A VARIATIONAL FORMULATION OF NONLINEAR VISCOELASTIC MODELS IN FINITE STRAIN REGIME

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Abstract. *The purpose of this work is to present a variational approach for constitutive viscoelastic models based on the mathematical background proposed in [9]. The approach is called variational since the constitutive updates obey a minimum principle within each load increment. The set of internal variables is strain-based and employs a multiplicative decomposition of strain in elastic and viscous components. The formulation has the particular characteristic of being able to accommodate, into simple analytical expressions, a wide set of specific models. Moreover, appropriate choices of the constitutive potentials allow the reproduction of other formulations in literature. At the end of the paper, some numerical examples contribute to show the characteristics of the proposed formulation.*
Keywords: *finite viscoelasticity, variational formulation, constitutive updates*

1. Introduction

The main characteristic in viscoelastic models is the existence of a rate of deformations, usually called creep, in the presence of non-zero states of stress. Moreover, inelastic strains occur in the presence of stresses, no matter their intensity. This behavior is in contrast with other models like plasticity or viscoplasticity, where a certain level of stress must be reached before inelastic deformations appear.

Many different models for viscoelastic materials in finite deformation regime are found in literature. However, in contrast with what we see in small deformation models, the choice of convenient internal variables and evolution laws is not trivial nor unique, leading to different formulations. We may distinguish two possible approaches. From one side, we recall the the work of Simo [13] in which an additive decomposition of stresses in which an additive decomposition of stresses in equilibrium and non equilibrium contributions is stated and the evolution law is defined as a linear differential equation on the non-equilibrium stresses. This approach was later followed, among many others, by [5], [3] or, more recently, [4],[1]. Multiplicative decomposition of strains applied to viscoelastic constitutive equations goes back to the work of Sidoroff [12] and later to [7], [10], [6], among others. In [11] the ability of different models to reproduce nonlinear viscous behavior is discussed and a model is proposed which is not restricted to small perturbations away from thermodynamic equilibrium.

The goal of this work is to provide a general framework for constitutive viscoelastic models based on the mathematical background proposed in [9],[14]. Thus, the approach is qualified as variational since the constitutive updates obey a minimum principle within each load increment. The set of internal variables is strain-based and thus employs, according to the specific model chosen, multiplicative decomposition of strain in elastic and viscous components.

Inserted in the same theoretical framework, this particularization and that for plasticity or viscoplasticity, share the same technical procedures to deal with the local nonlinear constitutive problem, i.e. the solution of a minimization problem to identify inelastic updates and the use of exponential mapping for time integration [2, 8]. However, instead of using the classic decomposition of inelastic strains into “size” and “direction”, we take profit of a spectral decomposition that provides additional facilities to accommodate, into simple analytical expressions, a wide set of viscous models. It is also possible to show that an appropriate choice of constitutive potentials allows to retrieve other models in literature.

2. Variational form of constitutive equations

Using conventional notation, let us call $\mathbf{F} = \nabla_0 \mathbf{x}$ the gradient of deformations, and $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ the Cauchy strain tensor, respectively. These values may be decomposed in volumetric and isochoric parts. The isochoric tensors are defined as follows:

$$\hat{\mathbf{F}} = \frac{1}{J^{1/3}}\mathbf{F}, \quad J = \det(\mathbf{F}), \quad \hat{\mathbf{C}} = \hat{\mathbf{F}}^T\hat{\mathbf{F}} = \frac{1}{J^{2/3}}\mathbf{F}^T\mathbf{F}, \tag{1}$$

We will work in the framework of irreversible thermodynamics, with internal variables. Thus, we define a general set $\mathcal{E} = \{\mathbf{F}, \mathbf{F}^i, \mathbf{Q}\}$ of external and internal variables, where $\mathbf{F}^i$ is the inelastic part of the (total) deformation, and $\mathbf{Q}$ contains all the remaining internal variables of the model. In addition, a multiplicative decomposition $\mathbf{F} = \mathbf{F}^e\mathbf{F}^i$ of the gradient of deformations is considered. We assume the existence of a free energy potential $W(\mathcal{E})$ and a dissipative potential $\phi(\dot{\mathbf{F}}; \mathcal{E})$, such that the Piola-Kirchhoff stress tensor, comprised of an equilibrium (elastic) and a dissipative (viscous) components, is derived as follows:

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}}(\mathcal{E}) + \frac{\partial \phi}{\partial \dot{\mathbf{F}}}(\dot{\mathbf{F}}; \mathcal{E}). \tag{2}$$

In addition, another dissipative potential $\psi(\dot{\mathbf{F}}^i, \dot{\mathbf{Q}}; \mathcal{E})$ is included to characterize the irreversible behavior related to the inelastic tensor $\mathbf{F}^i$, such that

$$\mathbf{T} = -\frac{\partial W}{\partial \mathbf{F}^i}(\mathcal{E}) = \frac{\partial \psi}{\partial \dot{\mathbf{F}}^i}(\dot{\mathbf{F}}^i, \dot{\mathbf{Q}}; \mathcal{E}), \qquad \mathbf{A} = -\frac{\partial W}{\partial \mathbf{Q}}(\mathcal{E}) = \frac{\partial \psi}{\partial \dot{\mathbf{Q}}^i}(\dot{\mathbf{F}}^i, \dot{\mathbf{Q}}; \mathcal{E}). \tag{3}$$

It was shown in [9] and [14] that an incremental version of the above equations, constituting an incremental update method for the material state, can be obtained from the following incremental potential:

$$\mathcal{W}(\mathbf{F}_{n+1}; \mathcal{E}_n) = \Delta t \phi\left(\mathring{\mathbf{F}}, \mathcal{E}_n\right) + \min_{\substack{\mathbf{F}^i_{n+1} \\ \mathbf{Q}_{n+1}}} \left\{ W(\mathcal{E}_{n+1}) - W(\mathcal{E}_n) + \Delta t \psi\left(\mathring{\mathbf{F}}^i, \mathring{\mathbf{Q}}; \mathcal{E}_n\right)\right\}, \tag{4}$$

where $\mathring{\mathbf{F}}\left(\mathbf{F}_{n+1}, \mathcal{E}_n\right)$, $\mathring{\mathbf{F}}^i\left(\mathbf{F}^i_{n+1}, \mathcal{E}_n\right)$ and $\mathring{\mathbf{Q}}\left(\mathbf{Q}_{n+1}, \mathcal{E}_n\right)$ are suitable incremental approximations of the rate variables $\dot{\mathbf{F}}, \dot{\mathbf{F}}^i$ and $\dot{\mathbf{Q}}$ respectively.

3. A group of visco-hyperelastic models

3.1 General form

A quite general group of viscoelastic materials can be modelled within the present variational framework. Due to the possibility of obtaining analytical or semi-analytical expression for the constitutive updates, only isotropic models will be considered now. However, no theoretical constraints to include more general behaviors are found. The rheological

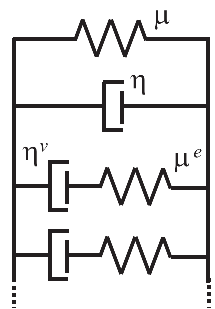

Figure 1. Generalized Kelvin-Maxwell model.

mechanism shown in Figure 1 is taken as a basis to include different potentials expressions in (4). The model is based on the following assumptions:

- The elastic part of the Kelvin branch is split in isochoric and volumetric energies. The isochoric part is an isotropic function of $\hat{\mathbf{C}} = \hat{\mathbf{F}}^T\hat{\mathbf{F}}$:

$$\varphi(\hat{\mathbf{C}}) = \varphi(c_1, c_2, c_3), \tag{5}$$

where c_j are the eigenvalues of $\hat{\mathbf{C}}$. The volumetric part may be defined using the usual expression $U(J) = \frac{K}{2}\left[\ln J\right]^2$, The viscous part of the Kelvin branch is an isotropic function of the symmetric part of the rate of deformation:

$$\phi(\mathbf{D}) = \phi(d_1, d_2, d_3) \quad \text{with} \quad \mathbf{D} = \operatorname{dev}\left(\operatorname{sym}\left(\dot{\mathbf{F}}\mathbf{F}^{-1}\right)\right), \tag{6}$$

where d_j are the eigenvalues of $\mathbf{D}$.

- The Maxwell branch, connected in parallel, is based on a multiplicative split of strains in an elastic and an isochoric inelastic (viscous) part:

$$\hat{\mathbf{F}} = \hat{\mathbf{F}}^e \mathbf{F}^v \Longrightarrow \hat{\mathbf{F}}^e = \hat{\mathbf{F}} \mathbf{F}^{v-1}, \quad \det \mathbf{F}^v = 1. \tag{7}$$

A flow rule for the internal variable $\mathbf{F}^v$ can be written as:

$$\dot{\mathbf{F}}^v = \mathbf{D}^v \mathbf{F}^v = \left(d_j^v \mathbf{M}_j^v \right) \mathbf{F}^v, \tag{8}$$

in which the spectral decomposition of $\mathbf{D}^v = \text{sym}(\dot{\mathbf{F}}^v \mathbf{F}^{v-1})$ in eigenvalues d_j^v and eigenprojections $\mathbf{M}_j^v$, $j = 1, 2, 3$, was used. The scalars d_j are chosen to be the internal variables contained in the set $\dot{\mathbf{Q}} = \{d_1, d_2, d_3\}$. In this case, it is important to note that (8) is a constraint relating the internal variables $\mathbf{F}^v$ and $\mathbf{Q}$. The elastic and viscous potentials associated to this branch are assumed to be isotropic functions of the elastic deformation and viscous stretching, and thus depend on their eigenvalues:

$$\varphi^e(\hat{\mathbf{C}}^e) = \varphi^e(c_1^e, c_2^e, c_3^e) \quad \text{and} \quad \psi(\mathbf{D}^v) = \psi(d_1^v, d_2^v, d_3^v), \tag{9}$$

where c_j^e are the eigenvalues of $\hat{\mathbf{C}}^e$.

- Viscous deformations are incrementally updated by exponential mappings:

$$\Delta \hat{\mathbf{F}} = \hat{\mathbf{F}}_{n+1} \hat{\mathbf{F}}_n^{-1} = \mathbf{R} \left(\exp[\Delta t \mathbf{D}] \right) \quad \Rightarrow \mathbf{D} = \frac{\Delta q_j}{\Delta t} \mathbf{M}_j = \frac{1}{2\Delta t} \ln \left(\Delta \hat{\mathbf{C}} \right). \tag{10}$$

$$\Delta \mathbf{F}^v = \mathbf{F}_{n+1}^v \mathbf{F}_n^{v-1} = \exp[\Delta t \mathbf{D}^v] \quad \Rightarrow \mathbf{D}^v = \frac{\Delta q_j^v}{\Delta t} \mathbf{M}_j^v = \frac{1}{2\Delta t} \ln \left(\Delta \mathbf{C}^v \right). \tag{11}$$

Expressions (10) and (11) show that $\mathbf{D}$ and $\mathbf{D}^v$ are approximated by incremental expressions of $\Delta \hat{\mathbf{C}}$ and $\Delta \mathbf{C}^v$ respectively. The exponential mapping has the particular convenient property of providing a isochoric tensor for any traceless argument [2, 8].

Taking into account (10) and (11), the minimizing variables $\mathbf{Q}_{n+1}, \mathbf{F}_{n+1}^v$ in (4) are replaced by the new incremental variables $\Delta q_j^v, \mathbf{M}_j^v$.:

$$\begin{aligned} &\mathcal{W}(\mathbf{F}_{n+1}; \mathcal{E}_n) = \mathcal{W}(\mathbf{C}_{n+1}; \mathcal{E}_n) = \Delta \varphi(\hat{\mathbf{C}}_{n+1}) + \Delta t \phi \left(\tfrac{\Delta q_j^v}{\Delta t} \right) + \Delta U(\theta_{n+1}) \\ &+ \min_{\mathbf{M}_j^v, \Delta q_j^v} \left\{ \Delta \varphi^e(\hat{\mathbf{C}}_{n+1}^e) + \Delta t \psi \left(\tfrac{\Delta q_j^v}{\Delta t} \right) \right\}, \end{aligned} \tag{12}$$

such that

$$\Delta q_j^v \in K_Q = \{ p_j \in \mathbb{R} : p_1 + p_2 + p_3 = 0 \}, \tag{13}$$

$$\mathbf{M}_j^v \in K_M = \{ \mathbf{N}_j \in Sym : \mathbf{N}_j \cdot \mathbf{N}_j = 1, \ \ \mathbf{N}_i \cdot \mathbf{N}_j = 0, \ \ i \neq j \}. \tag{14}$$

The set K_Q enforces the traceless form of $\mathbf{D}^v$, while the set K_M accounts for usual properties of eigenprojections. Moreover, it is easy to verify that both sets are convex on their respective variables. Given isotropic expressions for energy functions, the minimization in (12) can be performed analytically. A simple extension to this model can be obtained by considering a set of P Maxwell branches, as seen in Figure 1(a).

3.2 Hencky and Ogden models

Hencky models are based on quadratic forms of logarithmic strain tensors:

$$\varphi = \mu \sum_{j=1}^{3} \left(\epsilon_j \right)^2, \quad \phi = \eta \sum_{j=1}^{3} \left(d_j \right)^2, \tag{15}$$

$$\varphi^e = \mu^e \sum_{j=1}^{3} \left(\epsilon_j^e \right)^2, \quad \psi = \eta^v \sum_{j=1}^{3} \left(d_j^v \right)^2. \tag{16}$$

In this case, it is particularly convenient to obtain simple uncoupled linear expressions for the minimizing argument Δq_j^v. In spite of the facility offered by Hencky models in terms of analytical treatment, it is well known that these type of hyperelastic potentials do not fit well the behavior of rubber-like materials. For that case, a more adequate choice may

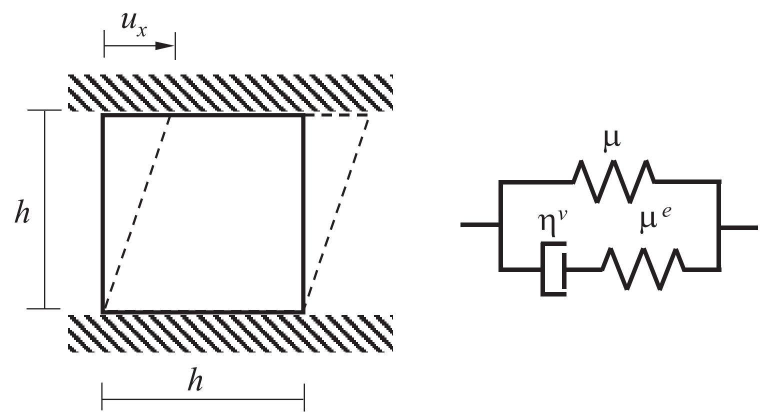

Figure 2. Cyclic shear test

be the Ogden model which has also the capability of generalizing other models like neo-Hookean and Mooney-Rivlin. Ogden models are based on the following potentials:

$$\varphi = \sum_{j=1}^{3}\sum_{p=1}^{N} \frac{\mu_p}{\alpha_p}\left([\exp(\epsilon_j)]^{\alpha_p} - 1\right), \quad \phi = \sum_{j=1}^{3}\sum_{p=1}^{N} \frac{\eta_p}{\alpha_p}\left([\exp(d_j)]^{\alpha_p} - 1\right), \tag{17}$$

$$\varphi^e = \sum_{j=1}^{3}\sum_{p=1}^{N} \frac{\mu_p^e}{\alpha_p}\left(\left[\exp(\epsilon_j^e)\right]^{\alpha_p} - 1\right), \quad \psi = \sum_{j=1}^{3}\sum_{p=1}^{N} \frac{\eta_p^v}{\alpha_p}\left(\left[\exp(d_j^v)\right]^{\alpha_p} - 1\right). \tag{18}$$

4. Numerical examples

4.1 Shear test

Consider a pure shear test of a single 3D element (Figure 2). Material parameters and load characteristics were taken from an equivalent example in [11] and the rheological model corresponds to that of figure 2, i.e. the potential $\phi = 0$.

The lateral displacement follows a sinusoidal law $u_x = U_x \sin wt$, where $w = 0.3\ s^{-1}$. The quotient u_x/h is numerically equal to the shear component $\mathbf{C}_{xy}$ (or $\mathbf{B}_{xy}$) in the cartesian coordinate system. The material was assumed to be almost incompressible through a convenient penalization value of K.Three cases are tested:

- Case 1: Hencky model for all potentials φ, φ^e and ψ.
- Case 2. Ogden model for φ and Hencky model for φ^e and ψ.
- Case 3. Ogden model for φ, φ^e and and Hencky model for ψ.

For Ogden models we used $p = 3$ with exponents $\alpha_1 = 1.8$; $\alpha_2 = -2$; $\alpha_3 = 7$. The relation $\mu_i^e = 2.57\mu_i$ and viscous coefficients η_i^v such that $\tau = \frac{\eta_i^v}{\mu_i^e} = 17.5$ has also been considered. For Hencky cases, the value $\mu = \sum_i \frac{1}{2}\mu_i\alpha_i$ was used, which is the consistent equivalent parameter for small deformations. Material parameters take, depending on the case, the following values:

Table 1. Material parameter values.

Potential	Ogden			Hencky
$\varphi(\mu_i)$	20	-7	1.5	$\mu = 30.25 = \sum_i \frac{1}{2}\mu_i\alpha_i$
$\varphi^e(\mu_i^e)$	51.4	-18	3.86	$\mu^e = 77.77 = \sum_i \frac{1}{2}\mu_i^e\alpha_i$
$\psi(\eta_i)$	899.5	-315	67.55	$\eta^v = 1361 = \sum_i \frac{1}{2}\eta_i^v\alpha_i$

Two different different shear amplitudes: 0.01 and 2, are shown in the following figures.

In the case of small strains shown in Figure 3, all models give identical results, as expected. For other amplitudes, quite different results are found.

The results of cases 1,2,3 for a displacement of 2mm are shown in Figures 4 compares the result of cases 1 and 3 while 5 compares the results form cases 2 and 3. From this figures it is clear the influence of the main spring on the average

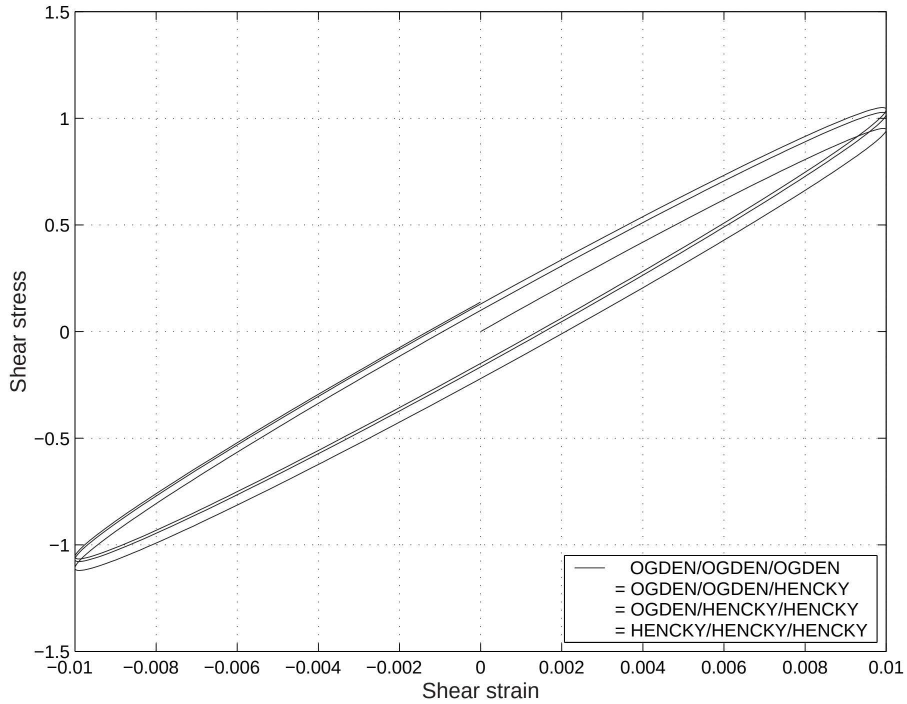

Figure 3. Cyclic shear test. Shear amplitude: 0.01

shape of the curve. The Ogden-type spring correctly follow the expected rubber-shape curves for large strains. However, hysteresis loops of case 2 (Hencky model in the elastic spring), are clearly "thinner" than that of the case 3 (Ogden model in the elastic spring). This behavior is in agreement with the fact that the Hencky model used in the Maxwell branch provides a contribution in stress much lower than a corresponding Ogden model for high deformations. Case 3 fit the non-linear case of [11], at least with eye accuracy.

4.2 Pinched cylinder

Figure 6 shows a thick viscoelastic cylinder and a rigid hemispherical tool performing the following path: first, it pinches the cylinder down to a cursor-end position where it remains during a relaxing period. Finally, it is removed at high velocity, which produces different final piece shapes and different times of separation, depending on its initial velocity. It is worth to remark that large displacements but moderate strains are present in this case.

Cylinder dimensions are $R_i = 12mm$, $R_e = 16mm$, $L = 20mm$. The tool has a radius of $2.5mm$ and its dispacement is equal to R_i. Ogden model with the same parameters was used for both springs potentials φ and φ^e : $\mu_1 = 2.758$; $\mu_2 = -1.725$; $\mu_3 = 0.704$ (MPa) and $\alpha_1 = 1.33$; $\alpha_2 = -3.05$; $\alpha_3 = 3.89$. The viscous potential ψ is of Hencky type with $\eta^v = 1.1394$ ($\tau = 2$ s). Potential ϕ is null.

Figure 7(a) shows the deformed configuration for the maximum tool displacement, while 7(b) illustrates the final configuration after the tool removing. The history of the Cauchy stress σ_x and displacement of the point situated on the cylinder interior surface just below the punch tip is plotted in figure 8. The history of the former point displacement is shown in 9. As expected, the faster the punch is applied, the higher stresses are developed and the higher the dissipated energy at the removal tool time $t = 3$s. Consequently, due to a higher relaxation in the system we see a faster separation from the tool during its back motion.

5. Conclusion

We presented in this article a general set of viscoelastic constitutive models based on a variational framework which provides appropriate mathematical structure for further applications like, for example, error estimation. The theoretical and numerical background is stated for general isotropic constitutive functions depending on eigenvalues of strains and strain-rates. As a consequence, most of the implementation effort, including stress updates and tangent matrix is done at generic level with no relation to a specific isotropic law (potential).

In the numerical examples we compared the capacity of Hencky and Ogden-type models to reproduce observed non-linear viscous behaviour. As expected, Ogden models perform better for the case of large strains and rubber-like materials, leading to non-elliptic hysteretic loops. Moreover, this choice do not represent, in the present formulation, any additional

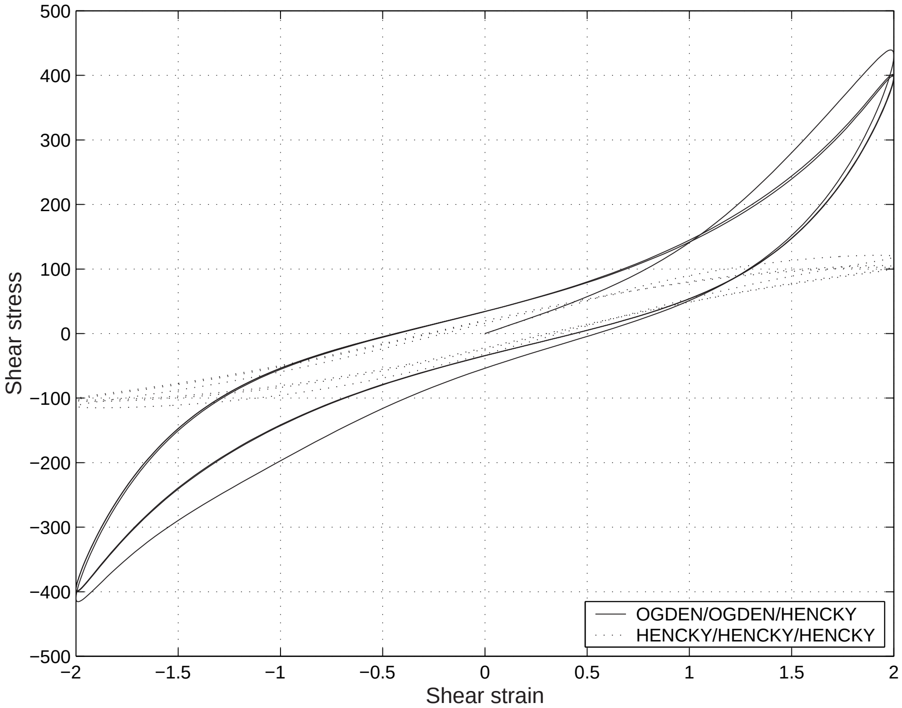

Figure 4. Cyclic shear test. Shear amplitude: 2

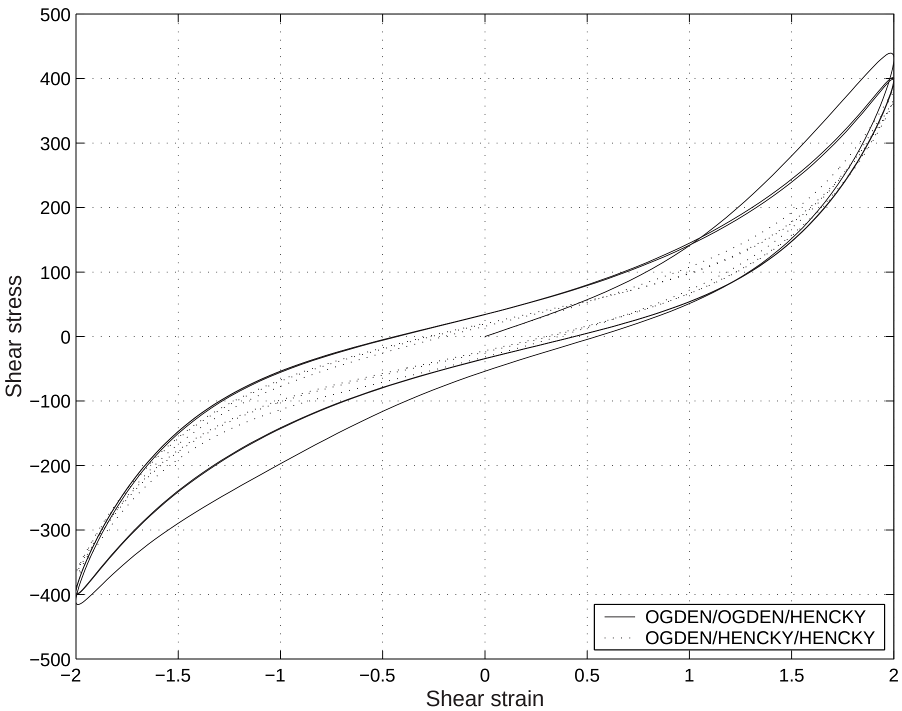

Figure 5. Cyclic shear test. Shear amplitude: 2

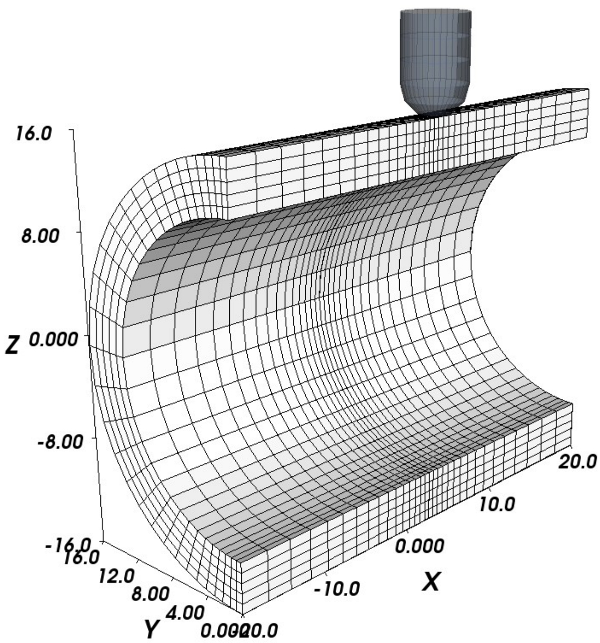

Figure 6. Pinched cylinder. Undeformed configuration.

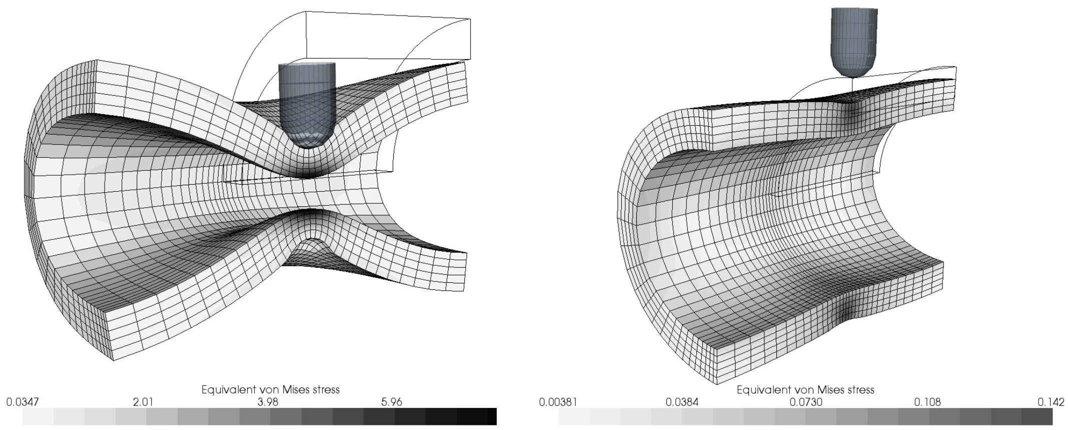

Figure 7. Pinched cylinder. (a)Intermediate and (b) final configurations.

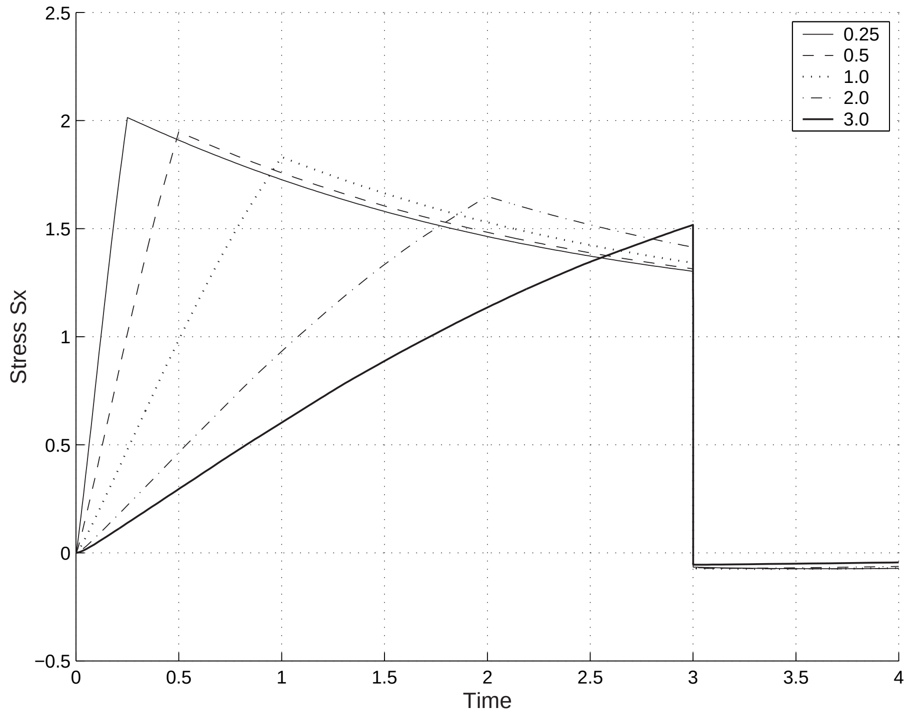

Figure 8. Cauchy stress σ_x versus time.

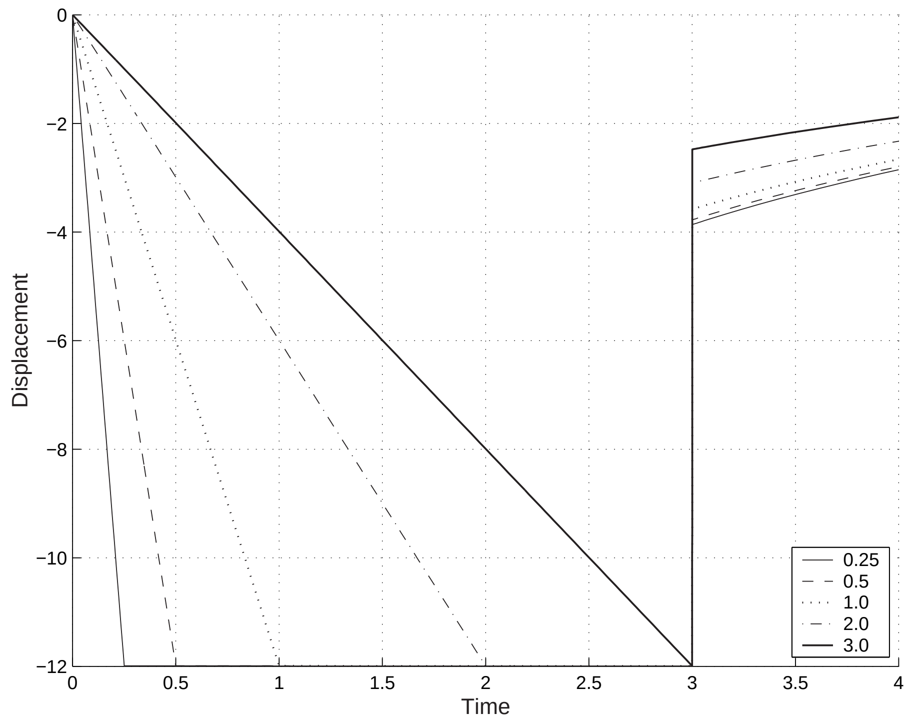

Figure 9. Displacement u_z of point interior point below punch tip.

complexity to the numerical implementation or additional cost on computations.

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