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OPTIMIZATION OF LOW-THRUST LIMITED-POWER TRANSFERS BETWEEN COPLANAR CIRCULAR ORBITS IN STRONG GRAVITY FIELD

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Abstract. *The main purpose of this work is to present a numerical analysis of optimal low-thrust limited-power trajectories for simple transfers (no rendezvous) between circular coplanar orbits in an inverse-square force field. Two different classes of algorithms in optimization of trajectories are applied. Optimal trajectories are computed through a direct approach of the optimization problem based on gradient techniques, and, through an indirect approach based on the solution of the two-point boundary value problem obtained from the set of necessary conditions for optimality. The fuel consumption is taken as the performance criterion and it is calculated for various radius ratios $\rho = r_f / r_0$, where r_0 is the radius of the initial circular orbit and r_f is the radius of the final circular orbit, and for various transfers durations $t_f - t_0$. Transfers with small and moderate amplitudes are studied and the numerical results are compared to the ones given by a linear theory expressed in non-singular orbital elements.*

Keywords*: Optimization of space trajectories, low-thrust limite-power trajectories, transfers between circular coplanar orbits.*

1. INTRODUCTION

The main purpose of this work is to present a numerical analysis of optimal low-thrust limited-power trajectories for simple transfers (no rendezvous) between circular coplanar orbits in an inverse-square force field. This analysis has been motivated by the renewed interest in the use of low-thrust propulsion systems in space missions verified in the last two decades due to the development and the successes of two space mission powered by ionic propulsion: Deep Space One and SMART 1. Several researchers have obtained numerical and analytical solutions for a number of specific initial orbits and specific thrust profiles (Kechichian, 1997; Sukhanov and Prado, 2001; Racca, 2003). Averaging methods are also used in such researches (Edelbaum, 1965; Marec and Vinh, 1977; Hassig et al, 1992).

Two idealized propulsion models have most frequently been used in the analysis of optimal space trajectories (Marec, 1979): the power-limited variable ejection velocity systems or, simply, LP systems, are characterized by a constraint concerning with the power (there exists a upper constant limit for the power); and the constant ejection velocity limited thrust systems or, simply, CEV systems, are characterized by a constraint concerning with the magnitude of the thrust acceleration which is bounded. In both cases, it is usually assumed that the thrust direction is unconstrained. In the analysis presented in the paper only LP systems are considered. The fuel consumption is taken as the performance criterion and it is calculated for various radius ratios $\rho = r_f / r_0$, where r_0 is the radius of the initial circular orbit O_0 and r_f is the radius of the final circular orbit O_f, and for various transfers durations $t_f - t_0$. The optimization problem associated to the space transfer problem is formulated as a Mayer problem of optimal control with Cartesian elements – components of position and velocity vectors – as state variables. Transfers with small and moderate amplitudes are studied and the numerical results are compared to the ones given by a linear theory expressed in non-singular orbital elements.

Two different classes of algorithms are applied in determining the optimal trajectories. They are computed through a direct approach of the trajectory optimization problem based on gradient techniques, and, through an indirect approach based on the solution of the two-point boundary value problem obtained from the set of necessary conditions for optimality. The direct approach involves a gradient-based algorithm which combines the main positive characteristics of the steepest-descent method and of a second order gradient method (Bryson, 1975). The indirect approach involves the solution of the two-point boundary value problem through two different algorithms of the neighboring extremals method. These algorithms involve the solution of a linearized two-point boundary-value problem through state transition matrix method and through a backward-sweep method based on a generalized Riccati transformation. For completeness, brief descriptions of the gradient-based and the neighboring extremals algorithms are presented in the next sections.

2. GRADIENT-BASED ALGORITHM

In this section, a brief description of a simplified gradient-based algorithm used in the analysis is presented. This algorithm has two distinct phases: in the first one, it uses a simplified version of the steepest-descent method developed for a Mayer problem of optimal control with free final state and fixed terminal times, in order to get great improvements of the performance index in the first iterates with satisfactory accuracy. In the second phase, the algorithm switches to a direct method based upon the second variation theory developed for a Bolza problem with fixed terminal times and constrained initial and final states, in order to improve the convergence as the optimal solution is approached. This

second order gradient method involves the solution of a linear two-point boundary value problem through a Riccati transformation.

2.1 STEEPEST-DESCENT METHOD

The steepest-descent method is an iterative direct procedure used for computing a *m*-vector of control variables $u(t), t_0 \le t \le t_f$, that minimizes a scalar performance index in an optimization problem. The most known version has been proposed by Bryson and Denham (1962). In this paper, a simplified version of the steepest descent method is presented for a Mayer problem of optimal control with free final state and fixed terminal times. Problems with constraints on the state variables at the final fixed time are treated by using the penalty function method. The algorithm of this simplified version is very simple, requiring a single numerical integration of the adjoint equations at each step.

The Mayer problem of optimal control is formulated as: Consider the system of differential equations:

$$\frac{dx_i}{dt} = f_i(x,u), \quad i = 1,...,n, \tag{1}$$

where *x* is an *n*-vector of state variables, *u* is an *m*-vector of control variables. It is assumed that there exist no constraints on the state or control variables. The optimal control problem consists in determining the control $u^*(t)$, that transfers the system (1) from the initial conditions:

$$x(t_0) = x_0, \tag{2}$$

to the final conditions at t_f:

$$x(t_f) - \text{free}, \tag{3}$$

and minimizes the performance index:

$$J[u] = g(x(t_f)). \tag{4}$$

The functions $f(.): R^n \times R^m \to R^n$, and $g: R^n \to R$ are assumed to be continuously differentiable with respect to their arguments. Initial and final times are specified.

The development of the algorithm is based on the classic Calculus of Variations (Gelfand and Fomin, 1963) and the step by step computing procedure used in the algorithm can be summarized as follows:

1. Choose a starting nominal control $u^0(t)$, $t_0 \le t \le t_f$, and integrate the differential equations (1) from t_0 to t_f, with the initial condition (2).

2. Integrate the adjoint equations $\frac{d\lambda}{dt} = -H_x^T$ from t_f to t_0, with the "initial" condition $\lambda(t_f) = -g_{x_f}^T$. *H* is the Hamiltonian function, $H(x,\lambda,u) = \lambda^T f(x,u)$, and, T denotes the transpose vector or matrix.

3. Compute the Lagrangian multiplier $\nu = \frac{1}{2K}\left\{\int_{t_0}^{t_f} H_u W^{-1} H_u^T dt\right\}^{1/2}$. *K* and *W* are defined through the control variation constraint $\int_{t_0}^{t_f} \delta u^T(\tau) W(\tau) \delta u(\tau) d\tau = K^2$, where $W(\tau)$ is an arbitrary, time-varying, positive-definite symmetric $m \times m$ matrix of weighting functions chosen to improve convergence of the steepest-descent method, and *K* is the step size in control space. Both, $W(\tau)$ and *K*, must be chosen by the user of the algorithm.

4. Compute the control "correction" $\delta u = \frac{1}{2\nu} W^{-1} H_u^T$, $t_0 \le t \le t_f$.

5. Obtain a new nominal control by using $u^1(t) = u^0(t) + \delta u(t)$, and repeat the process (1) through (4) until the integral $\int_{t_0}^{t_f} H_u W^{-1} H_u^T dt$ tends to zero or other convergence criterion is satisfied.

Remark 1: As the nominal control $u(t)$ approaches the optimal control $u^*(t)$, the integral $\int_{t_0}^{t_f} H_u W^{-1} H_u^T dt$ approaches zero and the Lagrangian multiplier ν tends to zero; thus the control "correction" $\delta u(t)$ can become too large and the process diverges. In order to avoid this drawback, the step size in control space *K* must be redefined.

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2.2 SECOND ORDER GRADIENT METHOD

The second order gradient method is also an iterative procedure used for computing a *m*-vector of control variables $u(t), t_0 \le t \le t_f$, that minimizes a scalar performance index in an optimization problem. In the paper, it is developed for a Bolza problem of optimal control with constrained final state and fixed terminal times. The generalized Riccati transformation (Longmuir and Bohn, 1969) is applied in solving the linear two-point boundary value problem associated to the accessory minimization problem obtained from the second variation of the augmented performance index of the original optimization problem.

The Bolza problem of optimal control is formulated as: Consider the system of differential equations:

$$\frac{dx_i}{dt} = f_i(x,u), \quad i = 1,...,n, \tag{5}$$

where *x* is an *n*-vector of state variables, *u* is an *m*-vector of control variables. It is assumed that there exist no constraints on the state or control variables. The problem consists in determining the optimal control $u^*(t)$, that transfers the system (5) from the initial conditions:

$$x(t_0) = x_0, \tag{6}$$

to the final conditions at t_f:

$$\psi(x(t_f)) = 0, \tag{7}$$

and minimizes the performance index:

$$J[u] = g(x(t_f)) + \int_{t_0}^{t_f} F(x,u)dt \ . \tag{8}$$

The functions $f(.): R^n \times R^m \to R^n$, $F(.): R^n \times R^m \to R$, $g: R^n \to R$ and $\psi: R^n \to R^q$, $q < n$, are assumed to be twice continuously differentiable with respect to their arguments. Furthermore, it is assumed that the matrix $[\partial\psi/\partial x]$ has maximum rank. Initial and final times are specified.

The second order gradient method is an extension of the steepest-descent method. The main difference is the inclusion of the second-order terms. The step by step computing procedure used in the second order gradient method can be summarized as follows:

1. Choose a starting nominal control $u^0(t), t_0 \le t \le t_f$ and integrate the differential equations (5) from t_0 to t_f, with the initial condition (6).
2. Choose a starting Lagrange multiplier μ^0 ($\mu \in R^q$).
3. Integrate the adjoint equations $\frac{d\lambda}{dt} = -H_x^T$ from t_f to t_0, with the "initial" condition $\lambda(t_f) = -\left(g_x + \mu^T \psi_x\right)^T$.
4. Compute the partial derivatives H_u, H_{uu}, H_{xu}, $H_{x\lambda}$, H_{xx} and $H_{\lambda u}$, of the Hamiltonian function $H(x,\lambda,u) = -F(x,u) + \lambda^T f(x,u)$.
5. Integrate backward, from t_f to t_0, the system of differential equations for the Riccati coefficients

$$-\dot{R} = RA + A^T R + RBR - C \quad -\dot{L} = (A^T + RB)L \quad -\dot{Q} = L^T BL \quad -\dot{s} = (A^T + RB)s + RD - E \quad -\dot{r} = L^T(D + Bs)$$

with the "initial" conditions

$$R(t_f) = -\Phi_{xx} \qquad L(t_f) = -\psi_x^T \qquad Q(t_f) = 0 \qquad s(t_f) = 0 \qquad r(t_f) = -k\,\psi$$

where $\Phi = g + \mu^T\psi$, *R* is an $n \times n$ symmetric matrix, *L* is an $n \times q$ matrix, *Q* is a $q \times q$ symmetric matrix, *s* is an $n \times 1$ matrix and *r* is a $q \times 1$ matrix, and, matrices *A*, *B*, *C*, *D*, *E* are given by

$$A = H_{\lambda x} - H_{\lambda u}H_{uu}^{-1}H_{ux} \quad B = -H_{\lambda u}H_{uu}^{-1}H_{u\lambda} \quad C = H_{xu}H_{uu}^{-1}H_{ux} - H_{xx} \quad D = -H_{\lambda u}H_{uu}^{-1}H_u \quad E = H_{xu}H_{uu}^{-1}H_u \ .$$

6. Compute $\delta\mu = -Q(t_0)^{-1} r(t_0)$.
7. Integrate the linear perturbation equation $\delta\dot{x} = (A + BR)\delta x + BL\delta\mu + Bs + D$.
8. Compute $\delta\lambda(t) = R(t)\,\delta x(t) + L(t)\,\delta\mu + s(t)$.
9. Compute $\delta u^*(t) = -H_{uu}^{-1}[H_u{}^T + H_{ux}\delta x(t) + H_{u\lambda}\delta\lambda(t)]$.
10. Compute the new control $u^1(t) = u^0(t) + \delta u(t), t_0 \le t \le t_f$, and the Lagrange multiplier $\mu^1 = \mu^0 + \delta\mu$.
11. Test the convergence. Repeat the process until it converges.

***Remark 2*:** The algorithm of the second order gradient method diverges if the Legendre condition $H_{uu} < 0$, computed for the nominal solution over the whole time interval $t_0 \le t \le t_f$, is not satisfied. By adding a term

$$\frac{1}{2}\|\delta u\|^2 = \frac{1}{2}\int_{t_0}^{t_f} \delta u^T W_2 \delta u dt ,$$

the Legendre condition can be satisfied if the $m \times m$ matrix W_2 is chosen large enough. Thus, H_{uu} must be replaced by $H_{uu} + W_2$ in the algorithm described above.

3. NEIGHBORING EXTREMALS ALGORITHMS

Consider the Bolza problem described in the preceding section, Eqns (5) – (8). The following two-point boundary value problem is obtained by applying the Pontryagin Maximum Principle (Pontryagin et al, 1962),

$$\frac{dx}{dt} = H_\lambda^T \qquad \frac{d\lambda}{dt} = -H_x^T \qquad H_u = 0 \qquad x(t_0) = x_0 \qquad \lambda^T(t_f) + g_{x_f} + \mu^T \psi_{x_f} = 0 \qquad \psi\big(x(t_f)\big) = 0 . \qquad (9)$$

All quantities in these equations are evaluated on the optimal solution.

The solution of the boundary value problem defined by (9) consists in determining the initial values of the adjoint variables $\lambda(t_0)$ and the Lagrange multipliers μ. The neighboring extremals method basically consists in iteratively determining these values and is based on two different procedures for solving the associated linearized two-point boundary value problem obtained through the linearization of Eq. (9) about an extremal solution (Breakwell et al, 1963): the state transition matrix method and backward-sweep method using a generalized Riccati transformation (Bryson, 1975; Longmuir and Bohn, 1969). The step by step computing procedure used in the neighboring extremals algorithm with the state transition matrix method, can be summarized as follows:

1. Guess the starting approximations for $\lambda(t_0)$ and μ; that is, $\lambda^0(t_0)$ and μ^0.
2. Integrate forward, from t_0 to t_f, the state and adjoint equations in Eq. (9), with the initial conditions $x(t_0) = x_0$ and $\lambda(t_0) = \lambda^0(t_0)$. The control is obtained solving $H_u = 0$.
3. Integrate forward, from t_0 to t_f, the state transition matrix differential equation

$$\frac{d\Phi}{dt} = \begin{bmatrix} A & B \\ C & -A^T \end{bmatrix}\Phi ,$$

with the initial condition $\Phi(t_0) = I$, where $A = H_{\lambda x} - H_{\lambda u}H_{uu}^{-1}H_{ux}$, $B = -H_{\lambda u}H_{uu}^{-1}H_{u\lambda}$ and $C = H_{xu}H_{uu}^{-1}H_{ux} - H_{xx}$. This step is made simultaneously with step 2.

4. Compute the matrices $T = \psi_{x_f}\Phi_{12}(t_f)$, $V = \psi_{x_f}^T$ and $U = \Phi_{22}(t_f) + \left(g_{x_f x_f}^T + \mu^T \psi_{x_f x_f}\right)\Phi_{12}(t_f)$.
5. Solve the linear algebraic system:

$$U\delta\lambda_0 + V\delta\mu = -\left(g_{x_f}^T + \psi_{x_f}^T\mu + \lambda_f\right) \qquad T\delta\lambda_0 = -\psi ,$$

and obtain the variations $\delta\lambda_0$ and $\delta\mu$.

6. Test the convergence. If it will not obtained, update the unknowns $\lambda(t_0)$ and μ; that is, compute the new values $\lambda^1(t_0) = \lambda^1(t_0) + \delta\lambda_0$ and $\mu^1 = \mu^0 + \delta\mu$.

7. Go back to step 2 and repeat the procedure until convergence is obtained.

***Remark 3*:** Only differential equations for the matrices Φ_{12} and Φ_{22} need to be integrate in step 3.

The step by step computing procedure used in the neighboring extremals algorithm with the backward-sweep method using a generalized Riccati transformation, can be summarized as follows:

1. Guess the starting approximation for $\lambda(t_0)$; that is, $\lambda^0(t_0)$.
2. Compute the control $u = u(x, \lambda)$ from $H_u = 0$.
3. Integrate forward, from t_0 to t_f , the state and adjoint equations in Eq. (9), with the initial conditions $x(t_0) = x_0$ and $\lambda(t_0) = \lambda^0(t_0)$ in order to obtain $x(t_f)$ and $\lambda(t_f)$.
4. Compute μ through the equation $\mu = -\left(\psi_x \psi_x^T\right)^{-1} \psi_x \left(\lambda(t_f) + g_x^T\right)$.
5. Integrate backward, from t_f to t_0 , the system of differential equations for the Riccati coefficients

$$-\dot{R} = RA + A^T R + RBR - C \qquad -\dot{L} = (A^T + RB)L \qquad -\dot{Q} = L^T BL$$

with the boundary conditions

$$R(t_f) = -(g_{xx} + \mu^T \psi_{xx}) \qquad L(t_f) = -\psi_x^T \qquad Q(t_f) = 0$$

and the state and adjoint equations in Eq. (9), with boundary conditions $x(t_f)$ and $\lambda(t_f)$.

6. Compute the variation $\delta\mu$ from $\delta\mu = -Q(t_0)^{-1} k\psi$.
7. Compute $\delta\lambda(t_0)$ from $\delta\lambda(t_0) = L(t_0)\delta\mu$.
8. Test the convergence. If it is not obtained, update the unknown $\lambda(t_0)$; that is, compute the new value $\lambda^1(t_0) = \lambda^0(t_0) + \delta\lambda(t_0)$.
9. Go back to step 2 and repeat the procedure until convergence is obtained.

***Remark 4*:** In order to assure the convergence of the algorithm described above, the procedure presented in Section 2.2 for the second order gradient method is applied. If the Legendre condition $H_{uu} < 0$, computed for the nominal solution over the whole time interval $t_0 \le t \le t_f$, is not satisfied we can take $H_{uu} + W_2$ in the place of H_{uu} .

Note that the neighboring extremals algorithms use different approaches to compute the Lagrangian multiplier μ.

4. OPTIMAL LOW-THRUST LIMITED POWER TRAJECTORIES

The algorithms described in the preceding section are applied in a numerical analysis of optimal low-thrust limited power trajectories for simple transfers (no rendezvous) between circular coplanar orbits in an inverse-square force field. The fuel consumption is taken as the performance criterion and it is calculated for various radius ratios $\rho = r_f / r_0$, where r_0 is the radius of the initial circular orbit O_0 and r_f is the radius of the final circular orbit O_f , and for various transfers durations $t_f - t_0$. Transfers with small and moderate amplitudes are studied and the numerical results are compared to the ones given by a linear theory expressed in non-singular orbital elements.

4.1 FORMULATION OF THE OPTIMIZATION PROBLEM

LP system is characterized by low-thrust acceleration level and a high specific impulse (Marec, 1979). The ratio between the maximum thrust acceleration and the gravity acceleration on the ground is between 10^{-4} and 10^{-2}. For LP system, the fuel consumption is described by the variable J ,

$$J = \frac{1}{2}\int_{t_0}^{t_f} \gamma^2 dt \tag{10}$$

where γ is the magnitude of the thrust acceleration vector $\boldsymbol{\gamma}$, used as control variable. The consumption variable J is a monotonic decreasing function of the mass m of the space vehicle, $J = P_{\max}\left(1/m - 1/m_0\right)$, where $P_{\max}$ is the maximum

power and m_0 is the initial mass. The minimization of the final value J_f is equivalent to the maximization of m_f or the minimization of the fuel consumption.

The optimization problem for simple transfers (no rendezvous) between coplanar orbits is formulated as: At time *t*, the state of a space vehicle *M* is defined by the radial distance *r* from the center of attraction, the radial and circumferential components of the velocity, *u* and *v*, and the fuel consumption *J*. In the two-dimensional formulation, the state equations are given by

$$\frac{du}{dt} = \frac{v^2}{r} - \frac{\mu}{r^2} + R \qquad \frac{dv}{dt} = -\frac{uv}{r} + S \qquad \frac{dr}{dt} = u \qquad \frac{dJ}{dt} = \frac{1}{2}\left(R^2 + S^2\right), \tag{11}$$

where μ is the gravitational parameter (it should not be confused with the Lagrangian multiplier in Sections 2 and 3), *R* and *S* are the radial and circumferential components of the thrust acceleration vector, respectively. The optimization problem is stated as: it is proposed to transfer a space vehicle *M* from the initial state at the time $t_0 = 0$,

$$u(0) = 0 \qquad v(0) = 1 \qquad r(0) = 1 \qquad J(0) = 0 \tag{12}$$

to the final state at the prescribed final time t_f:

$$u(t_f) = 0 \qquad v(t_f) = \sqrt{\frac{\mu}{r_f}} \qquad r(t_f) = r_f\,, \tag{13}$$

such that J_f is a minimum; that is, the performance index is

$$IP = J(t_f)\,. \tag{14}$$

For LP system, it is assumed that there are no constraints on the thrust acceleration vector (Marec, 1979). In the formulation described above, the variables are taken in canonical units.

4.2 LINEAR THEORY

A brief description of a first order analytical solution for the problem of optimal simple transfer (no rendezvous) between close quasi-circular coplanar orbits in an inverse-square force field is presented in this section. This approximate solution is expressed in non-singular orbital elements (Marec, 1979) and it is given by

$$\Delta x = A\lambda_0\,, \tag{15}$$

where $\Delta x = [\Delta\alpha \;\; \Delta h \;\; \Delta k\,]^{\mathrm{T}}$ denotes the imposed changes on non-singular orbital elements (state variables): $\alpha = a/\bar{a}$, $h = e\cos\omega$, $k = e\sin\omega$, where *a* is the semi-major axis, *e* is the eccentricity and ω is the argument of the pericenter; λ_0 is the 3 × 1 vector of initial value of the adjoint variables, and, *A* is a 3 × 3 symmetric matrix. The overbar denotes the reference orbit $\bar{O}$ about which the linearization is performed. In this first order solution, the adjoint variables associated to the non-singular elements are constant. The matrix A is given by:

$$A = \begin{bmatrix} a_{\alpha\alpha} & a_{\alpha h} & a_{\alpha k} \\ a_{h\alpha} & a_{hh} & a_{hk} \\ a_{k\alpha} & a_{kh} & a_{kk} \end{bmatrix}, \tag{16}$$

with: $a_{\alpha\alpha} = 4\sqrt{\frac{\bar{a}^5}{\mu^3}}\Delta\bar{\ell}$ $\qquad a_{\alpha h} = a_{h\alpha} = 4\sqrt{\frac{\bar{a}^5}{\mu^3}}\left(\sin\bar{\ell}_f - \sin\bar{\ell}_0\right)$ $\qquad a_{\alpha k} = a_{k\alpha} = -4\sqrt{\frac{\bar{a}^5}{\mu^3}}\left(\cos\bar{\ell}_f - \cos\bar{\ell}_0\right)$

$$a_{hh} = \sqrt{\frac{\bar{a}^5}{\mu^3}}\left[\frac{5}{2}\Delta\bar{\ell} + \frac{3}{4}\left(\sin 2\bar{\ell}_f - \sin 2\bar{\ell}_0\right)\right] \qquad a_{hk} = a_{kh} = -\frac{3}{4}\sqrt{\frac{\bar{a}^5}{\mu^3}}\left(\cos 2\bar{\ell}_f - \cos 2\bar{\ell}_0\right)$$

$$a_{kk} = \sqrt{\frac{\bar{a}^5}{\mu^3}}\left[\frac{5}{2}\Delta\bar{\ell} - \frac{3}{4}\left(\sin 2\bar{\ell}_f - \sin 2\bar{\ell}_0\right)\right], \tag{17}$$

where $\bar{\ell}_f = \bar{\ell}_0 + \bar{n}(t_f - t_0) = \bar{\ell}_0 + \Delta\bar{\ell}$ and $\bar{n} = \sqrt{\mu/\bar{a}^3}$ is the mean motion.

The optimal thrust acceleration $\boldsymbol{\gamma}^*$ and the variation of the consumption variable ΔJ during the maneuver are expressed by:

$$\boldsymbol{\gamma}^* = \frac{1}{\bar{n}\bar{a}}\left\{\left(\lambda_h \sin\bar{\ell} - \lambda_k \cos\bar{\ell}\right)\boldsymbol{e}_r + 2\left(\lambda_\alpha + \lambda_h \cos\bar{\ell} + \lambda_k \sin\bar{\ell}\right)\boldsymbol{e}_s\right\},$$

$$\Delta J = \frac{1}{2}\left\{a_{\alpha\alpha}\lambda_\alpha^2 + 2a_{\alpha h}\lambda_\alpha\lambda_h + 2a_{\alpha k}\lambda_\alpha\lambda_k + a_{hh}\lambda_h^2 + 2a_{hk}\lambda_h\lambda_k + a_{kk}\lambda_k^2\right\},$$

where a_{aa}, $a_{\alpha h,\ldots,}$ a_{kk} are given by Eqns (17), and, λ_α, λ_h, and λ_k are obtained from the solution of the linear algebraic system defined by Eq. (16); $\boldsymbol{e}_r$ and $\boldsymbol{e}_s$ are unit vectors extending along radial and circumferential directions in a moving reference frame, respectively.

The relationships between the Cartesian elements and the nonsingular orbit elements, valid for orbits with very small eccentricities, are given by

$$u = na\left(h\sin\bar{\ell} - k\cos\bar{\ell}\right) \qquad v = na\left(1 + h\cos\bar{\ell} + k\sin\bar{\ell}\right) \qquad r = \frac{a}{1 + h\cos\bar{\ell} + k\sin\bar{\ell}}. \quad (18)$$

For transfers between circular orbits, only $\Delta\alpha$ is imposed. If it is assumed that the initial and final positions of the vehicle in orbit are symmetric with respect to *x*-axis of the inertial reference system, that is, $\bar{\ell}_f = -\bar{\ell}_0 = \Delta\ell/2$, the solution of the system (15) is given by:

$$\lambda_\alpha = \frac{1}{2}\sqrt{\frac{\mu^3}{\bar{a}^5}}\left\{\frac{\Delta\alpha\left(5\Delta\bar{\ell} + 3\sin\Delta\bar{\ell}\right)}{10\Delta\bar{\ell}^2 + 6\Delta\bar{\ell}\sin\Delta\bar{\ell} - 64\sin^2\left(\Delta\bar{\ell}/2\right)}\right\} \quad \lambda_h = -\sqrt{\frac{\mu^3}{\bar{a}^5}}\left\{\frac{8\Delta\alpha\ \sin\Delta\bar{\ell}/2}{10\Delta\bar{\ell}^2 + 6\Delta\bar{\ell}\sin\Delta\bar{\ell} - 64\sin^2\left(\Delta\bar{\ell}/2\right)}\right\} \quad \lambda_k = 0.$$

We note that the linear theory is applicable only for orbits which are not separated by large radial distance. The reference orbit has been defined with a semi-major axis given by $\bar{a} = (a_0 + a_f)/2$, in order to improve the accuracy in the calculations.

5. RESULTS

The results of the numerical analysis are presented for various radius ratios $\rho = r_f / r_0$ and for various time durations $t_f - t_0$ (in canonical units). Small amplitude transfers, $\rho =$ 0.727; 0.800; 0.900; 0.950; 1.050; 1.100; 1.200; 1.523, with time durations $t_f - t_0 = 2.0$; 3.0; 4.0; 5.0; 20.0; 30.0; 40.0 and 50.0, and moderate amplitude transfers, $\rho =$ 2.500; 3.750, 5.000 and 6.250, with time durations $t_f - t_0 = 20.0$; 30.0; 40.0; and 50.0, are considered. We note that the Earth-Mars transfers corresponds to $\rho = 1.523$ and the Earth-Venus transfers corresponds to $\rho = 0.727$.

For the gradient-based algorithm, the set of parameters defined by the user and the first approximation of the control law have been chosen to get a good performance of the algorithm to each maneuver. The terminal constraints have been obtained with an error lesser than 1.0 × 10^{-7}, that is, $\left\|\psi\left(x(t_f)\right)\right\| \le 1.0\times10^{-7}$, and the performance index has been calculated with a error $e = \left|J^{n+1} - J^n\right| < 5.0\times10^{-12}$ for small amplitude transfers and $e = \left|J^{n+1} - J^n\right| < 1.0\times10^{-10}$ for moderate amplitude transfers, where *n* denotes the iteration. For the first neighboring extremals algorithm, based on the state transition matrix, the criterion adopted for convergence is a tolerance of 5.0×10^{-10} in the computation of corrections (variations) of the performance index (consumption variable). In view of this convergence criterion, the terminal constraints are obtained with an error less than 5.0 × 10^{-6}, that means $\left\|\psi\left(x(t_f)\right)\right\| \le 5.0\times10^{-6}$. For the second neighboring extremals algorithm, based on the generalized Riccati transformation, the criterion adopted for convergence is the same described for the gradient-based algorithm with $\left\|\psi\left(x(t_f)\right)\right\| \le 1.0\times10^{-7}$.

Tables 1 – 4 show the consumption variable *J* for small amplitude transfers, computed through the numerical algorithms and the linear theory, and, the relative difference between the numerical and analytical results, with $d_{rel1} = \left|\left(J_{neigh1} - J_{linear}\right)/J_{neigh1}\right|\times100\%$, $d_{rel2} = \left|\left(J_{neigh1} - J_{grad}\right)/J_{neigh1}\right|\times100\%$, $d_{rel3} = \left|\left(J_{neigh1} - J_{neigh2}\right)/J_{neigh1}\right|\times100\%$.

Table 5 show similar results for moderate amplitude transfers, taking only the ones computed through the numerical algorithms; since the linear theory is not applicable to this case. The results given by the neighboring

extremals algorithm based on the state transition matrix (denoted by number 1) have been chosen as the exact solution for each maneuver, in view of the accuracy obtained in fulfillment of the terminal constraints.

For small duration transfers, Tables 1 and 2 show that the maximum absolute relative difference d_{rel1} occur for the most of transfers with $t_f - t_0 = 5$, and it is about 2 % for $\rho > 1$ and 5.5 % for $\rho < 1$. The exception, $\rho = 1{,}5236$ and $t_f - t_0 = 3$, can be related to the choice of the parameters of the gradient-based algorithm. For moderate duration transfers, Table 3 and 4 show that the maximum absolute relative difference d_{rel1} is about 7 % for $\rho > 1$ and 3.5 % for $\rho < 1$. In both cases, the maximum absolute relative differences d_{rel1} occur for transfers with large radial excursion.The results presented in Tables 1 – 4 show that the linear theory gives a good approximation for the solution of optimal transfer problem between close circular coplanar orbits: for the small amplitude transfers with $|\rho - 1| \leq 0.100$, $d_{rel} < 0.5\%$ for the most of the maneuvers, and, for the shorter duration transfers with $t_f - t_0 = 2$ and $t_f - t_0 = 3$, $d_{rel} < 2.0\%$. In general, $7.0\,\% > d_{rel} > 1.0\,\%$ for transfers with moderate amplitude, $\rho = 0.727$ and $\rho = 1{,}5236$, corresponding to Earth-Venus and Earth-Mars transfers respectively. The results presented in the paper are better than the ones presented in da Silva Fernandes e Golfetto, 2005, in which only small duration transfers are analysed through the gradient-based algorithm. The difference between the results is due to the greater computational effort (number of iterations, accuracy of the terminal constraints) required to obtain the new results.

Table 1 – Consumption variable J ($\rho > 1$) for small duration transfers

ρ	t_f-t_0	J_{anal}	J_{grad}	J_{Neigh1}	J_{Neigh2}	d_{rel1}	d_{rel2}	d_{rel3}
1.0500	2.0	1.4463×10^{-3}	1.4463×10^{-3}	1.4459×10^{-3}	1.4459×10^{-3}	0.03	0.03	0.00
	3.0	3.4169×10^{-4}	3.4166×10^{-4}	3.4164×10^{-4}	3.4175×10^{-4}	0.02	0.01	0.03
	4.0	1.2533×10^{-4}	1.2538×10^{-4}	1.2537×10^{-4}	1.2563×10^{-4}	0.03	0.00	0.20
	5.0	6.7541×10^{-5}	6.7611×10^{-5}	6.7598×10^{-5}	6.7835×10^{-5}	0.08	0.02	0.35
1.1000	2.0	5.8778×10^{-3}	5.8741×10^{-3}	5.8716×10^{-3}	5.8716×10^{-3}	0.11	0.04	0.00
	3.0	1.3977×10^{-3}	1.3970×10^{-3}	1.3969×10^{-3}	1.3971×10^{-3}	0.06	0.00	0.01
	4.0	5.0619×10^{-4}	5.0666×10^{-4}	5.0664×10^{-4}	5.0720×10^{-4}	0.09	0.00	0.11
	5.0	2.6374×10^{-4}	2.6453×10^{-4}	2.6451×10^{-4}	2.6511×10^{-4}	0.29	0.01	0.23
1.2000	2.0	2.4187×10^{-2}	2.4097×10^{-2}	2.4097×10^{-2}	2.4097×10^{-2}	0.37	0.00	0.00
	3.0	5.8370×10^{-3}	5.8200×10^{-3}	5.8199×10^{-3}	5.8203×10^{-3}	0.29	0.00	0.01
	4.0	2.0813×10^{-3}	2.0845×10^{-3}	2.0844×10^{-3}	2.0855×10^{-3}	0.15	0.00	0.05
	5.0	1.0260×10^{-3}	1.0346×10^{-3}	1.0345×10^{-3}	1.0345×10^{-3}	0.83	0.00	0.00
1.5236	2.0	1.7743×10^{-1}	1.7434×10^{-1}	1.7434×10^{-1}	1.7433×10^{-1}	1.77	0.00	0.00
	3.0	4.4947×10^{-2}	4.4067×10^{-2}	4.4066×10^{-2}	4.4068×10^{-2}	1.99	0.00	0.00
	4.0	1.6051×10^{-2}	1.5889×10^{-2}	1.5889×10^{-2}	1.5893×10^{-2}	1.02	0.00	0.03
	5.0	7.2498×10^{-3}	7.3352×10^{-3}	7.3351×10^{-3}	7.3356×10^{-3}	1.17	0.00	0.01

Table 2 – Consumption variable J ($\rho < 1$) for small duration transfers

ρ	t_f-t_0	J_{linear}	J_{grad}	J_{Neigh1}	J_{Neigh2}	d_{re1l}	d_{rel2}	d_{rel3}
0.7270	2.0	3.7654×10^{-2}	3.7299×10^{-2}	3.7298×10^{-2}	3.7298×10^{-2}	0.95	0.00	0.00
	3.0	8.9269×10^{-3}	9.0261×10^{-3}	9.0259×10^{-3}	9.0255×10^{-3}	1.10	0.00	0.00
	4.0	4.0482×10^{-3}	4.2133×10^{-3}	4.2131×10^{-3}	4.2154×10^{-3}	3.91	0.00	0.05
	5.0	2.8941×10^{-3}	3.0573×10^{-3}	3.0572×10^{-3}	3.0646×10^{-3}	5.33	0.00	0.24
0.8000	2.0	2.0951×10^{-2}	2.0842×10^{-2}	2.0842×10^{-2}	2.0842×10^{-2}	0.52	0.00	0.00
	3.0	4.9040×10^{-3}	4.9173×10^{-3}	4.9172×10^{-3}	4.9168×10^{-3}	0.27	0.00	0.01
	4.0	2.0703×10^{-3}	2.1047×10^{-3}	2.1046×10^{-3}	2.1049×10^{-3}	1.63	0.00	0.01
	5.0	1.3838×10^{-3}	1.4198×10^{-3}	1.4197×10^{-3}	1.4227×10^{-3}	2.53	0.00	0.21
0.9000	2.0	5.4740×10^{-3}	5.4672×10^{-3}	5.4671×10^{-3}	5.4671×10^{-3}	0.13	0.00	0.00
	3.0	1.2771×10^{-3}	1.2772×10^{-3}	1.2771×10^{-3}	1.2769×10^{-3}	0.00	0.01	0.02
	4.0	5.0063×10^{-4}	5.0198×10^{-4}	5.0198×10^{-4}	5.0174×10^{-4}	0.27	0.00	0.05
	5.0	3.0496×10^{-4}	3.0653×10^{-4}	3.0652×10^{-4}	3.0685×10^{-4}	0.51	0.00	0.11
0.9500	2.0	1.3958×10^{-3}	1.3955×10^{-3}	1.3955×10^{-3}	1.3955×10^{-3}	0.02	0.00	0.00
	3.0	3.2649×10^{-4}	3.2649×10^{-4}	3.2647×10^{-4}	3.2636×10^{-4}	0.01	0.01	0.03
	4.0	1.2451×10^{-4}	1.2459×10^{-4}	1.2458×10^{-4}	1.2440×10^{-4}	0.06	0.00	0.14
	5.0	7.2585×10^{-5}	7.2671×10^{-5}	7.2667×10^{-5}	7.2647×10^{-5}	0.11	0.01	0.03

Table 3 – Consumption variable J ($\rho > 1$) for moderate duration transfers

ρ	t_f-t_0	J_{linear}	J_{grad}	J_{Neigh1}	J_{Neigh2}	d_{rel1}	d_{rel2}	d_{rel3}
1.0500	20.0	1.4520×10^{-5}	1.4536×10^{-5}	1.4533×10^{-5}	1.4531×10^{-5}	0.09	0.02	0.01
	30.0	9.7411×10^{-6}	9.7482×10^{-6}	9.7480×10^{-6}	9.7460×10^{-6}	0.07	0.00	0.02
	40.0	7.2599×10^{-6}	7.2659×10^{-6}	7.2660×10^{-6}	7.2636×10^{-6}	0.08	0.00	0.03
	50.0	5.8158×10^{-6}	5.8199×10^{-6}	5.8200×10^{-6}	5.8182×10^{-6}	0.07	0.00	0.03
1.1000	20.0	5.4007×10^{-5}	5.4168×10^{-5}	5.4167×10^{-5}	5.4165×10^{-5}	0.30	0.00	0.00
	30.0	3.6278×10^{-5}	3.6390×10^{-5}	3.6389×10^{-5}	3.6387×10^{-5}	0.30	0.00	0.01
	40.0	2.7003×10^{-5}	2.7083×10^{-5}	2.7078×10^{-5}	2.7077×10^{-5}	0.27	0.02	0.00
	50.0	2.1653×10^{-5}	2.1719×10^{-5}	2.1718×10^{-5}	2.1716×10^{-5}	0.30	0.00	0.01
1.2000	20.0	1.8980×10^{-4}	1.9172×10^{-4}	1.9154×10^{-4}	1.9154×10^{-4}	0.91	0.09	0.00
	30.0	1.2543×10^{-4}	1.2695×10^{-4}	1.2693×10^{-4}	1.2693×10^{-4}	1.18	0.02	0.00
	40.0	9.4416×10^{-5}	9.5396×10^{-5}	9.5391×10^{-5}	9.5390×10^{-5}	1.02	0.01	0.00
	50.0	7.5157×10^{-5}	7.5976×10^{-5}	7.5928×10^{-5}	7.5927×10^{-5}	1.02	0.06	0.00
1.5236	20.0	8.6591×10^{-4}	9.3232×10^{-4}	9.3151×10^{-4}	9.3158×10^{-4}	7.02	0.09	0.01
	30.0	5.7537×10^{-4}	6.1074×10^{-4}	6.1071×10^{-4}	6.1073×10^{-4}	5.79	0.00	0.00
	40.0	4.2991×10^{-4}	4.5311×10^{-4}	4.5296×10^{-4}	4.5299×10^{-4}	5.09	0.03	0.01
	50.0	3.4273×10^{-4}	3.6096×10^{-4}	3.6093×10^{-4}	3.6095×10^{-4}	5.04	0.01	0.01

Table 4 – Consumption variable J ($\rho < 1$) for moderate duration transfers

ρ	t_f-t_0	J_{linear}	J_{grad}	J_{Neigh1}	J_{Neigh2}	d_{rel1}	d_{rel2}	d_{rel3}
0.7270	20.0	7.2355×10^{-4}	7.4857×10^{-4}	7.4856×10^{-4}	7.4856×10^{-4}	3.34	0.00	0.00
	30.0	4.8236×10^{-4}	4.9863×10^{-4}	4.9862×10^{-4}	4.9862×10^{-4}	3.26	0.00	0.00
	40.0	3.6177×10^{-4}	3.7385×10^{-4}	3.7384×10^{-4}	3.7383×10^{-4}	3.22	0.00	0.00
	50.0	2.8942×10^{-4}	2.9903×10^{-4}	2.9901×10^{-4}	2.9897×10^{-4}	3.20	0.01	0.01
0.8000	20.0	3.4529×10^{-4}	3.5015×10^{-4}	3.5011×10^{-4}	3.5010×10^{-4}	1.37	0.01	0.00
	30.0	2.2973×10^{-4}	2.3313×10^{-4}	2.3316×10^{-4}	2.3311×10^{-4}	1.47	0.01	0.02
	40.0	1.7196×10^{-4}	1.7467×10^{-4}	1.7465×10^{-4}	1.7465×10^{-4}	1.54	0.01	0.00
	50.0	1.3736×10^{-4}	1.3978×10^{-4}	1.3959×10^{-4}	1.3959×10^{-4}	1.59	0.13	0.00
0.9000	20.0	7.3862×10^{-5}	7.4146×10^{-5}	7.4146×10^{-5}	7.4144×10^{-5}	0.38	0.00	0.00
	30.0	4.8663×10^{-5}	4.8851×10^{-5}	4.8852×10^{-5}	4.8850×10^{-5}	0.39	0.00	0.00
	40.0	3.6467×10^{-5}	3.6588×10^{-5}	3.6589×10^{-5}	3.6587×10^{-5}	0.33	0.00	0.00
	50.0	2.9218×10^{-5}	2.9317×10^{-5}	2.9316×10^{-5}	2.9314×10^{-5}	0.33	0.00	0.01
0.9500	20.0	1.7023×10^{-5}	1.7042×10^{-5}	1.7040×10^{-5}	1.7038×10^{-5}	0.10	0.01	0.01
	30.0	1.1240×10^{-5}	1.1251×10^{-5}	1.1249×10^{-5}	1.1247×10^{-5}	0.08	0.02	0.02
	40.0	8.4569×10^{-6}	8.4642×10^{-6}	8.4640×10^{-6}	8.4620×10^{-6}	0.08	0.00	0.02
	50.0	6.7519×10^{-6}	6.7581×10^{-6}	6.7580×10^{-6}	6.7561×10^{-6}	0.09	0.00	0.03

Table 5 – Consumption variable J ($\rho > 1$) for large transfers

ρ	t_f-t_0	J_{grad}	J_{Neigh1}	J_{Neigh2}	d_{rel1}	d_{rel2}
2.500	20.0	3.7918×10^{-3}	3.7736×10^{-3}	3.7733×10^{-3}	0.48	0.49
	30.0	2.4069×10^{-3}	2.3900×10^{-3}	2.3900×10^{-3}	0.71	0.71
	40.0	1.7684×10^{-3}	1.7541×10^{-3}	1.7541×10^{-3}	0.81	0.81
	50.0	1.3892×10^{-3}	1.3859×10^{-3}	1.3859×10^{-3}	0.24	0.24
3.750	20.0	8.1990×10^{-3}	8.1729×10^{-3}	8.1729×10^{-3}	0.32	0.32
	30.0	4.5458×10^{-3}	4.5421×10^{-3}	4.5420×10^{-3}	0.08	0.08
	40.0	3.2329×10^{-3}	3.2236×10^{-3}	3.2235×10^{-3}	0.29	0.29
	50.0	2.5307×10^{-3}	2.5257×10^{-3}	2.5256×10^{-3}	0.20	0.20
5.000	20.0	1.2883×10^{-2}	1.2839×10^{-2}	1.2837×10^{-2}	0.34	0.35
	30.0	7.0584×10^{-3}	7.0466×10^{-3}	7.0442×10^{-3}	0.17	0.20
	40.0	4.8792×10^{-3}	4.8279×10^{-3}	4.8440×10^{-3}	1.06	0.72
	50.0	3.5496×10^{-3}	3.5462×10^{-3}	3.5469×10^{-3}	0.10	0.08
6.250	20.0	1.9213×10^{-2}	1.9210×10^{-2}	1.9205×10^{-2}	0.02	0.04
	30.0	1.0358×10^{-2}	1.0336×10^{-2}	1.0306×10^{-2}	0.21	0.50
	40.0	6.3041×10^{-3}	6.2781×10^{-3}	6.2839×10^{-3}	0.41	0.32
	50.0	4.6526×10^{-3}	4.6443×10^{-3}	4.6482×10^{-3}	0.18	0.09

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6. CONCLUSIONS

A numerical analysis of optimal low-thrust limited-power transfers between circular coplanar orbits in an inverse-square force field has been performed through a gradient-based algorithm and two neighboring extremals algorithms. The results have compared to the ones obtained through a linear theory expressed in non-singular orbital elements. For small amplitude transfers; that is, transfers between close orbits, the linear theory gives a good approximation for the optimal trajectories. For moderate amplitude transfers, the numerical algorithms give very similar results.

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