & 0 & 0 \\ 0 & 431.03 & -767.51 & 6.55 & 0 & 0 \end{bmatrix} \quad (22d)$$

$$D_{yu} = \begin{bmatrix} 0 & 0 & -0.92 \end{bmatrix}^T \quad (22e)$$

$$D_{yw} = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}^T \quad (22f)$$

4. MIXED H_2/H_∞ CONTROLLER DESIGN APPROACH

In this work, the controlled output vectors z_1 and z_2 determined by the H_∞ and the H_2 performance objectives, respectively, is formulated as follows:

$$z_1 = C_1 x \quad (23)$$

$$z_2 = C_2 x + D_{22} v_m \quad (24)$$

where the matrices C_1, C_2, and D_{22} are given by

$$C_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (25)$$

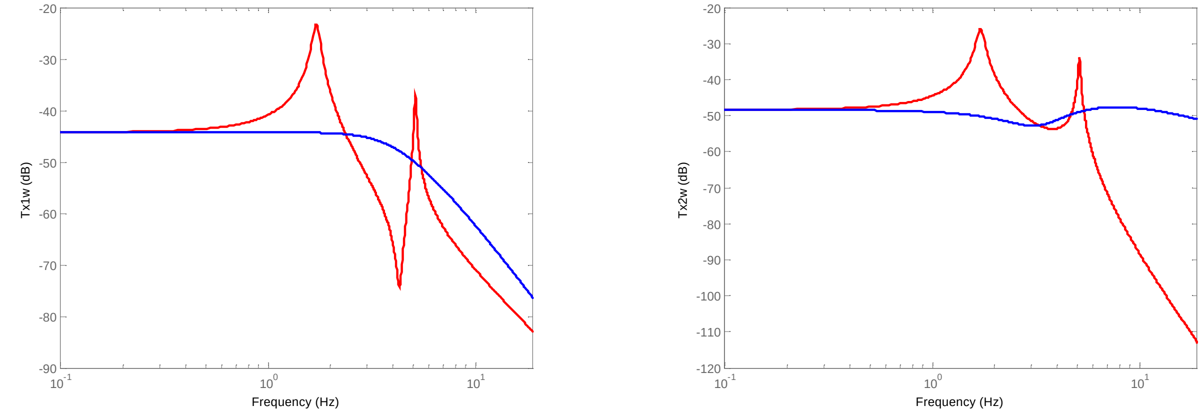
$$C_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (26)$$

$$D_{22} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T \quad (27)$$

It can be seen from Eqs. (23) and (24) that $z_1 = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ and $z_2 = \begin{bmatrix} x_1 & x_2 & \dot{x}_1 & \dot{x}_2 \end{bmatrix}^T + \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T v_m$. Hence, the H_∞ control objective is to minimize the H_∞ norm of the transfer function $T_{z_1 \ddot{x}_g}$ from disturbance $\ddot{x}_g$ to displacement of each floor, and the H_2 control objective is to minimize the H_2 norm of the transfer function $T_{z_2 \ddot{x}_g}$ from disturbance $\ddot{x}_g$ to displacement and velocity of each floor, and at the same time, to the control energy v_m. The resultant system guarantees certain robustness (H_∞ norm is bounded), limits the absolute displacement and velocity of each floor and minimizes the control energy (H_2 norm is optimized).

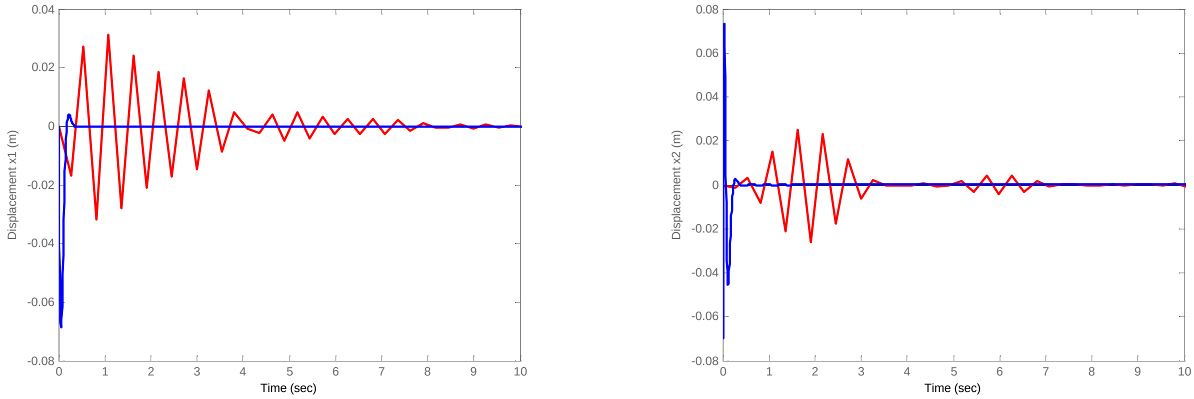
5. SIMULATION RESULTS

Figure 3 and 4 show the results of the frequency response of the system under seismic excitation ($w = \ddot{x}_g$) and the time history impulse response of x_1 and x_2 based on H_∞ control strategy – Eq. (18) – using the LMI control toolbox of Matlab®. The optimum value g_{opt} found is equal to 0.0072043.



(a) (b)

Figure 3. Open (red) and closed loop (blue) frequency response of T_{x_1w} (a) and T_{x_2w} (b) using H_∞ controller



(a) (b)

Figure 4. Open (red) and closed loop (blue) impulse response of x_1 (a) and x_2 (b) using H_∞ controller

In the mixed H_2/H_∞ control, the H_∞ performance index g has a great effect on the control effectiveness. The smaller g is, the better the control effectiveness is in theory. At that optimum value (g_{opt}), solving the mixed H_2/H_∞ control problem – Eq. (19) – using the LMI control toolbox of Matlab®, the optimal H_2 norm of the system is equal to 2387.

Furthermore, repeating the above procedure for a set of prescribed H_∞ performance values g, the optimal H_2 norm versus g is tabulated in Table 1.

Table 1. The optimal H_2 norm versus g

g	**Optimal H_2 norm**
0.00720430	2387
0.00720440	619
0.00720448	410
0.00720449	608
0.00720455	811

It can be observed from Table 1 that the optimal H_2 norm increases as the value of g decreases. At the optimum value (g_{opt}), the optimal H_2 norm of the system is very large (2387). This implies that improving disturbance attenuation level needs to be at the cost of the optimal H_2 norm.

The state feedback gain $K = [-0.0047 \quad 303.33 \quad 1228.15 \quad -2.81 \quad 20.34 \quad 17.67]$ obtained for g = 0.00720448 using Eq. (20) provides the best result between the H_2 and H_∞ objectives. For this optimization procedure, the search for the best frequency response characteristic is presented in Fig. 5.

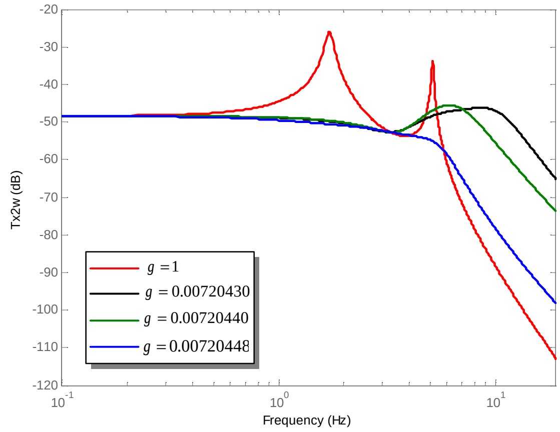
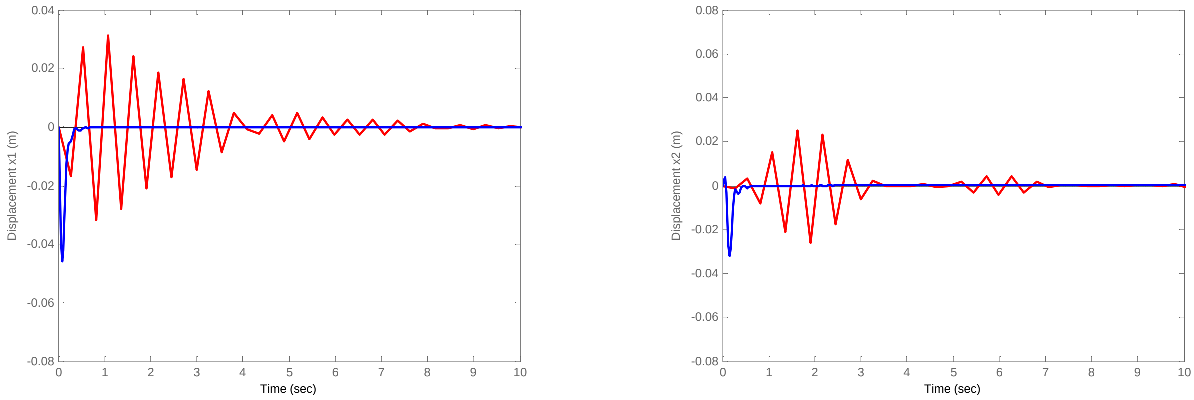


Figure 5. Search for the optimum frequency response of $T_{x_2 w}$ using mixed H_2/H_∞ controller

The optimum mixed H_2/H_∞ state feedback controller has achieved high damping in both modes. One practical aim of this design problem is to show the improvement of the time impulse response of the system due to the H_2 performance objective. The time history impulse response of x_1 and x_2 based on mixed H_2/H_∞ control strategy with and without control is shown in Fig. 6.



(a) (b)

Figure 6. Open (red) and closed loop (blue) impulse response of x_1 (a) and x_2 (b) using mixed H_2/H_∞ controller

As can be seen from Fig. 6, the impulse response of the mixed H_2/H_∞ control is much better than that of the H_∞ control (see Fig. 4) because the initial transient maximum amplitude in the mixed control is small. This is the result of the H_2 performance objective.

6. EXPERIMENTAL VERIFICATION

To verify the performance of the designed mixed H_2/H_∞ controller, shaking table test of the two-story building model with AMD introduced previously was conducted. An earthquake-type excitation was inputted to the shake-table system as the excitation source ($\ddot{x}_g$). The building test-bed on the shaking table was excited by the scaled El Centro earthquake signal shown in Fig. 7.

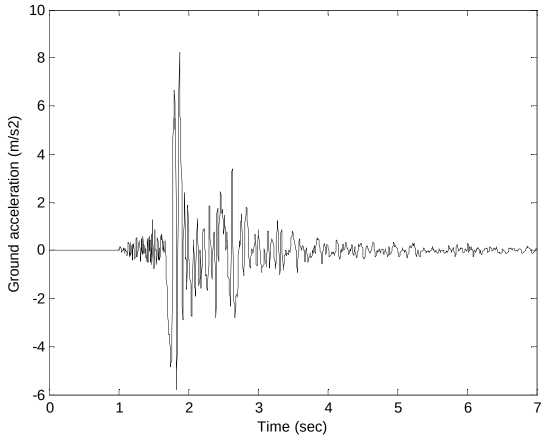


Figure 7. El Centro earthquake ground acceleration ($\ddot{x}_g$) used for seismic excitation

The controller is implemented using the Matlab Simulink® interface and executed in realtime using the QuaRC software. A schematic diagram of the control system is presented in Fig. 2. Figure 8 illustrates the block diagram developed for the seismic response control system. The state feedback controller is designed assuming that all of the states are measured exactly. As shown in Fig. 8, the full-order observer was then used to estimate the state from the actual measurements (y).

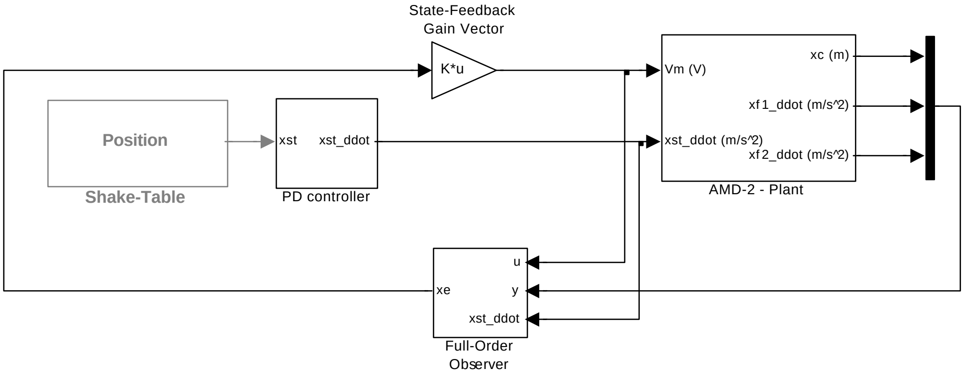
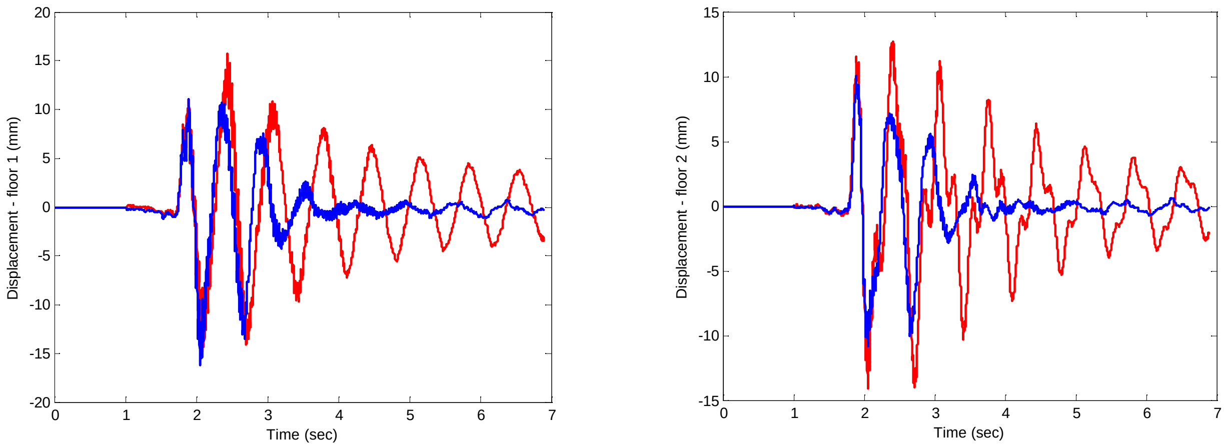


Figure 8. Block diagram for a seismic response control system

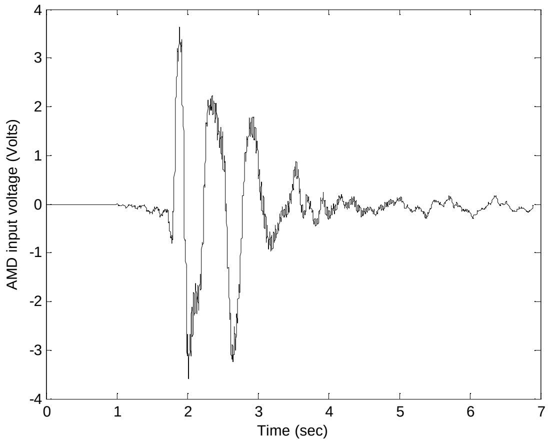
Figure 9 show the displacement of the first floor x_1 and the second floor x_2 of the bench-scale structure when excited by the scaled El Centro earthquake signal for the controlled and uncontrolled systems. From the results it can be noticed that the structural responses are reduced greatly. The reduction ratios of the displacement in the first floor and second floor are 30% and 37%, respectively.



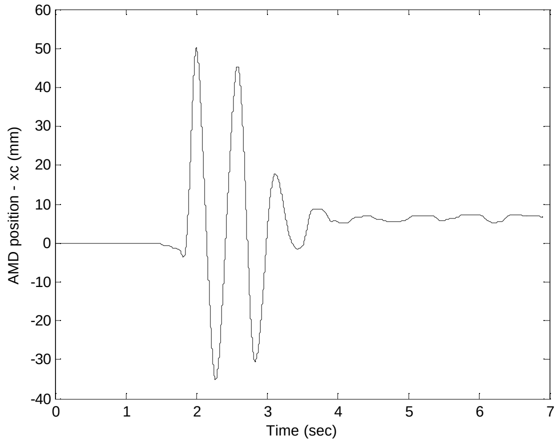
(a) (b)

Figure 9. Open (red) and closed loop (blue) acceleration responses of floors 1 (a) and 2 (b) under seismic excitation

The AMD input voltage (v_m) and its associated position (x_c) are illustrated in Fig. 10. As can be seen from Fig. 10 (b), note that the AMD do not reach its stroke limit (± 65 mm), i.e., the actuator saturation.



(a)



(b)

Figure 10. AMD input voltage v_m (a) and its associated position x_c (b)

7. CONCLUSION

The mixed H_2/H_∞ control strategy is formulated by means a system of LMIs to attenuate the vibrations of a two-floor building model under seismic excitation. The goal of this approach is to design a state feedback controller which guarantees not only a pre-specified H_∞ disturbance attenuation level, but also the minimum H_2 performance index. It is shown that when the mixed control strategy is used in the bench-scale structure, the experimental results show that the structural responses are reduced significantly with the proposed controller. The inclusion of uncertainties in the controller design constitutes the next implementation for this research.

8. ACKNOWLEDGEMENTS

The first author would like to thank the FAPESP (Nº 2008/05129-3) for the financial support of the reported research.

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