COB09 - 0684 - EVAPORATION EFFECTS IN RAREFIED GAS FLOWS

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Abstract. *In this work we consider a semi-infinite expanse of a rarefied gas bounded by its plane condensed phase on which evaporation takes place. The analysis is based on the BGK model derived from the Boltzmann equation. In particular, the strong evaporation problem is considered, where nonlinear aspects have to be taken into account. We present the complete development of a closed form solution for evaluating density and temperature perturbations. Numerical results are presented and discussed.*

Keywords: *Rarefied gas dynamics, strong evaporation, kinetic model, discrete ordinates method*

1. INTRODUCTION

Investigations on the kinetics of vapors close to inter-phase boundaries have been carried out over the last years (Aoki and Masukawa, 1994; Sone, Takata and Golse, 2001; Frezzotti and Ytrehus, 2006; Yano, 2008) due to the large number of problems of practical interest for which evaporation and condensation phenomena are relevant. As pointed out in Ytrehus (1976) work, problems of this type are encountered in several areas as: upper atmosphere meteorology, the sodium cooling of nuclear reactors, design of spacecraft experiments, petrochemical engineering, vacuum technology and the interaction of high-power laser radiation with metal surfaces. In addition, the evaporation problem is of theoretical interest, because it defines a general Knudsen layer problem in which non continuum boundary conditions are essential and where significant changes are associated with all the gas variables: density, velocity and temperature. For instance, Pao (1971b) studied the temperature and density jumps used to define boundary conditions for the fluid-dynamic set of equations when the gas is in contact with a condensed phase (Yasuda, Takata and Aoki, 2005).

When the evaporation/condensation is weak, according to the literature (Yasuda, Takata and Aoki, 2005), the problem of describing a rarefied gas flow can be treated by a linearized version of the Boltzmann equation. On the other hand, when evaporation or condensation is strong, either the nonlinear Boltzmann equation (Williams, 1971; Cercignani, 1988) or nonlinear models should be used to describe the problem. Because of the complexity of the models involved, studies have mostly been devoted to versions which are of the relevance to cases of weak evaporation and condensation only (Pao, 1971a; Siewert and Thomas Jr., 1973; Thomas Jr., Chang and Siewert, 1974; Thomas Jr. and Valougeorgis, 1985; Sone, Ohwada and Aoki, 1989; Loyalka, 1991; Siewert, 2003; Yasuda, Takata and Aoki, 2005; Scherer and Barichello, 2009).

In regard to the strong evaporation problem, in general, the distribution function, in the original nonlinear equation, is linearized around a downstream Maxwell distribution containing a drift velocity v_∞. This procedure is described in Ref. (Arthur and Cercignani, 1980) for the BGK model (Bhatnagar, Gross and Krook, 1954) with one degree of freedom. Still, in Ref. (Arthur and Cercignani, 1980) it is also showed the existence of a critical value of the drift velocity v_∞ for which the linearized version of the strong evaporation problem has solution. Numerical results for the linearized version of the strong evaporation problem can be found in Refs. (Siewert and Thomas Jr., 1981; Loyalka, Siewert and Thomas Jr., 1981; Siewert and Thomas Jr., 1982) where the method of elementary solutions and the F_N method are used. Nonlinear aspects are considered in Refs. (Ytrehus, 1976) and (Sone, Takata and Golse, 2001).

In this work, the ADO method (Barichello and Siewert, 1999) is used to solve the strong evaporation problem. Recently, in a concise way, this method has been used to solve in an unified manner a class of flow (Scherer, Prolo Filho and Barichello, 2009a) and heat-tranfer problems (Scherer, Prolo Filho and Barichello, 2009b) in rarefied gas dynamics. Here, firstly, the linearized version of the problem is solved to evaluate density, velocity and temperature perturbations of the gas. Then, the analytical ADO solution is used in the original nonlinear version of the model, in order to get a second set of results for the quantities of interest. Numerical results are presented for both approaches.

2. THE KINETIC MODEL

We consider here, the steady-state limit of the following dimensional, time dependent problem, as proposed by Ytrehus (1976): a liquid (or solid) is initially in equilibrium with its pure vapor which occupies the half-space $x \geq 0$ at uniform temperature and pressure T_0 and p_0, respectively. At time $t = 0$ the pressure level in the vapor changes discontinuously to the value p_∞ and it is kept at this (constant) value. Then, evaporation or condensation begins, through the phase boundary, according to whether the pressure level p_∞ is below or above the saturation pressure p_0, respectively.

Continuing to follow Ytrehus (1976) analysis, it is reasonable to assume that far downstream of the boundary, after a sufficiently long time, a steady state will be accomplished. The flow far from the phase boundary is then a uniform equilibrium flow with constant parameters ϱ_∞, v_∞ and T_∞. In this way, a kinetic boundary layer will form between the phase boundary and the downstream equilibrium region, in which, nonequilibrium effects may influence significantly the motion of the vapor (Ytrehus, 1976).

In this context, we describe here the state of gas by the nonlinear BGK model (Siewert and Thomas Jr., 1981), with one degree of freedom, which can be written as

$$\xi\frac{\partial}{\partial x}f(x,\xi)=\eta[\phi(x,\xi)-f(x,\xi)], \tag{1}$$

where $f(x,\xi)$ is the distribution function, ξ is the molecular velocity in the x direction, η is an appropriate collision frequency, $\phi(x,\xi)$ is a local Maxwell distribution,

$$\phi(x,\xi)=\frac{\varrho(x)}{\sqrt{2\pi RT(x)}}\exp\left\{-\frac{[\xi-v(x)]^2}{2RT(x)}\right\}, \tag{2}$$

and R is the specific gas constant. We continue to follow Siewert and Thomas Jr. (1981) to define the density $\varrho(x)$, mass velocity $v(x)$ and temperature $T(x)$ in Eq. (2), as

$$\varrho(x)=\int_{-\infty}^{\infty}f(x,\xi)\mathrm{d}\xi, \tag{3}$$

$$\varrho(x)v(x)=\int_{-\infty}^{\infty}\xi f(x,\xi)\mathrm{d}\xi \tag{4}$$

and

$$\varrho(x)RT(x)=\int_{-\infty}^{\infty}[\xi-v(x)]^2f(x,\xi)\mathrm{d}\xi. \tag{5}$$

We assume that far downstream the gas relaxes to an equilibrium distribution characterized by steady drift velocity v_∞, density ϱ_∞ and temperature T_∞,

$$f_\infty(\xi)=\lim_{x\to\infty}\phi(x,\xi)=\frac{\varrho_\infty}{\sqrt{2\pi RT_\infty}}\exp\left\{-\frac{(\xi-v_\infty)^2}{2RT_\infty}\right\}. \tag{6}$$

At this point, we follow Arthur and Cercignani (1980) and we linearize $f(x,\xi)$ and $\phi(x,\xi)$ around $f_\infty(\xi)$. In this manner, we write $f(x,\xi)$ as

$$f(x,\xi)=f_\infty(\xi)[1+h(x,\xi)], \tag{7}$$

where $h(x,\xi)$ is a perturbation to the absolute Maxwellian $f_\infty(\xi)$. Thus, we substitute Eq. (7) into Eq. (1) and linearize $\phi(x,\xi)$ around $f_\infty(\xi)$, to obtain the one-dimensional linearized equation written in terms of the perturbation function h,

$$c\frac{\partial}{\partial\tau}h(\tau,c)+h(\tau,c)=\pi^{-1/2}\int_{-\infty}^{\infty}\mathrm{e}^{-(c'-u)^2}K(c',c:u)h(\tau,c')\mathrm{d}c', \tag{8}$$

where

$$K(c',c:u)=1+2(c'-u)(c-u)+2\left[(c'-u)^2-1/2\right]\left[(c-u)^2-1/2\right], \tag{9}$$

is the scattering kernel and

$$\tau=\eta x(2RT_\infty)^{-1/2},\qquad c=\xi(2RT_\infty)^{-1/2}\qquad\text{and}\qquad u=v_\infty(2RT_\infty)^{-1/2}, \tag{10 a, b, c}$$

are the dimensionless variables. We note that u is the normalized downstream drift velocity.

2.1 Boundary Conditions

To obtain the boundary condition for the interface, in terms of the perturbation h, we follow Siewert and Thomas Jr. (1981) and set $x = 0$ in Eq. (7), to find (for $\xi > 0$)

$$h(0,\xi) = \frac{f(0,\xi) - f_\infty(\xi)}{f_\infty(\xi)}, \tag{11}$$

where $f(0,\xi)$ is the Maxwellian distribution, given by Eq. (2), evaluated at $x = 0$

$$f(0,\xi) = \frac{\varrho_0}{\sqrt{2\pi R T_0}} \exp\left\{-\frac{(\xi - v_0)^2}{2RT_0}\right\}. \tag{12}$$

We then linearize $f(0,\xi)$ around $f_\infty(\xi)$ to obtain the dimensionless boundary condition (for $c > 0$)

$$h(0,c) = \Delta\varrho_0 + 2(c-u)(u_0 - u) + \left[(c-u)^2 - 1/2\right]\Delta T_0, \tag{13}$$

with dimensionless variables defined by

$$u_0 = v_0(2RT_\infty)^{-1/2}, \qquad \Delta\varrho_0 = \frac{\varrho_0 - \rho_\infty}{\varrho_\infty} \qquad \text{and} \qquad \Delta T_0 = \frac{T_0 - T_\infty}{T_\infty}. \tag{14 a, b, c}$$

On the other hand, when $x \to \infty$, $f(x,\xi)$ approaches $f_\infty(\xi)$ and, looking back Eq. (7), we then find the condition

$$\lim_{\tau\to\infty} h(\tau,c) = 0. \tag{15}$$

2.2 Physical Quantities of Interest

We substitute Eq. (7) into Eqs. (3) to (5) to find, in terms of h, the density perturbation

$$\Delta\varrho(\tau) = \pi^{-1/2}\int_{-\infty}^{\infty} \mathrm{e}^{-(c-u)^2} h(\tau,c)\mathrm{d}c, \tag{16}$$

the velocity perturbation

$$\Delta v(\tau) = \frac{\pi^{-1/2}}{u}\int_{-\infty}^{\infty} \mathrm{e}^{-(c-u)^2}(c-u) h(\tau,c)\mathrm{d}c \tag{17}$$

and the temperature perturbation

$$\Delta T(\tau) = \pi^{-1/2}\int_{-\infty}^{\infty} \mathrm{e}^{-(c-u)^2}[2(c-u)^2 - 1] h(\tau,c)\mathrm{d}c. \tag{18}$$

3. A REFORMULATION

To develop an analytical solution to the problem defined by Eq. (8), it is convenient to introduce a new function

$$G(\tau,c) = \mathrm{e}^{-(c-u)^2} h(\tau,c) \tag{19}$$

such that, we rewrite Eq. (8) in the form

$$c\frac{\partial}{\partial\tau}G(\tau,c) + G(\tau,c) = \pi^{-1/2}\mathrm{e}^{-(c-u)^2}\int_{-\infty}^{\infty} K(c',c:u)G(\tau,c')\mathrm{d}c', \tag{20}$$

with boundary conditions given by

$$G(0,c) = \left\{\Delta\varrho_0 + 2(c-u)(u_0-u) + \left[(c-u)^2 - 1/2\right]\Delta T_0\right\}\mathrm{e}^{-(c-u)^2}, \quad c > 0 \tag{21}$$

and

$$\lim_{\tau\to\infty} G(\tau,c) = 0. \tag{22}$$

In the same way, based on the definition given in Eq. (19), we express the density perturbation as

$$\Delta\varrho(\tau) = \pi^{-1/2}\int_{-\infty}^{\infty} G(\tau,c)\mathrm{d}c, \tag{23}$$

the velocity perturbation as

$$\Delta v(\tau) = \frac{\pi^{-1/2}}{u} \int_{-\infty}^{\infty} (c-u) G(\tau, c) \mathrm{d}c \tag{24}$$

and the temperature perturbation as

$$\Delta T(\tau) = \pi^{-1/2} \int_{-\infty}^{\infty} [2(c-u)^2 - 1] G(\tau, c) \mathrm{d}c. \tag{25}$$

We develop the solution for the G problem: Eq. (20) supplemented by Eqs. (21) and (22), in the next section.

4. A DISCRETE ORDINATES SOLUTION

We seek solutions of Eq. (20) of the form

$$G(\tau, c) = \Phi(\nu, c) \mathrm{e}^{-\tau/\nu}. \tag{26}$$

If we substitute Eq. (26) into Eq. (20) we obtain

$$(1 - c/\nu)\Phi(\nu, c) = \pi^{-1/2} \mathrm{e}^{-(c-u)^2} \int_{-\infty}^{\infty} K(c', c : u) \Phi(\nu, c') \mathrm{d}c'. \tag{27}$$

We can still derive some normalization conditions (Siewert and Thomas Jr., 1981) to simplify Eq. (27). In fact, firstly, we integrate Eq. (27), over all c, to find

$$\int_{-\infty}^{\infty} c\Phi(\nu, c) \mathrm{d}c = 0. \tag{28}$$

Continuing, we can multiply Eq. (27) by $(c - u)$ and integrate the resultant equation over all c to find

$$\int_{-\infty}^{\infty} c^2 \Phi(\nu, c) \mathrm{d}c = 0. \tag{29}$$

Thus, looking back to Eqs. (28) and (29), we rewrite Eq. (27), as

$$(1 - c/\nu)\Phi(\nu, c) = \pi^{-1/2} \mathrm{e}^{-(c-u)^2} Q(c : u) \int_{-\infty}^{\infty} \Phi(\nu, c') \mathrm{d}c', \tag{30}$$

with

$$Q(c : u) = 1 + 2u^2 + 2(c^2 + u^2 - 1/2)(u^2 - 1/2) - 4cu^3. \tag{31}$$

Still, at this point we note that the exponential term, in Eq. (30) can be expressed as

$$\mathrm{e}^{-(c-u)^2} = \mathrm{e}^{-(c^2+u^2)} [\mathrm{senh}(2cu) + \cosh(2cu)] \tag{32}$$

and, in this manner, we write the final convenient form of Eq. (30)

$$(1 - c/\nu)\Phi(\nu, c) = \psi(c : u)[A(c : u) + B(c : u)] \int_{-\infty}^{\infty} \Phi(\nu, c') \mathrm{d}c', \tag{33}$$

where

$$\psi(c : u) = \pi^{-1/2} \mathrm{e}^{-(c^2+u^2)}, \tag{34}$$

$$A(c : u) = [1 + 2u^2 + 2(c^2 + u^2 - 1/2)(u^2 - 1/2)]\cosh(2cu) - 4cu^3 \mathrm{senh}(2cu) \tag{35}$$

and

$$B(c : u) = [1 + 2u^2 + 2(c^2 + u^2 - 1/2)(u^2 - 1/2)]\mathrm{senh}(2cu) - 4cu^3 \cosh(2cu). \tag{36}$$

Now we rewrite the integral term in Eq. (33),

$$(1 - c/\nu)\Phi(\nu, c) = \psi(c : u)[A(c : u) + B(c : u)] \int_{0}^{\infty} [\Phi(\nu, c') + \Phi(\nu, -c')] \mathrm{d}c'. \tag{37}$$

Then we introduce a (half-range) quadrature scheme $[0,\infty)$, to approximate the integral term of the above equation, such that

$$(1-c/\nu)\Phi(\nu,c)=\psi(c:u)[A(c:u)+B(c:u)]\sum_{k=1}^{N}w_k\left[\Phi(\nu,c_k)+\Phi(\nu,-c_k)\right]. \tag{38}$$

Here c_k and w_k are, respectively, the N nodes and weights of the (arbitrary) quadrature scheme. If we now evaluate Eq. (38) in $c=\pm c_i$, for $i=1,\dots,N$, and note that $\psi(c:u)$ and $A(c:u)$ are even functions,

$$\psi(c:u)=\psi(-c:u), \qquad A(c:u)=A(-c:u), \tag{39 a, b}$$

and $B(c:u)$ is an odd function,

$$B(c:u)=-B(-c:u), \tag{40}$$

we obtain the discrete-ordinates version of the Eq. (37) as

$$(1\mp c_i/\nu)\Phi(\nu,\pm c_i)=\psi(c_i:u)[A(c_i:u)\pm B(c_i:u)]\sum_{k=1}^{N}w_k\left[\Phi(\nu,c_k)+\Phi(\nu,-c_k)\right]. \tag{41}$$

We express now Eq. (41) in a matrix form, as

$$(\mathbf{I}-\mathbf{M}/\nu)\,\mathbf{\Phi}_+(\nu)=\mathbf{\Psi}[\mathbf{A}+\mathbf{B}]\mathbf{W}[\mathbf{\Phi}_+(\nu)+\mathbf{\Phi}_-(\nu)] \tag{42}$$

and

$$(\mathbf{I}+\mathbf{M}/\nu)\,\mathbf{\Phi}_-(\nu)=\mathbf{\Psi}[\mathbf{A}-\mathbf{B}]\mathbf{W}[\mathbf{\Phi}_+(\nu)+\mathbf{\Phi}_-(\nu)], \tag{43}$$

where $\mathbf{I}$ is the $N\times N$ identity matrix, $\mathbf{M}$, $\mathbf{\Psi}$, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{W}$ are $N\times N$ matrices defined by

$$\mathbf{M}=\text{diag}\,\{c_1,\dots,c_N\}, \tag{44}$$

$$\mathbf{\Psi}=\text{diag}\,\{\psi(c_1:u),\dots,\psi(c_N:u)\}, \tag{45}$$

$$\mathbf{A}=\text{diag}\,\{A(c_1:u),\dots,A(c_N:u)\}, \tag{46}$$

$$\mathbf{B}=\text{diag}\,\{B(c_1:u),\dots,B(c_N:u)\} \tag{47}$$

and

$$\mathbf{W}_{ij}=[w_j], \tag{48}$$

for $i,j=1,\dots,N$. Continuing, here, $\mathbf{\Phi}_\pm(\nu)$ are $N\times 1$ vectors, such that

$$\mathbf{\Phi}_\pm(\nu)=\begin{bmatrix}\Phi(\nu,\pm c_1) & \cdots & \Phi(\nu,\pm c_N)\end{bmatrix}^T, \tag{49}$$

where T denote the transpose operation.

We now add and subtract Eqs. (42) and (43) to find the equations

$$\mathbf{U}-\frac{1}{\nu}\mathbf{M}\mathbf{V}=2\mathbf{\Psi}\mathbf{A}\mathbf{W}\mathbf{U} \qquad \text{and} \qquad \mathbf{V}-\frac{1}{\nu}\mathbf{M}\mathbf{U}=2\mathbf{\Psi}\mathbf{B}\mathbf{W}\mathbf{U}, \tag{50 a, b}$$

with

$$\mathbf{U}=\mathbf{\Phi}_+(\nu)+\mathbf{\Phi}_-(\nu) \qquad \text{and} \qquad \mathbf{V}=\mathbf{\Phi}_+(\nu)-\mathbf{\Phi}_-(\nu). \tag{51 a, b}$$

$\mathbf{\Phi}_+(\nu)$ and $\mathbf{\Phi}_-(\nu)$ are the vectors defined in Eq. (49). Substituting Eq. (50b) into Eq. (50a) we find a quadratic eigenvalue problem

$$[\mathbf{I}\lambda^2+2\mathbf{M}^{-\mathbf{1}}\mathbf{\Psi}\mathbf{B}\mathbf{W}\lambda+2\mathbf{M}^{-\mathbf{2}}\mathbf{\Psi}\mathbf{A}\mathbf{W}-\mathbf{M}^{-\mathbf{2}}]\mathbf{U}=\mathbf{0}, \tag{52}$$

where the eigenvalues are given by $\lambda=\nu^{-1}$. Following Datta (1995), a quadratic eigenvalue problem, as the one given by Eq. (52), can be transformed in the standard eigenvalue problem

$$\begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{G} & -\mathbf{F}\end{bmatrix}\begin{bmatrix}\mathbf{U}\\ \lambda\mathbf{U}\end{bmatrix}=\lambda\begin{bmatrix}\mathbf{U}\\ \lambda\mathbf{U}\end{bmatrix}, \tag{53}$$

where

$$\mathbf{F} = 2\mathbf{M}^{-1}\boldsymbol{\Psi}\mathbf{B}\mathbf{W} \quad \text{and} \quad \mathbf{G} = 2\mathbf{M}^{-2}\boldsymbol{\Psi}\mathbf{A}\mathbf{W} - \mathbf{M}^{-2}. \tag{54 a, b}$$

From Eq. (53) we obtain the set of $2N$ eigenvectors $\mathbf{U}(\nu_j)$, associated with the separation constants ν_j, that we use in Eqs. (50) to evaluate the $N \times 1$ elementary solutions

$$\boldsymbol{\Phi}_+(\nu_j) = \frac{1}{2}\left[\frac{1}{\nu_j}\mathbf{M} + \mathbf{I} + 2\boldsymbol{\Psi}\mathbf{B}\mathbf{W}\right]\mathbf{U}(\nu_j) \tag{55}$$

and

$$\boldsymbol{\Phi}_-(\nu_j) = -\frac{1}{2}\left[\frac{1}{\nu_j}\mathbf{M} - \mathbf{I} + 2\boldsymbol{\Psi}\mathbf{B}\mathbf{W}\right]\mathbf{U}(\nu_j). \tag{56}$$

In this way, we are ready to write the general solution of the discrete-ordinates version of the G problem, given by Eq. (20), as

$$G(\tau, \pm c_i) = \sum_{j=1}^{2N} A_j \Phi(\nu_j, \pm c_i)\mathrm{e}^{-\tau/\nu_j}. \tag{57}$$

Since this is a conservative problem, we have to deal with the issue of having degenerate eigenvalues, that approach zero (separation constants going to infinity) as N tends to infinity. Because of that, we need to look for exact solutions of the problem given by Eq. (20) to add to the general discrete ordinates solution. For this specific problem we are solving in this work, the number of eigenvalues with this behavior (number of exact solutions) depends of the value of the parameter u, in Eq. (9), associated with the downstream drift velocity. Thus, we express the general discrete ordinates solution of Eq. (20) in the form

$$G(\tau, \pm c_i) = A_1^* G_1(\pm c_i) + A_2^* G_2(\pm c_i) + A_3^* G_3(\pm c_i) + A_4^* G_4(\tau, \pm c_i) + \sum_{j=1}^{2N-4} A_j \Phi(\nu_j, \pm c_i)\mathrm{e}^{-\tau/\nu_j} \tag{58}$$

for $u = 0$ and $u^2 = 3/2$, and

$$G(\tau, \pm c_i) = A_1^* G_1(\pm c_i) + A_2^* G_2(\pm c_i) + A_3^* G_3(\pm c_i) + \sum_{j=1}^{2N-3} A_j \Phi(\nu_j, \pm c_i)\mathrm{e}^{-\tau/\nu_j} \tag{59}$$

for $0 < u^2 < 3/2$ and $u^2 > 3/2$, where the introduced exact solutions are given by (Siewert and Thomas Jr., 1981)

$$G_1(c) = \mathrm{e}^{-(c-u)^2}, \qquad G_2(c) = (c-u)\mathrm{e}^{-(c-u)^2}, \qquad G_3(c) = (c-u)^2\mathrm{e}^{-(c-u)^2} \tag{60 a, b, c}$$

and (only for $u = 0$ and $u^2 = 3/2$)

$$G_4(\tau, c) = (\tau - c)Q(c : u)\mathrm{e}^{-(c-u)^2}. \tag{61}$$

The next step is to determine the arbitrary constants present in the solution (Eq. (58) or (59)). We use the boundary conditions for doing that. We then substitute the general solution, Eqs. (58) and (59), into Eq. (22) to obtain, for $u^2 < 3/2$,

$$G(\tau, \pm c_i) = \sum_{j=1}^{N-2} A_j \Phi(\nu_j, \pm c_i)\mathrm{e}^{-\tau/\nu_j} \tag{62}$$

and for $u^2 \geq 3/2$

$$G(\tau, \pm c_i) = \sum_{j=1}^{N-3} A_j \Phi(\nu_j, \pm c_i)\mathrm{e}^{-\tau/\nu_j}, \tag{63}$$

where, here, ν_j are the positive separations constants. In addition, the discrete-ordinates version of the interface boundary condition, Eq. (21), is

$$G(0, c_i) = \left\{\Delta\varrho_0 + 2(c_i - u)(u_0 - u) + \left[(c_i - u)^2 - 1/2\right]\Delta T_0\right\}\mathrm{e}^{-(c_i-u)^2}, \tag{64}$$

for $i = 1, \ldots, N$. In this way, if we substitute Eq. (62) into Eq. (64), we obtain for $u^2 < 3/2$ the square linear system $N \times N$

$$\sum_{j=1}^{N-2} A_j \Phi(\nu_j, c_i) - \Delta\varrho_0 \mathrm{e}^{-(c_i-u)^2} - [(c_i-u)^2 - 1/2]\Delta T_0 \mathrm{e}^{-(c_i-u)^2} = 2(c_i-u)(u_0-u)\mathrm{e}^{-(c_i-u)^2}, \tag{65}$$

for $i = 1, \ldots, N$. If we substitute Eq. (63) into Eq. (64), we obtain for $u^2 \geq 3/2$ the rectangular linear system $N \times N-1$

$$\sum_{j=1}^{N-3} A_j \Phi(\nu_j, c_i) - \Delta\varrho_0 \mathrm{e}^{-(c_i-u)^2} - [(c_i-u)^2 - 1/2]\Delta T_0 \mathrm{e}^{-(c_i-u)^2} = 2(c_i-u)(u_0-u)\mathrm{e}^{-(c_i-u)^2}, \tag{66}$$

for $i = 1, \ldots, N$. Once we solve Eqs. (65) and (66) we find the coefficients A_j and the quantities $\Delta\varrho_0$ and ΔT_0 defined in Eqs. (14b) and (14c).

As we mentioned before in this text, it was shown by Arthur and Cercignani (1980) the existence of solution for this problem only for specific values of the drift velocity, specifically, for $u^2 < 3/2$. This result was somehow confirmed by Siewert and Thomas Jr. (1981) calculations. Here, we note that, although, in principle, we can deal with the system given by Eq. (66), with the least squares method, we have not found convergence, when we increase N, to suggest a possible numerical reliable result. So, we consider this fact as an indication of the known theoretical result (Arthur and Cercignani, 1980) and assume Eq. (62) as our general solution for the strong evaporation problem, where the arbitrary constants are given by the solution of the square linear system defined by Eq. (65).

Thus, we substitute Eq. (62) into Eqs. (23) to (25) and we use the normalization conditions given by Eqs. (28) and (29) to express the final form of the density perturbation

$$\Delta\varrho(\tau) = \pi^{-1/2} \sum_{j=1}^{N-2} A_j \mathrm{e}^{-\tau/\nu_j} \sum_{k=1}^{N} w_k [\Phi(\nu_j, c_k) + \Phi(\nu_j, -c_k)], \tag{67}$$

velocity and temperature perturbations, respectively,

$$\Delta v(\tau) = -\Delta\varrho(\tau) \qquad \text{and} \qquad \Delta T(\tau) = (2u^2 - 1)\Delta\varrho(\tau). \tag{68 a, b}$$

5. A NONLINEAR APPROACH

Once the discrete-ordinates solution for the linearized version of the strong evaporation problem is completely established, we proceed to define what we call a "post-processing (PP)" procedure. In this sense, we consider the proposed nonlinear model, given by Eq. (1) to (5), with boundary conditions defined in Eqs. (6) and (12), rewritten in terms of the dimensionless variables given in Eqs. (10). We then use the quantities evaluated by the ADO method, Eqs. (67) and (68), into Eq. (2), which defines the Maxwellian distribution. Continuing, we substitute this distribution in the right-hand side of Eq. (1), which is then solved for a known distribution $\phi(x, \xi)$. The solution defines the original f distribution, which is then used to evaluated again Eqs. (3) to (5) – the macroscopic quantities for the gas. We do not write explicit derivations here, for this procedure, because of the requested length of this paper. We present, however, in the next section some numerical results and comparisons between this procedure and the linear version.

6. COMPUTATIONAL ASPECTS AND NUMERICAL RESULTS

To start the computational procedure, the first step is to define the quadrature scheme. Then, once we have the N quadrature points c_k and the weights w_k defined, the solution is concise and easy to implement. We proceed with:

- the solution of an eigenvalue problem, Eq. (53), to obtain the separation constants ν_j and the elementary solutions $\mathbf{\Phi}_{\pm}(\nu_j)$;
- the solution of a linear system, given by Eq. (65);
- the evaluation of the density, velocity and temperature perturbations, Eqs. (67) and (68). Still, from the solution of Eq. (65) we are able to get the quantities $\Delta\varrho_0$ and ΔT_0, Eqs. (14b) and (14c).
- The quantities listed above are then used, in what we called "post-processing" procedure, in Eqs. (1) to (5).

The numerical results showed here where obtained by a FORTRAN program, using, in general, $N = 200$ quadrature points. The computational time required for generating all quantities of interest for one value of u is less than one second in a Pentium IV (2.66GHz, 1.5GB RAM). If we increase N up to $N = 400$, all digits listed in the tables are preserved (plus or minus one in the last digit). Still, in regard to our quadrature scheme, as we have explained in previous works, we define a half-range scheme, in $[0,\infty)$. We use the transformation

$$u(c) = \mathrm{e}^{-c} \tag{69}$$

to map the interval $[0,\infty)$ into $[0,1]$, where we are then able to make use of the usual Gauss-Legendre quadrature scheme, after using a new change of variable

$$v(u) = 2u - 1. \tag{70}$$

In regard to the numerical results, we first checked some previous results available in the literature (Siewert and Thomas Jr., 1981), for the linearized problem, for ϱ_∞/ϱ_0 and T_∞/T_0. We obtained agreement with all digits listed in that reference. Results we generate from the linearized version are listed in Tab. 1, referred as "Linear" case. In Tab. 1 we also show the results we generate with the "Post-processing (PP)" approach. Both are compared with Ytrehus (1976) results for a numerical treatment of a set of moment equations – where the Boltzmann equation seems to be satisfied in an average sense – which seem to take into account nonlinear terms. These results (Ytrehus, 1976) seem to be in agreement with experimental works.

Table 1. ϱ_∞/ϱ_0 and T_∞/T_0 for $u_0 = 0.0$

	ϱ_∞/ϱ_0			T_∞/T_0		
u	Linear	PP	Ytrehus (1976)	Linear	PP	Ytrehus (1976)
0.0	1.000000	1.000000	1.0000	1.000000	1.000000	1.0000
0.1	0.881170	0.873324	0.8494	0.919458	0.920433	0.9567
0.2	0.796123	0.772378	0.7283	0.842106	0.845652	0.9152
0.3	0.732184	0.690327	0.6303	0.769530	0.776876	0.8756
0.4	0.682430	0.622488	0.5501	0.702368	0.714499	0.8378
0.5	0.642727	0.565542	0.4841	0.640748	0.658453	0.8016
0.6	0.610439	0.517090	0.4292	0.584522	0.608403	0.7671
0.7	0.583801	0.475381	0.3834	0.533391	0.563854	0.7342
0.8	0.561584	0.439135	0.3447	0.486987	0.524234	0.7028
0.9	0.542908	0.407414	0.3120	0.444912	0.488933	0.6729
1.0	0.527130	0.379535		0.406764	0.457345	
1.1	0.513774	0.355009		0.372150	0.428882	
1.2	0.502498	0.333496		0.340680	0.402986	

We found more significant difference, when comparing results from the linearized model with the PP approach, for the ratio ϱ_∞/ϱ_0 than the temperature ratio, as showed in Figs. 1 and 2. As expected, major variation is noted when u increases and the nonlinear modeling should be more effective. However, we also see that the PP procedure seems generate results in better agreement with Ytrehus (1976) results.

Still, in Tab. 2, we list results, for $u = 1.1$, for the distribution profiles, which were not provided in previous references, where only the ratios showed in Tab. 1 were evaluated.

Table 2. Density, Velocity and Temperature Perturbations for $u = 1.1$ and $u_0 = 0.0$

	$\Delta\varrho(\tau)$		$\Delta v(\tau)$		$\Delta T(\tau)$	
τ	Linear	PP	Linear	PP	Linear	PP
0.0	8.160593(–1)	9.6964(–1)	–8.160593(–1)	–4.9229(–1)	1.158804	1.1256(–1)
0.1	7.659667(–1)	8.5562(–1)	–7.659667(–1)	–4.6109(–1)	1.087672	1.4023(–1)
0.2	7.331004(–1)	7.8797(–1)	–7.331004(–1)	–4.4070(–1)	1.041002	1.5578(–1)
0.3	7.056325(–1)	7.3609(–1)	–7.056325(–1)	–4.2399(–1)	1.001998	1.6702(–1)
0.4	6.813778(–1)	6.9362(–1)	–6.813778(–1)	–4.0955(–1)	9.675565(–1)	1.7565(–1)
0.5	6.593716(–1)	6.5759(–1)	–6.593716(–1)	–3.9671(–1)	9.363077(–1)	1.8246(–1)
0.6	6.390771(–1)	6.2631(–1)	–6.390771(–1)	–3.8511(–1)	9.074896(–1)	1.8794(–1)
0.7	6.201558(–1)	5.9871(–1)	–6.201558(–1)	–3.7449(–1)	8.806212(–1)	1.9238(–1)
0.8	6.023761(–1)	5.7403(–1)	–6.023761(–1)	–3.6468(–1)	8.553742(–1)	1.9600(–1)
0.9	5.855710(–1)	5.5173(–1)	–5.855710(–1)	–3.5555(–1)	8.315108(–1)	1.9895(–1)
1.0	5.696143(–1)	5.3142(–1)	–5.696143(–1)	–3.4701(–1)	8.088523(–1)	2.0134(–1)
2.0	4.418633(–1)	3.9186(–1)	–4.418633(–1)	–2.8154(–1)	6.274460(–1)	2.0796(–1)
5.0	2.255993(–1)	2.0160(–1)	–2.255993(–1)	–1.6778(–1)	3.203510(–1)	1.7012(–1)
7.0	1.478535(–1)	1.3546(–1)	–1.478535(–1)	–1.1930(–1)	2.099521(–1)	1.3496(–1)

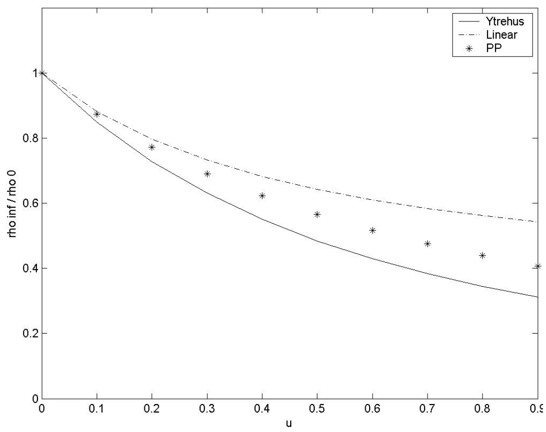

Figure 1. Density ratio: ϱ_∞/ϱ_0

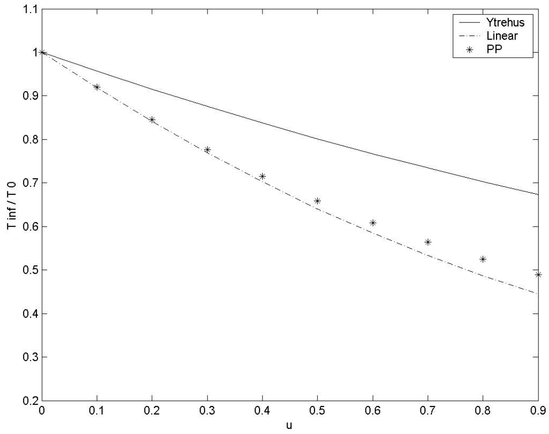

Figure 2. Temperature ratio: T_∞/T_0

7. CONCLUDING REMARKS

The ADO method was used to develop a closed form solution for the nonlinear BGK version of the strong evaporation problem in rarefied gas dynamics. The analytical discrete-ordinates solution obtained for the linearized version of the problem was associated with a re-evaluation of the quantities of interest, in order to take into account the nonlinear effects inherent to the problem. The new approach seemed to improve the results of the linearized version, mainly when the values of the drift velocity increase, when compared with results available in the literature.

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