VISCOUS FLOW SIMULATION THROUGH SPLIT AND BÄCKLUND TRANSFORMATIONS

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Abstract. *This work proposes a new methodology to solve the incompressible Navier-Stokes equations. It employs an analytical method based on the split of the original equations, followed by a genesis of differential equations. The problem is solved in two steps. The first step consists of applying a split on the Navier-Stokes equations, which result in a set of three inhomogeneous partial differential equations, each containing the time derivative and the viscous terms. These equations are solved through mapping them into first order ones, which provides the velocity field. Using this result we obtain a variety which defines the stream function prescribed in the genesis and, ultimately, to plot the corresponding streamlines. This sequence allows the use of portable computers to achieve simulation results for engineering problems in small enough processing time. These advantages can be put to use in problems such as unsteady turbulent flow calculations or pollution dispersion simulations in water bodies. Simulations were performed for incompressible viscous flows around a sphere for Reynolds numbers ranging from* 10^3 *to* 10^5.

Keywords*: turbulence, analytical method, Navier-Stokes, viscous flow*

1. INTRODUCTION

Obtaining solutions for the velocity field in turbulent tridimensional flows usually require very high processing times. The hybrid method proposed here is based on symmetries (Ibragimov, 1995) admitted by Partial Differential Equations (PDEs) which describe the velocity and vorticity fields around Submersed Bodies (S.B.). One of the advantages of the proposed method resides in reducing the simulation processing time compared to the usual numerical methods (Bluman and Kumei, 1989) employed. The obtained solutions are particular, although consist in varieties containing enough arbitrary elements to satisfy the restrictive conditions of many related problems.

For many years analytical methods were not employed to solve Transport Phenomena problems due to two important factors: first it is difficult, if not impossible, to obtain closed-form solutions for most nonlinear problems in this area; in second place comes the great impulse received by numerical methods from the development in computational resources during the last half century. Recent attempts to obtain closed-form solutions for the Navier-Stokes equations for internal flows, although without the use of symbolic computational solvers, were made by Lyberg (Lyberg and Tryggerson, 2007). In some geometricaly complex cases, for which computational times are still too long for most numerical methods, symbolic computational solvers regained the viability for some analytical methods where they are more convenient and more efficient if properly applied. Numerical tools like Direct Numerical Simulation (DNS) (Freire et al., 2002) are used today to generate benchmarks to validate other methods, but its use to solve engineering problems is yet at this time not viable with the computational power available to most researchers, especially if one needs to use mobile computers to make real-time decisions (White, 1991, pg.397). Some great advances in the solution of turbulent flows were made through the use of Large Eddy Simulation (LES) which, excluding transition and other complex regions, is already a viable solution as we will comment further ahead.

2. USING HYBRID METHODS

Analytical solutions for viscous flows traditionally follow this order of execution:

a') Obtain a closed-form solution for the problem;
b') Particularize the solution through the initial and classical boundary conditions.

This basic and almost obvious sequence virtually prevents obtaining solutions for most boundary value problems. This also takes for granted that the equations in their traditionally employed forms can fully represent all the physical phenomena involved. These traditional forms seem somewhat troublesome when dealing with turbulence, either by restricting their use to less complex geometrical scenarios, or by the need to combine them with turbulence models. To overcome these shortcomings LES applies Sub-Grid Models, which replace the direct solution of the physical phenomena by energy generation and dissipation assumptions. Those models allowed LES to deal with most geometrical scenarios, except a few challenging ones, but they don't really address the mathematical interpretation of the physical phenomena.

The proposed method divides the solution in two steps directly related to those classic ones previously stated:

a’’) Obtain a variety that can be applied to the targeted scenarios;

b’’) Constraint this variety through realistic initial and boundary conditions, which will not mask the mechanisms that generate the physical phenomena involved.

These particular solutions are those which can be more easily obtained through the use of symbolic computational software, but are still ample enough to describe the proposed physical scenarios. The method can be tailored to obtain solutions valid to specific regions of the flow, such as the upstream or entrance region, the detachment region, the wake region and the undisturbed region. These regions need to have their solutions bounded by auxiliary sets of equations which ensure that they and their derivatives are continuous through the interfaces. The advantage of having a small set of solutions is that we can particularize each for its specific region, but still have just a handful of regions instead of thousands of infinitesimal grid subdivisions to solve. This “analythical grid” contrasts with a typical tetrahedrical LES mesh around a sphere containing 1.2×10^6 nodes and 6.6×10^6 elements (Geuzaine).

In order to obtain our solutions for a viscous flow around a sphere we will use both Navier-Stokes and Helmholtz equations (Beck, 2005). This choice relies in the fact that the former equations still doesn’t have all its symmetries available in the literature for turbulent flows, and that the floating components of these flows can also be modeled by the solutions of the later set of equations, because they belong to the null space of the rotational operator.

3. SOLVING THE FLOW AROUND A SPHERE

We will first make a split in the equations to generate an inhomogeneous PDE set. This first step will take us to a possible solution containing arbitrary parameters and functions, which will then be used in a genesis in order to particularize it to accommodate physically consistent solutions for turbulent viscous flows.

3.1. Stream Function and Three-dimensional Velocity Field

The usual stream function definition found in literature is two-dimensional. In order to plot the streamlines around a sphere in a section of the flow, we will need to redefine the stream function for the three-dimensional case, because the velocity field renders post-processing harder to achieve. We will now define a particular stream function for our case and the correspondent representation for the velocity field.

The Continuity Equation for incompressible flows is represented by a null divergence of the velocity vector, and the trajectories defined by the isosurfaces of the stream function represent the continuous net of points where the velocity vector is tangent to it. A completely vectorial form of tridimensional stream function was implemented by Elshabka (Elshabka and Chung, 1999). Though this vectorial form is physically correct, it doesn’t help the ploting process. Instead, we will use a stream function which has its amplitude defined by a vectorial field obtained from a scalar function ψ, given by $F(\psi,\psi,\psi)$. This function has fixed direction, but allows us to easily reproduce the vortex structures. Let’s make an axial vector F represent the velocity vector’s solenoidal portion, the potential portion being given by the polar gradient of *ø*. The complete definition of the velocity vector is

$$\underset{\sim}{V} = \nabla \times F(\psi,\psi,\psi) + \nabla\phi \qquad (1)$$

which implicitly satisfies the equation of continuity and produces the cartesian velocity components in the form

$$\underset{\sim}{V} = \begin{pmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \psi & \psi & \psi \end{pmatrix} + \nabla\phi = \begin{cases} u = \dfrac{\partial\phi}{\partial x} + \dfrac{\partial\psi}{\partial y} - \dfrac{\partial\psi}{\partial z} \\ v = \dfrac{\partial\psi}{\partial z} + \dfrac{\partial\phi}{\partial y} - \dfrac{\partial\psi}{\partial x} \\ w = \dfrac{\partial\psi}{\partial x} - \dfrac{\partial\psi}{\partial y} + \dfrac{\partial\phi}{\partial z} \end{cases} \qquad (2)$$

Using the definition $\underset{\sim}{\omega} = \nabla \times \underset{\sim}{V}$, we obtain the cartesian vorticity components, where the potential terms are, by definition, eliminated. The most important difference is the treatment of the stream function as a scalar field, through which we generate a vector one. This choice is not the only possible representation, but it allows physically realistic conditions to be implemented and helps to solve the problem at the same time.

3.2. Split and Genesis of PDE’S

A genesis of PDE’s will now be apllied to solve the problem through the use of a mathematical condition which part of the equation will have to respect. Let’s start by analyzing the Navier-Stokes and Hemholtz equations’ structures. By conveniently arranging the equations' terms in vectorial notation, we obtain for the Navier-Stokes equations the following form

$$\frac{\partial \underset{\sim}{V}}{\partial t} - \upsilon \nabla^2 \underset{\sim}{V} = -\underset{\sim}{V}.\nabla \underset{\sim}{V} - \frac{1}{\rho} \nabla p \tag{3}$$

and for the Helmholtz equations

$$\frac{\partial \underset{\sim}{\omega}}{\partial t} - \upsilon \nabla^2 \underset{\sim}{\omega} = -\underset{\sim}{V} \nabla \underset{\sim}{\omega} - \underset{\sim}{\omega}.\nabla \underset{\sim}{V} \tag{4}$$

Both equations can be expressed using the form

$$\frac{\partial \underset{\sim}{F}}{\partial t} - \upsilon \nabla^2 \underset{\sim}{F} = N \tag{5}$$

where N represents the remaindered part of each equation, and the vector function $\underset{\sim}{F}$ stands for the *velocity field* in Navier-Stokes equations and to the *vorticity vector* in Helmholtz equations. The linear operator appearing in the left hand side of (5) will from now on be called A, whose formal definition is given by

$$A = \frac{\partial (\bullet)}{\partial t} - \upsilon \nabla^2 (\bullet) \tag{6}$$

The mathematical property that allows us to make a *split* is that the linear portion belongs to the null space of the divergent operator, or

$$\nabla . \left(\frac{\partial \underset{\sim}{F}}{\partial t} - \upsilon \nabla^2 \underset{\sim}{F} \right) = 0 \tag{7}$$

This also means that $\nabla.(N) = 0$, which shows that N results from the the rotational operator being applied over a vector field $\underset{\sim}{r}$, such as

$$N = \nabla \times \underset{\sim}{r} \tag{8}$$

where a potential field which is usually added to the solenoidal part was intentionally omitted. The *split* will result, both for Navier-Stokes and for Helmholtz equations, in a system expressed in the form

$$\begin{cases} A[\underset{\sim}{F}] = Q \\ N = Q \end{cases} \tag{9}$$

where Q is, for the time being, an unknown source. Now we can start to solve the linear portion of the *split*, which will be made easier by using a differential operator capable of mapping exact solutions into new and more ample solutions of this equation. This operator, which we will call B, when applied over a vector function F_k will transform it into another vector function F_{k+1} that also belongs to the space of solutions of $A[F]=Q$. This means that the operator B adds to our system the auxiliary equation

$$A\left[B[\underset{\sim}{F}]\right] = Q \tag{10}$$

Hence, operator B induces auto-Bäcklund transformations over exact solutions of $A[F]=Q$. For sake of simplicity, the structure of this operator is prescribed as

$$B = a\frac{\partial (\bullet)}{\partial x} + b\frac{\partial (\bullet)}{\partial y} + c\frac{\partial (\bullet)}{\partial z} + g\frac{\partial (\bullet)}{\partial t} + hI \tag{11}$$

where *a, b, c, g* and *h* are linear functions in *x, y, z* and *t*, being *I* the identity operator. Next we apply operator *B* over Eq.(10) in order to obtain its functions *a, b, c, g* and *h* from the differential equations resulting from function $\underset{\sim}{F}$ derivatives' coefficients, as will be explained next.

3.3. Operator *B* and source *Q* models

The linear equation of system (9) is given by *A*[*f*]=*Q*, being *Q* an arbitrary vector field that obeys to the Continuity Equation. Isolating the time derivative, the equation appears in the form

$$\frac{\partial f}{\partial t} = -\upsilon\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) + Q \tag{12}$$

Replacing the correspondent expression in *B*[*f*] the time derivative is eliminated before we substitute the resulting expression in the auxiliary equation *A*[*B*[*f*]]=*Q*.

An equivalent way of simultaneously imposing the restrictions *A*[*f*]=*Q* and *A*[*B*[*f*]]=*Q* resides in applying the comutativity condition over operators *A* and *B*. This implies that *B* operates Symmetries admissible by *A*[*f*]=*Q*, in other words, it transforms previous exact solutions in new exact solutions of this equation. Besides that, expanding *A*[*B*[*f*]]=*Q* makes clear that the time derivatives of *f(x,y,z,t)* are null. By making each individual coefficient of the function derivatives null, we are able to keep *f* as an arbitrary function. Once we apply over *A*[*f*]-*Q=0* and *A*[*B*[*f*]]-*Q=0* all the restrictions from the coefficients, all auxiliary equations are identically satisfied, remaining only the following differential restriction for the source

$$\left(A_{0000} + A_{0001}t + B_{000}y\right)\frac{\partial Q}{\partial x} + \left(B_{0000} + B_{0001}t + B_{000}x + B_{003}z\right)\frac{\partial Q}{\partial y} + \left(C_{002} + B_{003}y\right)\frac{\partial Q}{\partial z} + \left(\frac{A_{0001}x + B_{0001}y}{2\upsilon} - 1\right)Q = 0 \tag{13}$$

which is equivalent to the fixed-point equation *B*[*Q*]=*Q*. Solving this equation using the method of characteristics we obtain for *Q* two arguments $\mu 1$ and $\mu 2$. To use a convenient notation we can rename the remaining constants thus: $A_{0000}=A_0$, $A_{0001}=A_1$, $B_{000}=B_0$, $B_{0001}=B_1$, $B_{003}=B_2$,$B_{0000}=B_3$ e $C_{002}=C_0$, expressing source *Q* in this form:

$$Q = g\left(A_0 y + A_1 t y - \left(\frac{\left(x^2+y^2\right)B_0}{2}\right) - B_3 x - B_1 t x - B_2 x z, \left(A_0 + A_1 t - B_0 y\right)z - \left(C_0 - B_2 y\right)x\right). e^{\left(2\left(-k_{42} - \frac{A_1 x y + \frac{B_1 y^2}{2}}{2\upsilon}\right)\upsilon\left(-C_0 + B_2 y\right)A_1\left(A_1 x + B_1 y\right)\right)} \tag{14}$$

Since the structure of the source *Q* is also valid for the unknown funtion, its form must be valid for every function *f* that satisfies the *split*. This means that it can be used to express the solenoidal portion of the velocity field, as well as the vorticity field.

3.4. Obtaining the Stream Function from the Velocity Field

The form obtained in (14) posesses enough arbitrary constants and functions to allow us to impose all restrictive conditions we need. Though manipulating it through symbolic solution software is more convenient using a more suitable generic variety, such as

$$V_N = g_N(\alpha_N, \beta_N)\, e^{h(x,y,z,t)} \tag{15}$$

where the index *N* is used to show that each of the velocity vector components depends upon a different funtion $g(\alpha,\beta)$ distinct from those of the other components. Function *h(x,y,z,t)*, however, is the same for all three cartesian components of velocity. This comes from the fact that the velocity field variety is equally valid for the vorticity field. If we try to obtain the vorticity field by applying the rotational over the velocity field, we will conclude that, if function *h(x,y,z,t)* was different for each velocity component, we would have a vorticity field variety with many terms instead of the single-termed form previously obtained. Otherwise, if the exponential function *h(x,y,z,t)* is identical for all velocity components, the form for the *x* vorticity component would be given by

$$\omega_1 = \left(\frac{\partial g_3}{\partial y} + g_3 \frac{\partial h}{\partial y} - \frac{\partial g_2}{\partial z} - g_2 \frac{\partial h}{\partial z} \right) e^h \tag{16}$$

where the sum between parenthesis can be regarded as a new function $i_N\ (g_{N+1},, g_{N+2},,g'_{N+1},,g'_{N+2},,h')$, corresponding now to the generic form previously prescribed. Next, we force the stream function isosurfaces to match the local velocity field tangents, by imposing that its material derivative is null

$$\frac{D\psi}{Dt} = \frac{\partial \psi}{\partial t} + V_1 \frac{\partial \psi}{\partial x} + V_2 \frac{\partial \psi}{\partial y} + V_3 \frac{\partial \psi}{\partial z} = 0 \tag{17}$$

Expression (17) defines a hipersurface over which the stream function space and time variations compensate each other mutually, keeping its value constant. In order to the stream function be an arbitrary function of two arguments $\eta\ (x,y,z,t)$ and $\theta\ (x,y,z,t)$ we should zero all the coefficients of its derivatives in Eq.(17). Solving the resulting system we reach a relationship between these two arguments of the stream function. To do so, first we replace the velocity components by a model based on Eq.(15), as well as the stream function by $\psi\ (\eta\ ,\theta)$, and then aplying the chain rule to recast its derivatives, yielding the following expression:

$$\begin{aligned}
&\frac{D\psi(\eta,\theta)}{Dt} = \frac{\partial \eta(x,y,z,t)}{\partial t}\frac{\partial \psi(\eta,\theta)}{\partial \eta} + \frac{\partial \theta(x,y,z,t)}{\partial t}\frac{\partial \psi(\eta,\theta)}{\partial \theta} + \\
&\left(g_1(\alpha_1,\beta_1)h(x,y,z,t)\right)\left(\frac{\partial \eta(x,y,z,t)}{\partial x}\frac{\partial \psi(\eta,\theta)}{\partial \eta} + \frac{\partial \theta(x,y,z,t)}{\partial x}\frac{\partial \psi(\eta,\theta)}{\partial \theta}\right) + \\
&\left(g_2(\alpha_2,\beta_2)h\right)(x,y,z,t)\left(\frac{\partial \eta(x,y,z,t)}{\partial y}\frac{\partial \psi(\eta,\theta)}{\partial \eta} + \frac{\partial \theta(x,y,z,t)}{\partial y}\frac{\partial \psi(\eta,\theta)}{\partial \theta}\right) + \\
&\left(g_3(\alpha_3,\beta_3)h(x,y,z,t)\right)\left(\frac{\partial \eta(x,y,z,t)}{\partial z}\frac{\partial \psi(\eta,\theta)}{\partial \eta} + \frac{\partial \theta(x,y,z,t)}{\partial z}\frac{\partial \psi(\eta,\theta)}{\partial \theta}\right) = 0
\end{aligned} \tag{18}$$

The system to be solved includes two aditional equations. The first is the continuity equation for incompressible flows, which implies that the divergence of the velocity vector is null:

$$\nabla \bullet \underset{\sim}{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} = 0 \tag{19}$$

The second is the condition of independence between both arguments of the stream function:

$$\nabla \eta \bullet \nabla \theta = \frac{\partial \eta}{\partial x}\frac{\partial \theta}{\partial x} + \frac{\partial \eta}{\partial y}\frac{\partial \theta}{\partial y} + \frac{\partial \eta}{\partial z}\frac{\partial \theta}{\partial z} = 0 \tag{20}$$

This restriction is necessary to ensure that the stream function really depends on two variables. By equaling each individual derivative coefficient to zero in (18) we obtain an auxiliary system to solve. We start by isolating one of the velocity components in one of the coefficient equations and replacing the equivalent expression in the other two velocity equations. These definitions will then be applied in Eq.(19), cancelling the derivative coefficients of g_1, g_2 and g_3 in order to keep them arbitrary. Simplifying the resulting system we obtain the following relations between the derivatives of η and θ

$$\frac{\partial \theta}{\partial y} = \frac{\partial \eta}{\partial y}\frac{\partial \theta}{\partial x}\Big/\frac{\partial \eta}{\partial x} \quad \text{and} \quad \frac{\partial \theta}{\partial z} = \frac{\partial \eta}{\partial z}\frac{\partial \theta}{\partial x}\Big/\frac{\partial \eta}{\partial x} \tag{21}$$

If we isolate the time derivative of θ in (18), replacing the resulting expression and relations (21) in Eq.(19), this last equation is automaticaly satisfied, leaving only relations (21) to restrict the stream function arguments final form. It is worth noticing that these remaining restrictions depend only upon the velocity components having the form expressed in (15) and not forcibly those of (14), so in the following deduction *V* will not be restricted to the form initially obtained for the source *Q*.

Now, using relations (21), we can set an arbitrary form for one of the arguments in order to obtain the other, and then we can apply the boundary conditions. If we would like to do both steps together it would be convenient to arbitrate a form for η and θ which could work as ortogonal coordinates in a space with axes η, θ and $\psi(\eta,\theta)$. The most convenient way to apply the boundary conditions would contemplate η and θ that could express as a set of coordinates which

a) for constant values of ψ reproduce hypersurfaces parallel to the S.B. like those in a potential flow;

b) reflected the way the disturbances propagate from the surface, passing the boundary layer and beyond that through the wake towards the undisturbed flow.

Both objectives can be achieved arbitrating one of the arguments, for example η, in this form

$$\eta(x,y,z) = R_T(x,y,z)(1 - f_C(x,y,z)) \tag{22}$$

where $f_C(x,y,z)$, which we will call from now on the *contour function*, serves to describe the position of each point relatively to the S.B. boundary position. Specifying $f_C(x,y,z)$ in such way that it will be unitary where $\eta=0$, that would be over the isosurface where the stream function ψ is null and over the entire S.B. surface. The higher than the unit the value of η becomes, the farther the point is located from the surface. Function R_T is, in the region the flow passes by the S.B., a *transversal reference relative to the S.B. surface*. In the region before and after the S.B. R_T is refers to the line or symmetry plane of the main flow direction. For a sphere, centered in the origin and with the main flow in the direction of the *X* axis, in its surroundings $R_T = \sqrt{x^2+y^2+z^2} - R_0$, which is the distance from the point to the surface, and in the rest of the flow $R_T = \sqrt{y^2+z^2}$, representing the distance from the point to the *X* axis. When we apply (22) to the relations (21), we easily obtain a solution in (symbolic software) command line to the form of θ, which is

$$\theta(x,y,z,t) = g1(R_T(x,y,z),(1 - f_C(x,y,z)),t) \tag{23}$$

This way η possesses a function $f_C(x,y,z)$ which allows us to change the S.B. geometry as necessary, and θ becomes an arbitrary time dependent function capable of reproduce other factors related to diffusion and turbulent disturbances.

3.5. Propagation of Disturbances

As we previously stipulated, both stream function arguments should have such forms that will allow us to express turbulence generating disturbances, as well as their propagation through the body of the flow. As soon as we eliminate η time dependence, we have transfered to θ the need to express the flow disturbances. To do so we will consider that the S.B. roughness can be expressed as a senoidal function. The *formal solution* of a differential equation such as

$$\frac{\partial f}{\partial t} = A\,f \tag{24}$$

where $f(r_1,...,r_n,t)$ is a function of space and time and *A* is a differential operator, exists and is

$$f = \left[e^{tA}\right] f_0 \tag{25}$$

where f_0 represents in function the initial instant $t=0$. To obtain an explicit solution, we expand the exponential of the differential operator in a Taylor series

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \tag{26}$$

Let operator *A* be such that it transforms (24) in an advective-diffusive equation, for simplicity sake unidimensional, like a Burgers-type equation, given by

$$\frac{\partial f}{\partial t} = f\,\frac{\partial f}{\partial x} + \upsilon \frac{\partial^2 f}{\partial x^2} \tag{27}$$

and the unknown function f be a senoidal disturbance of velocity such as $\delta V = \varepsilon\ sen(\phi\, x)$. The first terms of the series in the explicit solution would be

$$SF_{pert.} = \left(\varepsilon\, sen(\phi x)\right) + \upsilon\left(\varepsilon\phi^2\, sen(\phi x) + \varepsilon\phi\, sen(\phi x)\cos(\phi x)\right) + ... \tag{28}$$

The first term simply reproduces the disturbance, but the second term shows the effect which both advective and diffusive terms of the equation exerts over it. The first part of the second term represents the diffusion of the disturbance, which depends explicitly on the relation υ/ϕ^2. Since the order of magnitude of the viscosity for liquids is 10^{-6}[m^2/s], the higher frequencies, which have wave numbers ϕ smaller than 10^{-3}m, will be amplified by this term through most regions of the turbulent wake. The second part of this term represents the advection of the disturbance, but it also generates a new fluctuation with wave a number that is *twice* that of the original one. It is easy to recognize that

$$\varepsilon\phi\, sen(\phi x)\cos(\phi x) = \frac{\varepsilon\phi\, sen(2\phi x)}{2} \tag{29}$$

which means that the new disturbance has twice the original wave number and its amplitude multiplied by $\phi\,/\,2$. If this factor is higher than one, there will be amplification of this new disturbance. It is important to notice that, since viscosity doesn't change, the relation υ/ϕ^2 makes each advective part of the next term of the series geometricaly amplify the harmonic generated by the previous one. When any method *linearizes* some equation term, it is not only approximating the solution, but also eliminating the additional harmonics generated by it. This happens because a linear PDE formal solution is only capable to amplify or dampen the disturbance with the original wave number, without producing new disturbances with larger wave numbers.

Thinking about physical consistency, terms that generate disturbances with wave numbers smaller than the molecular scale of the fluid are not realistic, and the series must be truncated before them. That is why it is important to observe that *one should only apply a magnitude order analysis to eliminate terms of the solution, but never to neglect terms of the corresponding differential equation*. These arguments allows us to conclude that, when we simulate flows around boundaries of submersed bodies, if we reject the influence of some term due to its small order of magnitude we will probably be eliminating the geration, propagation and amplification of high frequency disturbances from this S.B. towards the bulk of the flow. Classical analysis based on the Reynolds Averaged Navier-Stokes equations (RANS) (Schlichting, 1979, pg. 450) usually make this mistake by eliminating quadratic disturbance terms, but at the same time they recognize the existence of high frequency disturbances through experimental observation. The immediate consequence of these simplifying attempts, caused by the elimination of some of the disturbance-generating terms, induced these same researchers to look for some origin of flow instability in the upstream region and even in the undisturbed flow (Schlichting, 1979, pg. 400). Those possible sources must not be underrated, but their inexistence should not prevent an eventual transition from laminar to turbulent flow inside the boundary layer. If we have the physical mechanisms responsible for generating turbulence present in our formulation, we should be able to simulate any flow from its initial undisturbed state all the way to turbulent flow. As important as this, is to notice that the classical boundary conditions are equally inadequate because they are not capable to reflect any mechanisms from dimensions close to molecular scale. Spacial and time averages, represented by second-kind boundary conditions are valid aproximations only for macroscopic phenomena, which are restricted by the continuum hypothesis and exclude some of the mechanisms responsible for the onset of turbulence. It would equally incoherent to consider that a small layer of fluid particles was attached to the boundary, thus respecting the classic conditions, because there is no plausible justification that forbids a second layer to slip over the first one: or both layers have some measure of slip or both are solidary to the boundary. The second hypothesis would not allow any flow to occur, so we are logically driven to admit that *physically there cannot exist a no-slip condition*, but *there must exist a partial slip condition* (Panton, 1996, pg.142) described by a boundary condition of third-kind.

Resuming all previous considerations, we can state that classical boundary conditions are not physically coherent with microscale-related phenomena. They can only be considered time and spatial coarsed-grained averages of the real boundary conditions, and their importance diminishes in the same measure as we approach molecular scale phenomena. Since turbulence involves such phenomena, we should naturally apply an adequate stream function definition at the boundaries. To do so, we will translate boundary roughness as a summation of senoidal components, which will generate a series of disturbances proporcional to the fluctuations of the velocity components over the S.B..

3.6. Restrictive Conditions in Turbulent Flows

A third-kind boundary condition simply makes the function derivatives proportional to the function itself. The less restrictive way to apply this condition is a differential constraint over the function, because it doesn't need to be associated solely to the boundary, but it can also reflect the way the physical phenomena generated in the boundary propagate throughout space. In this particular form it can represent the way a specific phenomenon relates to a specific flow variable, whether it is one axis, the time or even an auxiliary variable such as η or θ. If such condition is related for example to the *X* axis, it would have a form like this

$$\frac{\partial \psi(x,y,z,t)}{\partial x} = f_1(x,y,z,t).\psi \tag{30}$$

where f_1 is a function which depends upon all arguments in order that ψ won't be forcibly linear in *x*. Integrating Eq. (30) generates a stream function in the form

$$\psi(x,y,z,t) = g_1(y,z,t).e^{\int f_1(x,y,z,t)dx} \tag{31}$$

which we can restrict by stablishing asymptotic conditions for great distances, over the boundary, in the initial moment, and after the flow is fully developed. Observing (31) we can see that function g_1 depends upon all function arguments except those over which the differential restriction was imposed. In order to its form be general enough to fulfill the above requirements, we can say that function g_1, as well as the argument of the exponential, should be a sum of various terms. Each term will be related to some possible disturbance propagation or dampening mechanism, whether in the spatial variables or in time. So we will apply a differential restriction over η in the form of Eq. (30), making the disturbance at $t=0$ to be null. Since we have imposed a senoidal form disturbance, function $g_1(\theta)$ will be a sine whose argument vanishes for $t=0$. For the sake of simplicity it will be multiplied by *t*, or making θ such as

$$\theta = c_1\, t\, .\, g_2(R_T(x,y,z),(1-f_C(x,y,z))) \tag{32}$$

Over the boundary the exponential must be equal to one in order to $\eta=0$, so a form capable of satisfying both conditions would result in a stream function like

$$\psi = U_\infty\, e^{(f_2(\eta,\theta).\eta)} sen(g_3(\theta)) \tag{33}$$

where function f_2 in the argument of the exponential is the generic form of the integral of f_1 seen in (31), being valid for the downstream region of the flow. For the undisturbed and upstream regions the stream function is equivalent to $U_\infty.\eta$, which is the usual form of expressing it in two dimensions.

To reflect the disturbance dampening, both in the main flow direction and transversally, other terms are added to the argument of the exponential. This same stream function form can be obtained through a microscale physical analysis of the fluid behaviour close to the S.B. surface. This behaviour is a summation of the disturbances generated by each point of the surface, whose influence can be obtained through a convolution. The resulting form for the stream function would be

$$\psi = U_\infty \eta + g_1(\theta)\, sen(g_3(\theta))\, e^{\left(-e^{-C_3 R_L} - e^{C_1(R_L - C_4)} - \frac{\eta^2}{\ln\left(1+e^{C_2(\theta)}\right)}\right)} \tag{34}$$

where R_L is a *longitudinal reference* in the flow direction obtained in the same way we did for R_T , and coefficients C_1 to C_4 can be constants or depend upon the S.B. dimensions or velocity components in the undisturbed flow U_∞. Function g_1 represents the contribuitions of all other variables which are not included in the senoidal disturbance.

All the restrictive conditions applied so far still allow us to have θ as an arbitrary function, so we chose the following simple polinomial form for θ

$$\theta = \frac{U_\infty}{\sqrt{\upsilon}\, R_P} t \left[-\left(c_0 + R_T(1-f_C)\right) + \frac{\left(c_0 + R_T(1-f_C)\right)^2}{2} - \ldots \right] \tag{35}$$

where c_0 is an arbitrary constant and the fraction to its left is a proporcionality factor multiplied by time. This factor takes into consideration many physical features that can have influence over the disturbance, which increases with velocity, is inversely proportional to its distance to the S.B. surface, and is also dampened by fluid viscosity. We obtained results truncating this series in the first, second and third terms, but there was no perceptible difference between these three attempts, showing that the series converges very quickly. So in our simulations we used only the first term of the series.

4. RESULTS

The following plottings were obtained for a water flow with main direction over the *X* axis, using a unitary radius 3D sphere, in a section located over the *XY* plane. The streamlines resulting from the intersection of the cutting plane with the hypersurfaces of the stream function ψ clearly display the vortices generated in the surface of the sphere.

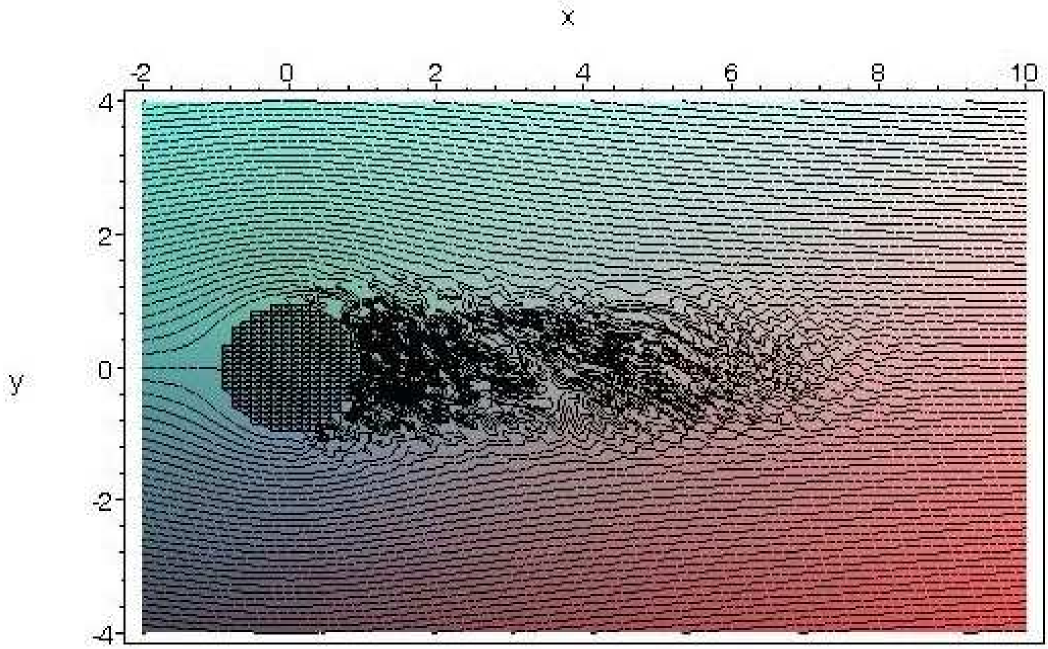

Figure 3.1 - flow around a sphere for *Re* =10,000

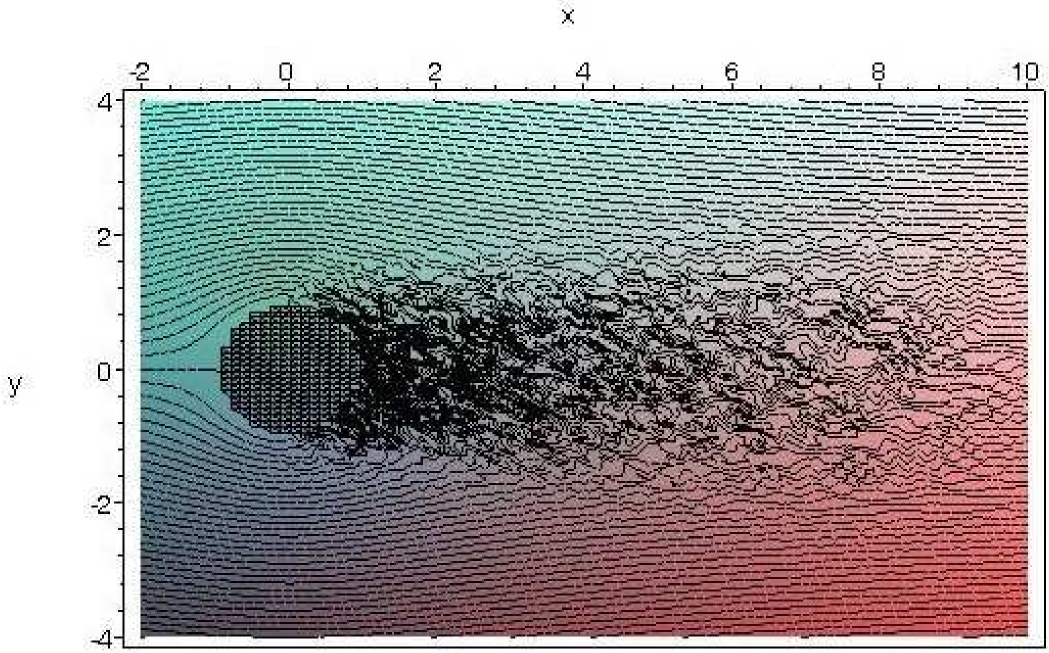

Figure 3.2 – flow around a sphere for *Re*=100,000

Qualitative characteristics of the wake, such as the widening of the wake angle around the body, as well as the increase in length downstream as *Re* increases (Landau and Lifshitz, 1987, pg.152), can be very clearly observed. Flow simulations with Reynolds numbers ranging from 1,000 to 100,000 reflect the evolution from alternating vortices to turbulent flow, showing that the model has good perspectives in flow analysis over a very wide range of velocity values. The more homogeneous distribution of small dimension vortices, which is the effect of the *vortex stretching* term in Helmholtz Equation, can be clearly seen in the simulation obtained for *Re*=100,000 of Fig. 3.2, which shows the flow approaching the transition from laminar to turbulent flow. Further development of the method will allow the simulation of flows with higher Reynolds numbers.

5. CONCLUSIONS

We have shown here that small flow disturbances capable of generating turbulence can be justified through a physically coherent interpretation of the phenomena which occour between the fluid and the S.B. surface. By associating this interpretation with the proposed method, and applying it over Navier-Stokes and Helmholtz equations, we obtained qualitatively realistic simulations for viscous incompressible flows.

The relative simplicity of the simulation algorithms allowed us to obtain stream function plottings in an average personal computer, in very small execution times when compared to those demanded by numerical simulations. The hardware used had an AMD Opteron 165 dual core 1.8 GHz processor, using a total of 2 GB DDR-400 memory, resulting in processing times varying between 25 seconds and 5 minutes. Equivalent hardware is today available at reasonably low cost portable computers. The hardware portability and the small enough time necessary to perform simulations are compatible with real-time decision making, both desirable qualities to solve engineering problems.

All the adequation of coefficients and arbitrary functions, as well as further adjusting the disturbance function, must be done taking into account experimentally observed disturbance frequencies. These improvements will push this method towards a more faithful representation of the disturbances generated by a S.B. over the flow, without the need to sacrifice its features, e.g., the computational performance and its easiness of implementation.

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7. RESPONSIBILITY NOTICE

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