GALERKIN METHOD APPLIED TO STOCHASTIC DIFFUSION PROBLEMS

Claudio R. Ávila da S. Jr., avila@utfpr.edu.br
Admilson T. Franco, admilson@utfpr.edu.br
Universidade Federal Tecnologica do Paraná, Departamento Acadêmico de Mecânica, Av. Sete de Setembro, 3165, 80230-901, Curitiba, PR, Brasil.

André T. Beck, atbeck@sc.usp.br
Engenharia de Estruturas, EESC, USP, Av. Trabalhador Sancarlense 400, 13566-590, São Carlos, SP, Brasil.

Abstract. *In this paper, the Askey-Wiener scheme and the Galerkin method are used to obtain approximate solutions to stochastic diffusion problems. The uncertainty in diffusion coefficient is represented as a parameterized stochastic process. The space of approximate solutions is built using results of density between spaces of continuous functions and Sobolev spaces. The random behavior of the problems response is modeled using the Askey-Wiener scheme. From the approximate solution, first and second order moments are obtained and compared with corresponding estimates obtained via Monte Carlo simulation. Results show very fast convergence to the exact solution, at excellent accuracies.*

Keywords*: Diffusion Equation, Galerkin method, Polynomial chaos, Monte Carlo simulation*

1. INTRODUCTION

Analysis and modeling of the thermal behavior of mechanical systems is a very important problem in engineering. The use of simulation to forecast the structural and thermal behavior of a system is essential if optimal performance is to be achieved. Numerical solutions to heat transfer problems have become increasingly common in the last decades, as result of new theoretical developments and ever-greater availability of computational power. Such advances stimulate scientists and engineers to look for more powerful and accurate methods to solve increasingly complex problems. Developments in constitutive models and improvements in consistency and robustness of mathematical models, however, are not sufficient to preview system behavior in the presence of uncertainties. Deterministic formulations will always be limited, because parameter and system uncertainties are grossly neglected. On the other hand, probabilistic modeling in continuum mechanics problems has increased in recent years.

The application of the Galerkin method to stochastic systems was first described by Spanos and Ghanem (1989). Heat transfer with uncertainties in the material properties has been studied extensively in recent years. Examples include the studies by Hien and Kleiber (1997), who investigated transient heat transfer using the perturbation technique; Kaminski and Hien (1999), who applied the perturbation method to study transient heat conduction in composite materials; Xiu and Karniadakis (2003), who used the Galerkin method to study transient heat conduction, and Emery (2004), who solved the transient heat conduction problems using a number of different methods. Jardak et al. (2002) used chaos polynomials to obtain approximate, numerical solutions for the advection problem, in stationary regime, with a stochastic transport velocity. Xiu and Shen (2007) obtained numerical solutions for the Helmholtz equation, modeling propagation of acoustic waves in a random surface. Jin and Zou (2008) studied an inverse heat conduction problem, in the stationary regime and with a Robin-type boundary condition. It is significant to note that none of the cited references presented any results about the existence and uniqueness of the solutions to the Helmholtz equation, in the case of uncertain parameters.

In this paper, numerical solutions are constructed, via Galerkin method, for stationary stochastic diffusion problems, subject to Neumann boundary conditions. Uncertainty in the diffusion coefficient is modeled via parameterized stochastic processes. The approximate solution space is constructed by tensor product between Sobolev spaces and a probability measure space. Generalized chaos polynomials, generated via the Askey-Wiener scheme (Xiu and Karniadakis, 2003), are used to represent the solution (random response). The developed methodology is evaluated by comparing first and second moments, derived from the Galerkin solution, with the same moments obtained via Monte Carlo simulation.

2. THE STOCHASTIC DIFFUSION PROBLEM

The steady-state stochastic diffusion equation is given by

$$\begin{cases} -\nabla.\left(\kappa.\nabla u\right) = q, & \forall\left(\boldsymbol{x},\omega\right)\in D\times\Omega; \\ \kappa.\dfrac{\partial u}{\partial \eta} = h, & \forall\left(\boldsymbol{x},\omega\right)\in \Gamma\times\Omega; \end{cases} \tag{1}$$

where u is the response, κ is the thermal conductivity coefficient, q is the source term, $D \subset \mathbb{R}^2$ is a bounded and closed domain with regular boundary Γ, h is a function defined in the boundary, ω is an element of the sample domain Ω. Equation (1) is the strong form of the stochastic diffusion problem. In order to ensure existence and uniqueness of the strong solution, the following hypotheses are necessary:

$$\begin{aligned} &\text{H1}: \exists \underline{\kappa}, \overline{\kappa} \in (0, +\infty),\ P\left((\boldsymbol{x}, \omega) \in \overline{D} \times \Omega : \kappa(\boldsymbol{x}, \omega) \in [\underline{\kappa}, \overline{\kappa}]\right) = 1; \\ &\text{H2}: q \in L^{\infty}\left(\Omega, \mathcal{F}, P; L^2(D)\right) \text{ and } h \in L^2\left(\Omega, \mathcal{F}, P; L^{\infty}(\Gamma)\right); \\ &\text{H3}: \int_{\Omega}\int_{D} q(\boldsymbol{x}, \omega) d\boldsymbol{x} dP(\omega) + \int_{\Omega}\int_{\Gamma} h(\boldsymbol{x}, \omega) d\boldsymbol{x} dP(\omega) = 0. \end{aligned} \tag{2}$$

Hypothesis H1 ensures coercivity and continuity of the bi-linear form associated to the heat conduction problem. Hypothesis H2 ensures limitation of the source terms in $\overline{D}$. Hypothesis H1, H2 and H3 are necessary to ensure existence and uniqueness of the solution to the stochastic diffusion problem.

3. MODELLING OF THE UNCERTAINTY

Explicit mathematical modeling of the uncertainty is necessary in order to obtain numerical solutions to stochastic problems. Uncertainty models must be compatible with hypothesis H1 presented in Eq. (2). In this paper, the uncertainty in conductivity coefficient is represented as a parameterized stochastic process (Grigoriu, 1995), $\tau : \overline{D} \times \Omega \to \mathbb{R}$, defined in probability space $(\Omega, \mathcal{F}, P)$, and having the following form:

$$\tau(\boldsymbol{x}, \omega) = \mu_{\tau}(\boldsymbol{x}) + \sum_{i=1}^{N} \varphi_i(\boldsymbol{x}).\xi_i(\omega), \forall (\boldsymbol{x}, \omega) \in \overline{D} \times \Omega; \tag{3}$$

where τ is the property to be modeled and $\{\xi_i(\omega)\}_{i=1}^{N}$ is a vector of random variables, uniformly distributed in $\{-1, 1\}$, such that:

$$\begin{cases} \prec \xi_i \succ = \int_{-1}^{+1} \xi_i \ \frac{1}{2} d\xi_i = 0, \forall i \in \{1, ..., N\}; \\ \prec \xi_i, \xi_j \succ = \frac{1}{3}.\delta_{ij}, \forall (i, j) \in \{1, ..., N\} \times \{1, ..., N\}. \end{cases} \tag{4}$$

In eq. (4), $\prec \cdot \succ$ is the mathematical expectation operator. Hence, the mean value of τ is: $\prec \tau(\boldsymbol{x}, \cdot) \succ = \mu_{\tau}(\boldsymbol{x})$ and the variance of τ is $\prec \left[\tau(\boldsymbol{x}, \cdot) - \mu_{\tau}(\boldsymbol{x})\right]^2 \succ = \sigma_{\tau}^2(\boldsymbol{x})$.

3.1 The Askey-Wiener scheme

The Askey-Wiener scheme is a generalization of chaos polynomials, also known as Wiener-chaos. Chaos polynomials were proposed by Wiener (1938) to study statistical mechanics of gases. Xiu and Karniadakis (2003) have shown the close relationship between results presented by Wiener (1938) and Askey and Wilson (1985) for the representation of stochastic processes by orthogonal polynomials. Xiu and Karniadakis (2003) extended the studies of Spanos and Ghanem (1989) and Ogura (1972) for polynomials belonging to the Askey-Wiener scheme.

The Cameron-Martin theorem (1947) shows that Askey-Wiener polynomials form a base for a dense subspace of second order random variables $L^2(\Omega, \mathcal{F}, P)$. As shown by Jason (1997), an element $f \in L^2(\Omega, \mathcal{F}, P)$ can be represented as

$$f(\omega) = \sum_{\alpha \in \Im} c_{\alpha} \psi_{\alpha}(\omega), \tag{5}$$

where $\{\psi_{\alpha}\}_{\alpha \in \Im}$ is a sequence of polynomials, such that $\overline{span\left\langle \{\psi_{\alpha}\}_{\alpha \in \Im} \right\rangle}^{L^2(\Omega, \mathcal{F}, P)} = L^2(\Omega, \mathcal{F}, P)$, and $c_{\alpha} \in \mathbb{R}^N, \forall \alpha = (\alpha_1, \alpha_1, ...) \in \Im$ with $\alpha_i \in \mathbb{N}$.

The tensor product between polynomials ψ_i and ψ_j in $L^2(\Omega,\mathcal{F},P)$ is defined as,

$$\left(\psi_i,\psi_j\right)_{L^2(\Omega,\mathcal{F},P)} = \int_{\mathbb{R}^N} \left(\psi_i.\psi_j\right)\left(\boldsymbol{\xi}(\omega)\right)dP(\omega), \tag{7}$$

where dP is a probability measure. These polynomials form a complete ortho-normal system with respect to the probability measure, with the following properties:

$$\psi_0 = 1, \qquad \left(\psi_i,\psi_j\right)_{L^2(\Omega,\mathcal{F},P)} = \delta_{ij}, \quad \forall i,j \in \mathbb{N}. \tag{8}$$

It is important to observe that in Eq. (8) the polynomials are orthogonal with respect to the probability measure *dP*.

The Askey-Wiener scheme represents a family of sub-spaces generated by orthogonal polynomials obtained from ordinary differential equations (Xiu and Karniadakis, 2003). Among them, the Hermite, Laguerre, Jacobi and Legendre polynomials can be cited. The families generated by these polynomials form a complete system in L^2. The orthogonality between the polynomials is defined with respect to a weight function, which is identical to the probability density function of a certain random variable.

4. NOTATION AND SPACE OF FUNCTIONS

In this section, some definitions and notations that will be used along the study are presented. The principle of causality says that, for problems with uncertainty in the source term or in system parameters, system response will necessarily show stochastic behavior. For these problems, the solution space should contain functions to represent this random behavior. In this study, the solution space is constructed via tensor product between Sobolev and probability spaces. This originates the so-called Stochastic Sobolev Spaces.

4.1 Stochastic Sobolev Spaces

The association between the theories of probability, product tensor and Sobolev spaces originate the Stochastic Sobolev Spaces. The convergence of numerical solutions obtained in these spaces, to the theoretical solution, is based on the isomorphism between stochastic Sobolev spaces and Sobolev spaces defined in more complex measure spaces (Babuška et al., 2005 and Frauenfelder et al., 2005). The theoretical solution to the stochastic diffusion problem is defined in $V = L^2\left(\Omega,\mathcal{F},P;H^1(D)\right)$, with,

$$V = \left\{ u : D\times\Omega \to \mathbb{R} \,\middle|\, u \text{ is measurable and } \int_\Omega \left\|u(\omega)\right\|^2_{H^1(D)} dP(\omega) < +\infty \right\}. \tag{9}$$

Noticing that $L^2(\Omega,\mathcal{F},P;H^1(D)) \simeq L^2(\Omega,\mathcal{F},P)\otimes H^1(D)$, one obtains $V \simeq L^2(\Omega,\mathcal{F},P)\otimes H^1(D)$. It is also necessary to redefine the differential operator for the space obtained via tensorial product. The operator $\nabla_\omega : V \to L^2(\Omega,\mathcal{F},P)\otimes L^2(D)$, (Matthies and Keese, 2005), acts over an element $u\in V$ the following way,

$$\nabla_\omega u : (\nabla v)(x).w(\omega). \tag{10}$$

For $u,v\in V$, the following internal product is defined,

$$(u,v)_V = \int_\Omega \int_D \left(\nabla_\omega u \cdot \nabla_\omega v\right)(\boldsymbol{x},\omega)\,\boldsymbol{dx}\,dP(\omega), \tag{11}$$

this internal product induces the *V*-norm $\|u\|_V = (u,u)_V^{1/2}$.

The abstract variational problem, or weak form associated to Eq. (1), is defined as follows,

$$\begin{cases} \text{Find } u\in V \text{ such that,} \\ a(u,v) = \langle f,v\rangle, \forall v\in V; \end{cases} \tag{12}$$

where $a: V \times V \to \mathbb{R}$ is a bi-linear form, given by,

$$a(u,v) = \int_{\Omega}\int_{D} (\kappa . \nabla_{\omega} u \cdot \nabla_{\omega} v)(\boldsymbol{x},\omega)\, d\boldsymbol{x} dP(\omega), \tag{13}$$

and $\langle \cdot,\cdot \rangle$ is a duality between V and V', defined as,

$$\langle f, v \rangle = \int_{\Omega}\int_{D} (q.v)(\boldsymbol{x},\omega)\, d\boldsymbol{x} dP(\omega) + \int_{\Omega}\int_{\Gamma} (h.v)(\boldsymbol{x},\omega)\, d\boldsymbol{x} dP(\omega). \tag{14}$$

Numerical solutions are obtained for the abstract variational problem defined in Eq. (12).

5. METHOD OF GALERKIN

The Galerkin method is used in this paper to solve the stochastic diffusion problem defined in Eq. (1). It is proposed that approximate solutions for the stochastic response be obtained as

$$u(\boldsymbol{x},\omega) = \sum_{i=1}^{\infty} u_i \delta_i(\boldsymbol{x},\omega), \tag{15}$$

where $u_i \in \mathbb{R}, \forall i \in \mathbb{N}$ are coefficients and $\delta_i \in V$. Numerical solutions to the abstract variational problem defined in Eq. (12) will be obtained. Hence, it becomes necessary to define spaces less abstract than those defined earlier, but without compromising the existence and uniqueness of the solution. Consider two complete orthogonal systems $\Phi = span\left[\{\phi_i\}_{i=1}^{\infty}\right]$ and $\Psi = span\left[\{\psi_i\}_{i=1}^{\infty}\right]$, such that $\overline{\Phi}^{H^1(D)} = H^1(D)$, $\overline{\Psi}^{L^2(\Omega,\mathcal{F},P)} = L^2(\Omega,\mathcal{F},P)$ and define the tensorial product between Φ and Ψ as

$$(\phi \otimes \psi)_j(\boldsymbol{x},\omega) = \phi_k(\boldsymbol{x}).\psi_l(\omega), \text{ with } k,l \in \mathbb{N}. \tag{16}$$

To simplify the notation, we will use $\delta_j = (\phi \otimes \psi)_j$. Since approximated numerical solutions are derived in this paper, the solution space has finite dimensions. This implies truncation of the complete orthogonal systems Φ and Ψ. Hence one has $\Phi_m = span\left[\{\phi_i\}_{i=1}^{m}\right]$ and $\Psi_n = span\left[\{\psi_i\}_{i=1}^{n}\right]$, which results in $V_M = \Phi_m \otimes \Psi_n$. With the above definitions and results, it is proposed that numerical solutions are obtained from truncation of the series expressed in Eq. (18) at the *M*-th term

$$u_M(\boldsymbol{x},\omega) = \sum_{i=1}^{M} u_i \delta_i(\boldsymbol{x},\omega). \tag{17}$$

Replacing Eq. (17) in Eq. (12), one arrives at the approximated variational problem

$$\begin{cases} \text{Find } u_M \in V_M \text{ such that} \\ a(u_M, \delta_j) = \langle f, \delta_j \rangle, \forall \delta_j \in V_M. \end{cases} \tag{18}$$

Eq. (18) can be represented in vector form as,

$$\mathbf{K}\,\mathbf{U} = \mathbf{F}, \tag{19}$$

with $\mathbf{K} \in \mathrm{M}_M(\mathbb{R})$, $M = \dim(\mathbf{K}) = \mathrm{m.n}$. From Eqs. (13) and (16) the entries of matrix $\mathbf{K}$ are obtained as,

$$\mathbf{K} = \left[k_{ij}\right]_{M \times M}, \quad k_{ij} = \int_{\Omega}\int_{\Gamma} \left(\kappa . (\nabla\phi \otimes \psi)_i \bullet (\nabla\phi \otimes \psi)_j\right)(\boldsymbol{x},\omega)\, d\boldsymbol{x} dP(\omega); \tag{20}$$

and the elements of vector $\mathbf{F}$ are given by,

$$\mathbf{F} = \{f_i\}_{i=1}^{M}, \quad f_i = \int_{\Omega}\int_{D}\left(q.\left(\phi \otimes \psi\right)_i\right)(\boldsymbol{x},\omega)\boldsymbol{dx}dP(\omega) + \int_{\Omega}\int_{\Gamma}\left(h.\left(\phi \otimes \psi\right)_i\right)(\boldsymbol{x},\omega)\boldsymbol{dx}dP(\omega). \tag{21}$$

6. STATISTICAL MOMENTS

Numerical solutions to be obtained are defined in $V_M \subset V$. Interest lies in the statistical moments of the stochastic response. In this section, it is shown how the first and second order moments are evaluated from the numerical solution.

The statistical moment of k[th] order of a random variable $u(\boldsymbol{x},\cdot)$ is obtained, for a fixed point $\boldsymbol{x} \in D$, by taking the k[th] power of the stochastic response process and integrating with respect to it's probability measure,

$$\mu_u^k(\boldsymbol{x}) = \int_{\Omega} u_M^k(\boldsymbol{x},\omega)dP(\omega) = \overbrace{\sum_{i_1}\cdots\sum_{i_k}}^{k\ times} u_{i_1}\ldots u_{i_k}\int_{\Omega}\left(\delta_{i_1}\ldots\delta_{i_k}\right)(\boldsymbol{x},\boldsymbol{\xi}(\omega))dP\left(\boldsymbol{\xi}(\omega)\right). \tag{22}$$

The integration term $dP(\cdot)$ is a probability measure defined as,

$$dP\left(\boldsymbol{\xi}(\omega)\right) = \prod_{i=1}^{N}\rho_i\left(\xi_i(\omega)\right)d\xi_i(\omega), \tag{23}$$

where $\rho_i : \xi_i \to \mathbb{R}$ is the probability density function of random variable ξ_i. From the measure and integration theory, one knows that the probability measure defined in Eq. (23) is the product measure obtained from the product between probability measure spaces associated to the random variables $\boldsymbol{\xi}(\omega) = \{\xi_i(\omega)\}_{i=1}^{N}$, with $\xi_i : \Omega \to [a_i, b_i]$. With the probability measure defined in Eq. (23) one has,

$$\mu_u^k(\boldsymbol{x}) = \overbrace{\sum_{i_1}\cdots\sum_{i_k}}^{k-times} u_{i_1}\ldots u_{i_k}\int_{a_1}^{b_1}\cdots\int_{a_N}^{b_N}\left(\delta_{i_1}\ldots\delta_{i_k}\right)(\boldsymbol{x},\boldsymbol{\xi}(\omega))\rho_1(\xi_1)\ldots\rho_N(\xi_N)d\xi_1(\omega)\ldots d\xi_N(\omega). \tag{24}$$

The integrals in Eq. (21) are called iterated integrals. The first order statistical moment, or expected value, of the stochastic displacement process evaluated at a point $\boldsymbol{x} \in D$ is,

$$\mu_u(\boldsymbol{x}) = \sum_{i=1}^{m} u_{(i-1).n+1}\phi_i(\boldsymbol{x}). \tag{25}$$

The second-order statistical moment (the variance) is,

$$V_u(\boldsymbol{x}) = \sum_{i=1}^{M}\sum_{j=1}^{M} u_i u_j \int_{a_1}^{b_1}\cdots\int_{a_N}^{b_N}\left(\delta_i.\delta_j\right)(\boldsymbol{x},\boldsymbol{\xi}(\omega))\rho_1(\xi_1)\ldots\rho_N(\xi_N)d\xi_1(\omega)\ldots d\xi_N(\omega) - \sum_{i=1}^{m}\sum_{j=1}^{m} u_{(i-1).n+1}u_{(j-1).n+1}\left(\phi_i.\phi_j\right)(\boldsymbol{x}). \tag{26}$$

In the numerical example to follow, the statistical moments defined in Eqs. (22 and 26) are evaluated and compared with the same moments obtained via Monte Carlo simulation, using ten thousand samples (N_S = 10000). To evaluate the error of approximated solutions, the relative error functions in expected value and variance (ε_{μ_u} and ε_h, respectively), are defined as

$$\varepsilon_{\mu_u}(x,y) = \begin{cases} 100\% \times \left|1 - \left(\frac{\mu_{u_M}}{\mu_u}\right)(x,y)\right|, & \forall\ (x,y) \in D \\ 0, & \forall\ (x,y) \in \Gamma; \end{cases} \wedge\ \varepsilon_h(x) = \begin{cases} 100\% \times \left|1 - \left(\frac{h}{\hat{h}}\right)(x)\right|, & \forall\ x \in (0, L_x) \\ 0, & \forall\ x \in \{0, L_x\}; \end{cases} \tag{27}$$

where μ_{u_M} and μ_u are the expected value obtained via Galerkin and Monte Carlo solutions, respectively. The functions $h,\widehat{h}:\left(0,\frac{L_x}{2}\right)\to\mathbb{R}$ are defined from the covariance function, either based on the Galerkin (Cov_{u_M}) or Monte Carlo (Cov_u) solutions, as follows,

$$h(x)=Cov_{u_M}(x_1,y_1,x,y) \quad \text{and} \quad \widehat{h}(x)=Cov_u(x_1,y_1,x,y),\ \forall(x_1,y_1,x,y)\in\{x_1\}\times\{y_1\}\times(0,L_x)\times\left(0,L_y\right). \tag{28}$$

7. NUMERICAL RESULTS

In this section, numerical results to a stochastic diffusion problem are presented. The problem domain is defined as $D=\left\{(x,y)\in\mathbb{R}^2\,\middle|\,0<x<L_x,\,0<y<L_y\right\}$, with $L_x=L_y=1\,\text{m}$. The source term is constant and given by $q(x,y,\omega)=x.(x-L_x).y.(y-L_y)-\int_0^{L_y}\int_0^{L_x}x.(x-L_x).y.(y-L_y)\,dxdy$, $\forall(x,y,\omega)\in D\times\Omega$. Flux is prescribed in the boundaries:

$$\begin{cases} h(x,y,\omega)=10.\cos\left(\frac{\pi y}{L_y}\right),\ \forall(x,y)\in\Gamma_1\times\Omega; \\ h(x,y,\omega)\equiv 0, \qquad\qquad \forall(x,y)\in(\Gamma\setminus\Gamma_1)\times\Omega; \end{cases}$$

with $\Gamma_1=\left\{(x,y)\in\mathbb{R}^2\,\middle|\,x=L_x,\,0\le y\le L_y\right\}$. The numerical solutions are obtained for $m=1$, $(m=\dim\langle\Phi_m\rangle)$, and for different dimensions of Ψ_n, $n\in\{5,15,35\}$. Results for the expected value and variance of the stochastic response, and the corresponding relative errors, are computed. Numerical results were obtained in a personal computer, running a MATLAB computational code.

The diffusion coefficient is represented as a parameterized stochastic process,

$$\begin{aligned}\kappa(x,y,\omega)=\mu_\kappa(x,y)+\frac{\sigma_\kappa}{2}\Big[&\xi_1(\omega).\cos\left(\tfrac{\pi x}{2.L_x}\right)+\xi_2(\omega).\sin\left(\tfrac{\pi x}{2.L_x}\right)\\ &+\xi_3(\omega).\cos\left(\tfrac{\pi y}{2.L_y}\right)+\xi_4(\omega).\sin\left(\tfrac{\pi y}{2.L_y}\right)\Big],\quad \forall(x,y,\omega)\in D\times\Omega;\end{aligned} \tag{29}$$

where μ_κ and σ_κ are the expected value and standard deviation, respectively, and $\boldsymbol{\xi}=\{\xi_i\}_{i=1}^4$ is a vector of independent uniform random variables, with properties given in Eq. (4). For the present example, $\mu_\kappa(x,y)=2461\ \text{W/K},\forall(x,y)\in\overline{D}$ and $\sigma_\kappa=\left(\frac{1}{10}\right).\mu_\kappa$. In the present paper, no investigation of convergence of the approximated solutions is presented.

Figure 1 shows ten thousand samples $(\text{N}_S=10000)$, obtained via Monte Carlo simulation, of the random response for this problem, evaluated at $u_0(x)=u\left(x,\frac{2.L_y}{5}\right)$.

Figure 2a shows the mean of the random response in the problem domain. Figure 2b shows the "covariance" function $h(x,y)$, evaluated for $(x_1,y_1,x,y)=\left(\frac{2.L_x}{5},\frac{2.L_y}{5},x,y\right)$.

Figure 3 shows the relative error functions, defined in Eq. (27), in terms of the order of approximate Galerkin solutions, for $p\in\{1,2,3\}$. Figure 3a shows the error in the mean and figure 3b shows the error in function $h(x,y)$, both evaluated from the temperature response $u_0(x)=u\left(x,\frac{2.L_y}{5}\right)$. It is noted that the error in expected value is very small, even for $p = 1$. The error in "variance" is significant for $p = 1$, but is largely reduced for $p \ge 2$. It is also noted that the errors for $p = 2$ and $p = 3$ accumulate over each other.

It is not shown in this paper, but the proposed solution of the diffusion problem via chaos polynomials can be used to estimate moments of any order of the response. The experience of the authors, however, tells that the approximations loose quality when the order of the approximated moment increases. This is observed in Fig. 3, were for approximated solutions of same order, the error was shown to be larger for the variance than for the expected value.

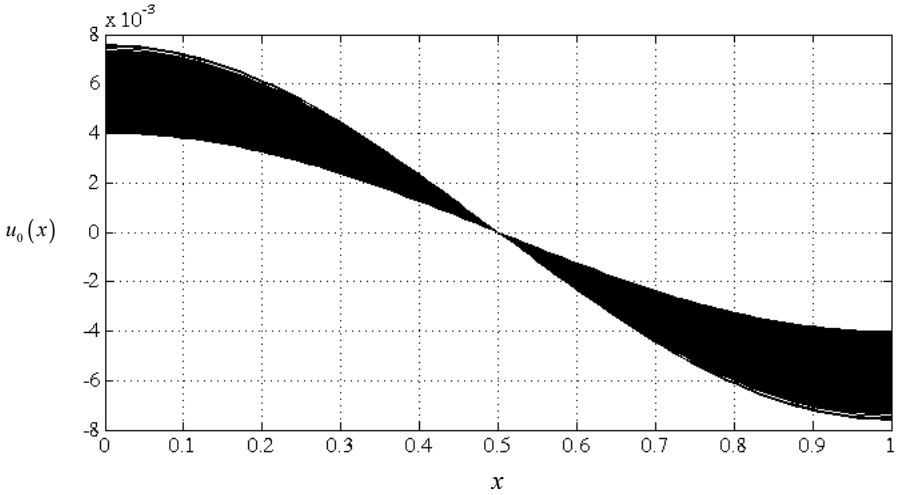

Figure 1: Realizations of the response field.

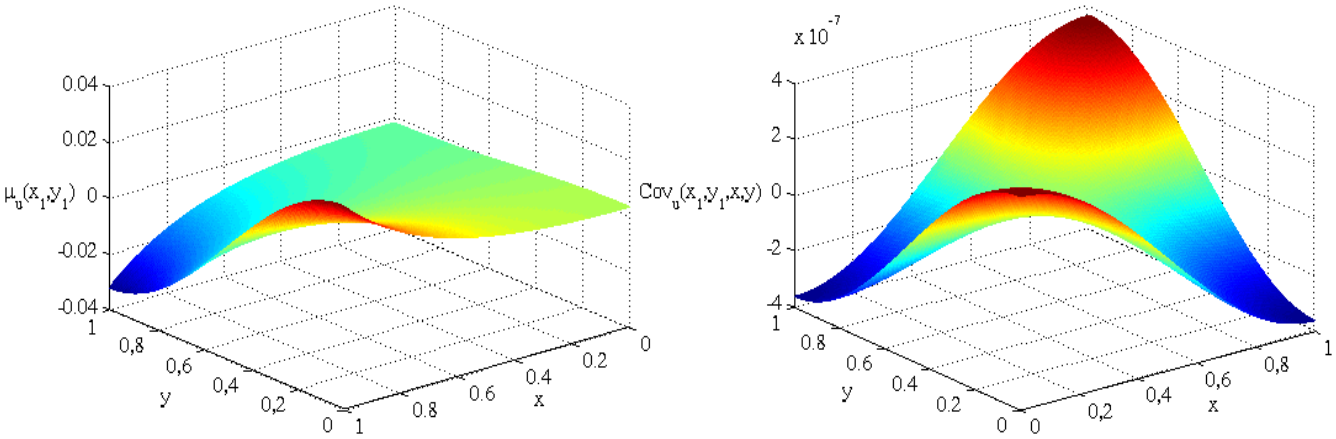

Figure 2: a) mean of the random response;
b) covariance of random response evaluated at $(x_1, y_1, x, y) = \left(\frac{2L_x}{5}, \frac{2L_y}{5}, x, y\right)$.

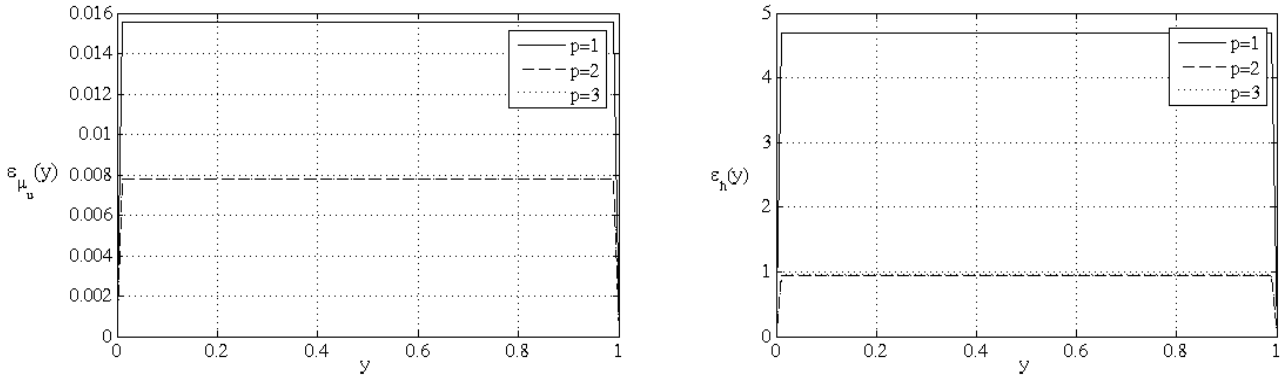

Figure 3: Relative error in expected value (a) and in “variance” (b) of random response.

8. CONCLUSIONS

In this paper, the Galerkin method was used to obtain approximate solutions to stochastic diffusion problems. The uncertainty in diffusion coefficient was represented as a parameterized stochastic process. The space of approximate solutions was constructed from the tensorial product between Sobolev spaces and Legendre polynomials, derived via Askey-Wiener scheme. The proposed solution scheme was applied to an example problem. Numerical solutions were obtained for increasing orders of chaos polynomials. From the approximate solutions, first and second order moments were computed and compared with corresponding estimates obtained via Monte Carlo simulation. Results show the good performance of chaos polynomials in estimating the first and second order moments of the response field. Results also show that the approximated solutions loose quality as the order of the computed moment increases. The Chaos-Galerkin scheme developed herein is shown to be a theoretically sound and efficient method for the solution of stochastic problems in engineering

ACKNOWLEDGEMENTS

The authors kindly acknowledge financial support of this research project by the agencies CNPq (National Council for Research and Development) and FAPESP (São Paulo State Foundation for Research).

REFERENCES

Askey, R.; Wilson, J., 1985, Some Basic Hypergeometric Polynomials that Generalize Jacobi Polynomials, Mem. Amer. Math. Soc. 319, AMS, Providence, RI.

Babuska, I.; Tempone, R.; Zouraris, G. E., 2005, Solving elliptic boundary value problems with uncertain coefficients by the finite element method: the stochastic formulation, Computer Methods in Applied Mechanics and Engineering, v. 194, n. 12-16, p. 1251-1294.

Cameron, R.H.; Martin, W.T., 1947, The orthogonal development of nonlinear functionals in series of Fourier-Hermite functionals, Annals Mathematics, n. 48, pp. 385-392.

Emery A.F., 2004, Solving stochastic heat transfer problems, Engineering Analysis with Boundary Elements, n. 28, pp. 279–291.

Frauenfelder P.; Schwab C. and Todor R. A., 2005, Finite elements for elliptic problems with stochastic coefficients, Computer Methods in Applied Mechanics and Engineering, v. 194, n. 2-5, pp. 205-228.

Grigoriu, M., 1995, Applied Non-Gaussian Processes: Examples, Theory, Simulation, Linear Random Vibration, and Matlab Solutions, Prentice Hall, p. 442.

Hien T. D.; Kleiber M., 1997, Stochastic finite element modeling in linear transient heat transfer, Computer Methods in Applied Mechanics and Engineering, n. 144, pp. 111-124.

Jason S., 1997, Gaussian Hilbert Spaces, Cambridge University Press, pp. 337.

Jardak M.; Su C. H.; Karniadakis G. E., 2002, Spectral Polynomial Chaos Solutions of the Stochastic Advection Equation, Journal of Scientific Computing, v. 17, n. 1-4, pp. 319-338.

Jin B.; Zou J., 2008, Inversion of Robin coefficient by a spectral stochastic finite element approach, Journal of Computational Physics, v. 227, n. 6, pp. 3282-3306.

Kaminski M.; Hien T. D., 1999, Stochastic finite element modeling of transient heat transfer in layered composites, International Communication Heat Mass Transfer, v. 26, n. 6, pp. 801-810.

Matthies, H. G.; Keese A., 2005, Galerkin methods for linear and nonlinear elliptic stochastic partial differential equations, Computer Methods in Applied Mechanics and Engineering, v. 194, n. 12-16, p. 1295-1331.

Ogura, H., 1972, Orthogonal functionals of the Poisson process, IEEE Trans. Inform. Theory, v. 18, n. 4, p. 473–481.

Spanos, P. D.; Ghanem, R., 1989, Stochastic finite element expansion for media random, Journal Engineering Mechanics, v. 125, n. 1, pp. 26-40.

Wiener, N., 1938, The homogeneous chaos, American Journal Mathematics, n. 60, pp. 897-936.

Xiu D.; Karniadakis G.E., 2003, A new stochastic approach to transient heat conduction modeling with uncertainty, International Journal of Heat and Mass Transfer, n. 46, pp. 4681–4693.

Xiu D.; Shen J., 2007, An Efficient Spectral Method for Acoustic Scattering from Rough Surfaces, Comm. Comput. Phys., v. 1, n. 2, pp. 54-72.

RESPONSIBILITY NOTICE

The authors are the only responsible for the printed material included in this paper.