LIMITING THERMAL ENERGY RATE INSIDE NUCLEAR FUEL RODS WITH VARIABLES SOURCES

Marcelo Ferreira Pelegrini
Departamento de Engenharia Mecânica, UNESP/Ilha Solteira, Av. Brasil, 56, Centro, CEP 15385-000, Ilha Solteira, SP, Brasil
marcelo@dem.feis.unesp.br

Thiago Antonini Alves
antonini@dem.feis.unesp.br

Cassio Roberto Macedo Maia
cassio@dem.feis.unesp.br

Ricardo Alan Verdú Ramos
ramos@dem.feis.unesp.br

Abstract. *The study of diffusive problems with variable sources in its domain has important application in the project of fuel cells in nuclear reactors. Such cells have small dimension and liberate high rates of thermal energy as result of the fission nuclear of the material. A design parameter of such reactors is related to the control of the maximum power generated in a unity, because must be taken into account the temperature limit of the fissile material. In this case, it is important to consider the geometrical characteristics and the thermophysical properties of the fuel rods. In this work cylindrical fuel rods are studied to verify the influence of the shape of the fuel rod cross-section in the limitation of the thermal energy generation rate. In this analysis, the variable source term, is a function of the neutron diffusion within of the rods. Common fuel rods having circular cross-section are taken as reference for the analysis of another cylindrical fuel rods having cross-section with rectangular or elliptical shapes. Due to larger difference temperature inside fuel rods it is assumed that the thermal conductivity is dependent of the temperature. Consequently, the differential equation that models the heat diffusion process has a non-linear structure. The temperature distribution is obtained by the application of the Kirchhoff transformation and of the integral transformation technique onto the diffusion equation. Thus, physical parameters of interest were determined for some cases and the obtained results were compared with the results available in the literature.*

Keywords: *diffusive problems, generalized integral transform technique, Kirchhoff's transformation, fuel rods.*

1. Introduction

The fuel rods of nuclear reactors have a peculiar characteristic because they are designed to produce high density of thermal power, as a result of the nuclear fissile reaction of the fissionable element. For these reactors, the maximum specific power delivered is related to some making factors of the fuel cell-core and, especially, to the temperature limit that a fuel rod is allowed to reach. The critical limit corresponds to the fuel fusion temperature and it is a fundamental condition for the operation, according to Glasstone and Sesonske (1994). Therefore, the process for the maximum transference of the energy generated in the nuclear fuel to the reactor request, among other actions, the need for research development about fuel rods that obey such limiting condition. The main factor that limits the power density in the fuel cell is related to its size and to its geometrical characteristics. Fuels rods with large dimension have a bigger thermal resistance to the thermal diffusion and, consequently, they will deliver a smaller extraction of specific power in order to obey the limiting temperature affordable by the fissile material and, due to that, such elements are made with small dimension. The shape of the cell also plays an important role in the heat diffusion process. So, as bigger the heat exchange surface for an even fuel volume, smaller will be the thermal resistance. For this case it is possible to obtain a higher power density.

In this context, this work deals with the influence of the fuel cell shape on the maximum generation rate of thermal energy. Taking as reference the cylindrical cells with circular cross-section, it is investigated the cells with rectangular and elliptical cross-section. As the difference temperature in the fuel is very larger, it will be considered also a general dependency of thermal conductivity with the temperature. In particular, the present challenge is to explore the potential of the Generalized Integral Transform Technique - GITT, developed by Cotta (1998), which is being used with success to solve several kinds of diffusive problems such those ones involving irregular domains (Aparecido *et al.*, 1989); diffusive problems with moving boundaries (Diniz *et al.*, 1990); non linear diffusive problems (Alves *et al.*, 2004; Maia *et al.*, 2002; Maia *et al.*, 2003 and Pelegrini *et al.*, 2004). For the present work it will be applied a proper coordinate system change to the domain in order to facilitate the geometrical representation of the proposed shapes as well as to facilitate the application of boundary conditions. The non-linear diffusion equation will be conveniently linearized by using a change of variable known as Kirchhoff's transformation (Özisik, 1993). So, it is obtained proper conditions to apply the GITT to the energy equation to obtain the analytical solution to the temperature distribution within the cell and, consequently, to determine the limiting specific power for several shapes of the fuel rods.

2. Analysis

For the proposed cells it will be considered that the source term is proportional to the neutron diffusion flux through the fuel rods. The neutron diffusion within a nuclear fuel rod is a complex phenomenon and is difficult to solve (Glasstone, 1994). Thus, in this work, this problem will be studied by using Fick's Diffusion Law. This procedure was used in the project of the first fission power reactors and nowadays it is utilized for a first approximation.

The temperature drop in the fuel housing as well as in the gap between fuel rod and fuel housing will not be treated here. All analysis will be done only for the diffusion in the fuel rod that presents a thermal conductivity dependent of the temperature, $k = k(T)$. As the temperature gradient along the fuel housing is relatively small when compared to the temperature gradients in the whole cell, it will be assumed here that the temperature is constant along the boundary of the fuel rod. In this model, the diffusion equation for cylindrical cells with cross-section domain Ω and boundary Γ, in steady state, is given by:

$$\nabla \cdot \left[k(T) \nabla T(x,y) \right] + \bar{q}''' g(x,y) = 0 , \qquad \{(x,y) \in \Omega\}, \tag{1}$$

$$T(x,y) = T_P , \qquad \{(x,y) \in \Gamma\}. \tag{2}$$

where $\bar{q}'''$ represents the average source term, $g(x,y)$ is the non-dimensional and normalized neutron flux in the cell and T_P is the temperature on the fuel rod surface.

2.1. Linearization of diffusion equation

For the facilitation of the analytical procedure, the differential equation will be adequately linearized by using Kirchhoff's transformation and defining a variable T^* as follows:

$$T^*(x,y) = \frac{1}{k_0} \int_{T_0}^{T(x,y)} k(T') dT' , \tag{3}$$

where $k_0 = k(T_0)$, being T_0 a reference temperature. From this definition follows that:

$$dT^* = \frac{1}{k_0} k(T) dT , \tag{4}$$

which permits to write the gradient of the T^* as:

$$\nabla T^*(x,y) = \frac{1}{k_0} k(T) \nabla T(x,y). \tag{5}$$

With this new variable definition the diffusion equation is then linearized to become:

$$\nabla^2 T^*(x,y) + \frac{\bar{q}'''}{k_0} g(x,y) = 0 , \qquad \{(x,y) \in \Omega\}, \tag{6}$$

$$T^*(x,y) = T_P^* = \frac{1}{k_0} \int_{T_0}^{T_P} k(T') dT' , \qquad \{(x,y) \in \Gamma\}. \tag{7}$$

where T_P^* is the Kirchhoff changed temperature at the boundary.

In its dimensionless form Eq. (6) can be rewritten as:

$$\nabla^2 \theta(X,Y) + g(X,Y) = 0 , \qquad \{(X,Y) \in \Omega\}, \tag{8}$$

$$\theta(X,Y) = 0 , \qquad \{(X,Y) \in \Gamma\}, \tag{9}$$

$$X = \frac{x}{L_{ref}} , \qquad Y = \frac{y}{L_{ref}} , \qquad \theta(X,Y) = \frac{\left[T^*(X,Y) - T_P \right] k_0}{L_{ref}^2 \, \bar{q}'''} , \tag{10a, b, c}$$

where the parameter L_{ref} is a characteristic length for each considered shape. The shape and geometrical parameters for proposed cell shapes are shown in Fig. 1. The cell shapes present symmetry to the axes X and Y, therefore it is sufficient to consider for solution domain just one quadrant of the original domain, represented by gray areas in Fig. 1.

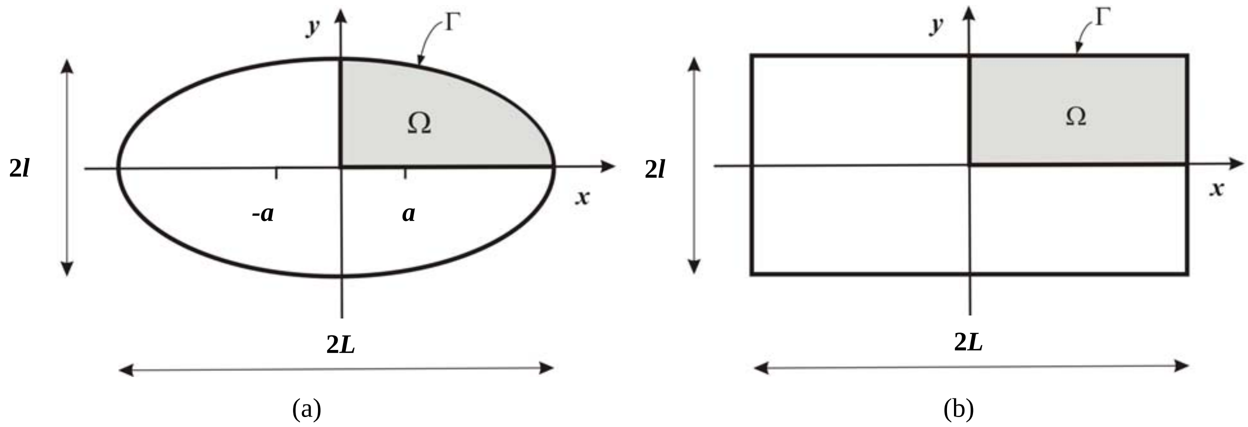

Figure 1. Representation of the cells considered: (a) Elliptical cell cross-section and (b) Rectangular cell cross-section.

2.2. Changing the coordinate system

For cells with elliptical cross-section the cartesian coordinate system does not facilitate its shape representation. Therefore, it is adequate to proceed a adequate change in the coordinate system in order to facilitate the application of the boundary conditions.

2.2.1. Cell with elliptical cross-section

The orthogonal elliptical coordinate system is used to change from the original domain with boundary shaped as an ellipsis on the plane (X,Y) to a new domain with boundary shaped as a rectangle defined on the new plane (u,v):

$$X = a^* \cos(u)\cosh(v), \qquad Y = a^* \sin(u)\sinh(v), \tag{11a, b}$$

with

$$a^* = \frac{a}{L_{ref}}, \qquad a = \frac{L}{\cosh(v_o)}, \qquad v_o = arc\ tanh\left(\frac{l}{L}\right), \qquad L_{ref} = \frac{2\,A_S}{Per}. \tag{12a, b, c, d}$$

where a is the focal distance, L_{ref} is the reference length, A_S is the superficial area, Per is the perimeter and v_o is the parameter that defines the domain boundary on the plane (u,v). The metric coefficients, transformation Jacobian and Laplacian operator are obtained by using the following equations:

$$h_u(u,v) = h_v(u,v) = a^*\left[\sin^2(u) + \sinh^2(v)\right]^{1/2}, \tag{13}$$

$$J(u,v) = \frac{\partial(X,Y)}{\partial(u,v)} = a^{*2}\left[\sin^2(u) + \sinh^2(v)\right], \tag{14}$$

$$\nabla^2\theta(u,v) = \frac{1}{J(u,v)}\left[\frac{\partial^2\theta(u,v)}{\partial u^2} + \frac{\partial^2\theta(u,v)}{\partial v^2}\right]. \tag{15}$$

For a domain having just one quadrant represented by $\{0 \le u \le u_o,\ 0 \le v \le v_o\}$ with $u_o = \pi/2$ and v_o given by Eq. (12c), the diffusion equations and boundary conditions in elliptical coordinate system are given by:

$$\left[\frac{\partial^2\theta(u,v)}{\partial u^2} + \frac{\partial^2\theta(u,v)}{\partial v^2}\right] + J(u,v)g(u,v) = 0, \qquad \{0 \le u \le u_o,\quad 0 \le v \le v_o\}, \tag{16}$$

$$\frac{\partial\theta(u,v)}{\partial u} = 0, \qquad \{u = 0,\quad u = u_o,\quad 0 \le v \le v_o\}, \tag{17}$$

$$\frac{\partial\theta(u,v)}{\partial v} = 0, \qquad \{0 \le u \le u_o,\quad v = 0\}, \tag{18}$$

$$\theta(u,v) = 0, \qquad \{0 \le u \le u_o,\quad v = v_o\}. \tag{19}$$

2.2.2. Cell with rectangular cross-section

For cell with rectangular cross section, the domain boundary matches naturally with the Cartesian coordinate system and for to keep the uniformity in the representation of the space variables and in the identity transformation is applied to this geometry the following transformation:

$$X = u\,, \qquad Y = v\,. \tag{20a, b}$$

Then, for a domain of a quarter, the diffusion equation and its boundary conditions are rewritten as:

$$\left[\frac{\partial^2\theta(u,v)}{\partial u^2} + \frac{\partial^2\theta(u,v)}{\partial v^2}\right] + g(u,v) = 0\,, \qquad \{0 \le u \le u_o, \quad 0 \le v \le v_o\}, \tag{21}$$

$$\frac{\partial\theta(u,v)}{\partial u} = 0\,, \qquad \{u = 0, \quad 0 \le v \le v_o\}, \tag{22}$$

$$\theta(u,v) = 0\,, \qquad \{u = u_o, \quad 0 \le v \le v_o\}, \tag{23}$$

$$\frac{\partial\theta(u,v)}{\partial v} = 0\,, \qquad \{0 \le u \le u_o, \quad v = 0\}, \tag{24}$$

$$\theta(u,v) = 0\,, \qquad \{0 \le u \le u_o, \quad v = v_o\}. \tag{25}$$

with

$$u_o = \frac{L}{L_{ref}}\,, \qquad v_o = \frac{l}{L_{ref}}\,, \qquad L_{ref} = \frac{A_S}{Per}\,. \tag{26a, b, c}$$

2.3. GITT development

To obtain temperature profiles the integral transform is applied onto the diffusion equation. Due to its two-dimensional characteristic, the potential $\theta(u,v)$ is written in terms of a series expansion using orthonormal eigenfunctions obtained from the solution of auxiliary eigenvalue problems for each space coordinate. In this way, it is done by parts for each one of eigenvalue problems proposed.

2.3.1. Application of GITT for a cell with elliptical cross-section

Consider the following auxiliary eigenvalue problem:

$$\frac{d^2\psi(u)}{du^2} + \mu^2\psi(u) = 0\,, \qquad \{0 < u < u_o\}, \tag{26}$$

$$\frac{d\psi(0)}{du} = 0\,, \qquad \frac{d\psi(u_o)}{du} = 0\,. \tag{27a, b}$$

The eigenvalues and corresponding eigenfunctions for this problem are:

$$\psi_i(u) = cos(\mu_i\, u)\,, \qquad \mu_i = 2(i-1)\,, \qquad i = 1,2,3\ldots\,. \tag{28a, b}$$

The orthogonality properties of the eigenfunctions above allow the development of the following transform-inverse pair:

$$\bar{\theta}_i(v) = \int_0^{u_o} K_i(u)\theta(u,v)du\,, \qquad (transform), \tag{29}$$

$$\theta(u,v) = \sum_{i=1}^{\infty} K_i(u)\bar{\theta}_i(v)\,, \qquad (inverse). \tag{30}$$

where $\bar{\theta}_i(v)$ is the transformed potential related to the axis u and $K_i(u)$ are the normalized eigenfunctions given by:

$$K_i(u) = \frac{\psi_i(u)}{N_i^{1/2}}\,, \qquad N_i = \int_0^{u_o} \psi_i^2(u)du = \begin{cases} u_o\,, & i = 1 \\ u_o / 2\,, & i > 1. \end{cases} \tag{31a, b}$$

To remove the partial derivaton related to the variable u, it is done the function dot product between the set of normalized eigenfunctions, K_i (u), and the diffusion equation, Eq. (19). Making use of the respective boundary conditions; of the boundary conditions of the eigenvalue problem; and of the eigenfunctions orthogonally property, it is achieved the first transformation of the partial differential equation that becomes:

$$A_i(v)+\mu_i^2\,\overline{\theta}_i(v)=\frac{d^2\overline{\theta}_i(v)}{dv^2}, \qquad A_i(v)=-\int_0^{u_o} K_i(u)\,J(u,v)\,g(u,v)\,du\,. \qquad (32a, b)$$

To proceed the integral transformation related to the coordinate v, consider the following eigenvalue problem:

$$\frac{d^2\phi(v)}{dv^2}+\lambda_j^2\,\phi(v)=0\,, \qquad \{0<\phi<v_o\}, \qquad (33)$$

$$\frac{d\phi(0)}{dv}=0\,, \qquad \phi(v_o)=0\,. \qquad (34a, b)$$

The eigenvalues and eigenfunctions for the new auxiliary problem are:

$$\phi_j(v)=cos(\lambda_j\,v), \qquad \lambda_j=\frac{(2j-1)\pi}{2v_o}\,, \qquad j=1,2,3.... \qquad (35a, b)$$

Eigenfunctions $\phi_j(v)$ are orthogonal and can be used to development of the following transform-inverse pair:

$$\tilde{\tilde{\theta}}_{ij}=\int_0^{u_o}\int_0^{v_o} K_i(u)\,Z_j(v)\,\theta(u,v)\,du\;dv\,, \qquad (\textit{transform}), \qquad (36)$$

$$\theta(u,v)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} K_i(u)\,Z_j(v)\,\tilde{\tilde{\theta}}_{ij}\,, \qquad (\textit{inverse}). \qquad (37)$$

where $\tilde{\tilde{\theta}}_{ij}$ is the transformed temperature and $Z_j(v)$ are the normalized eigenfunctions related to the v axis and given by:

$$Z_j(v)=\frac{\phi_j(v)}{M_j^{1/2}}, \qquad M_j=\int_0^{v_0}\phi_j^2(v)\,dv=\frac{v_o}{2}\,. \qquad (38a, b)$$

Removing of the partial derivation related to the variable v is done through the dot product of the normalized eigenfunctions, Z_j (v), with the one time transformed partial differential equation. Making use of the boundary conditions of the problem, of the boundary conditions of the second eigenvalue problem, and the orthogonality properties of the respective eigenfunctions, it is reached the integral transform for the diffusion equation:

$$B_{ij}+\left(\mu_i^2+\lambda_j^2\right)\tilde{\tilde{\theta}}_{ij}=0\,, \qquad B_{ij}=\int_0^{v_o} Z_j(v)\,A_i(v)\,dv=-\int_0^{v_o}\int_0^{u_o} K_i(u)\,Z_j(v)\,J(u,v)g(u,v)\,du\;dv\,. \qquad (39a, b)$$

where the coefficients B_{ij} are known after the integration. The system given by Eq. (39a) is algebraic, linear and decoupled, hence the transformed potential can be obtained explicitly as:

$$\tilde{\tilde{\theta}}_{ij}=-\frac{B_{ij}}{\left(\mu_i^2+\lambda_j^2\right)}, \qquad (40)$$

So, a closed analytical solution to the dimensionless temperature profile $\theta(u,v)$ can be obtained as:

$$\theta(u,v)=-\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{B_{ij}}{\left(\mu_i^2+\lambda_j^2\right)}K_i(u)\,Z_j(v). \qquad (41)$$

For computational purposes, the potential $\theta_{elp}(u,v)$ for the cell with elliptical cross-section can be achieved numerically, truncating the series given by Eq. (41) up to an order of N and M, for each index i and j, respectively:

$$\theta_{elp}(u,v)=-\sum_{i=1}^{N}\sum_{j=1}^{M}\frac{B_{ij}}{\left(\mu_i^2+\lambda_j^2\right)}K_i(u)\,Z_j(v). \qquad (42)$$

The previous equation will provide more accurate results increasing the truncation order N and M. Afterwards, by using the definition of θ and T^*, it is determined the temperature field over the whole domain.

2.3.2. Application of the GITT for cells with rectangular cross-section

The boundary condition of the rectangular cross-section cells differs from the previous problem. But, following the same procedure, the application of the GITT leads to a formula similar to the potential:

$$\theta_{ret}(u,v) = -\sum_{i=1}^{N}\sum_{j=1}^{M} \frac{B_{ij}}{\left(\mu_i^2+\lambda_j^2\right)} K_i(u)\, Z_j(v), \tag{43}$$

with

$$B_{ij} = -\int_0^{u_o}\int_0^{v_o} K_i(u)\, Z_j(v)\, g(u,v)\, du\, dv, \tag{44}$$

$$K_i(u) = \frac{\psi_i(u)}{N_i^{1/2}}, \qquad N_i = \int_0^{u_o} \psi_i^2(u) du = \frac{u_o}{2}, \tag{45a, b}$$

$$Z_j(v) = \frac{\phi_j(v)}{M_j^{1/2}}, \qquad M_j = \int_0^{v_o} \phi_j^2(v)\, dv = \frac{v_o}{2}, \tag{46a, b}$$

$$\psi_i(u) = cos(\mu_i\, u), \qquad \mu_i = \frac{(2i-1)\pi}{2u_o}, \qquad i = 1,2,3..., \tag{47a, b}$$

$$\phi_j(v) = cos(\lambda_j\, v), \qquad \lambda_j = \frac{(2j-1)\pi}{2\, v_o}, \qquad j = 1,2,3.... \tag{48a, b}$$

where the Jacobian of the coordinate system is $J(u,v) = 1$.

2.4. Neutron Diffusion Equation

How mentioned before, the source term of the energy equation is proportional to the thermal neutrons flux within the fuel cell. This particularity may be represented, in a first approximation, by neutron diffusion equation, obtained by Fick's Law and this equation and the boundary conditions are:

$$\nabla^2\varphi - \frac{1}{L_d^2}\varphi = 0, \qquad L_d^2 = \frac{D}{\Sigma_a}, \tag{49a, b}$$

$$\varphi = \varphi_P, \qquad \{(x,y)\in\Gamma\}. \tag{50}$$

where φ is the neutron flux, L_d is the length diffusion, D is the diffusion coefficient and the Σ_a is the absorption cross-section to thermal neutrons in a fuel cell rod.

Making use of the same procedure described above, the non-dimensional and homogenized diffusion equation and the boundary conditions for the fuel cell with elliptical cross-section are given by:

$$\frac{\partial^2\varphi(u,v)}{\partial u^2} + \frac{\partial^2\varphi(u,v)}{\partial v^2} + J(u,v)\left[\varphi(u,v)+1\right] = 0, \qquad \{0 \le u \le u_o,\quad 0 \le v \le v_o\}, \tag{51}$$

$$\frac{\partial\varphi(u,v)}{\partial u} = 0, \qquad \{u = 0,\quad u = u_o,\quad 0 \le v \le v_0\}, \tag{52a, b}$$

$$\frac{\partial\varphi(u,v)}{\partial v} = 0, \qquad \{0 \le u \le u_o,\quad v = 0\}, \tag{53}$$

$$\varphi(u,v) = 0, \qquad \{0 \le u \le u_o,\quad v = v_o\}. \tag{54}$$

where $\varphi(u,v)$ is the non-dimensional and normalized neutrons flux, $J(u,v)$ is the Jacobian of the transformation, $u_o = \pi/2$ and v_o is given by Eq. (12c).

For the fuel cell with rectangular cross-section the diffusion equation and the boundary conditions are presented in Eq. (55) to Eq. (59).

$$\frac{\partial^2\varphi(u,v)}{\partial u^2} + \frac{\partial^2\varphi(u,v)}{\partial v^2} + J(u,v)\left[\varphi(u,v)+1\right] = 0, \qquad \{0 \le u \le u_o,\quad 0 \le v \le v_o\}, \tag{55}$$

$$\frac{\partial\varphi(u,v)}{\partial u} = 0, \qquad \{u = 0,\quad 0 \le v \le v_o\}, \tag{56}$$

$$\varphi(u,v)=0\,, \qquad \{u=u_o,\quad 0\le v\le v_o\}, \tag{57}$$

$$\frac{\partial\varphi(u,v)}{\partial v}=0\,, \qquad \{0\le u\le u_o,\quad v=0\}, \tag{58}$$

$$\varphi(u,v)=0\,, \qquad \{0\le u\le u_o,\quad v=v_o\}. \tag{59}$$

where $J(u,v)$ = 1 is the Jacobian of the transformation and u_o and v_o are given by Eq. (26a, b).

The neutron diffusion problem in a nuclear fuel cell may be solved by GITT. The GITT is applied in the u e v coordinates by internal product of $K_i(u)$, $Z_m(v)$ and $\varphi(u,v)$. Making use of the boundary conditions, the following algebraic system can be obtained:

$$\sum_{j=1}^{\infty}\sum_{n=1}^{\infty} B_{i\,j\,m\,n}\,\tilde{\tilde{\varphi}}_{j\,n}+D_{i\,m}=\left(\mu_i^2+\lambda_m^2\right)\tilde{\tilde{\varphi}}_{i\,m}\,, \qquad i,\, m=1,2,3\,... \tag{60}$$

$$B_{i\,jmn}=-\int_0^{u_o}\int_0^{v_o} K_i(u)K_j(u)\,Z_m(v)Z_n(v)\,J(u,v)\,du\;dv\,, \tag{61}$$

$$D_{im}=-\int_0^{u_o}\int_0^{v_o} K_i(u)\,Z_m(v)\,J(u,v)\,du\;dv\,. \tag{61}$$

Despite all involved terms in the problem to be transformed, the equation system is not coupled due to absorption term in a resultant equation. Finally, the infinite equation system above may be calculate truncating the expansion for a given order $i=N$ and $j=M$ sufficiently larger to get the required accuracy.

3. Parameters of physical interest

The limiting thermal generation rate correspondent to the operating condition when the maximum temperature is reached of a given cell reaches its critical limit. Thus, from the Eq. (10c), results:

$$\bar{\dot{q}}'''_{lim}=\frac{\left(T^*_{lim}-T^*_P\right)k_0}{\theta_{max}\,l^2}\,. \tag{62}$$

where, T^*_{lim} and T^*_P are the terms obtained by integration of the Eq. (3) for the fuel fusion temperature and the cell boundary temperature, respectively. Here, the potential θ_{max} corresponds to the maximum value reached by temperature inside the domain.

To verify how the limiting thermal generation rate varies with the cell shape, the computation must be done for a constant cross-section area, guaranteeing that the fuel volume is fixed. In this way, taking as reference cells with cylindrical cross-section, it is defined the parameter non dimensional generation rate:

$$\zeta=\frac{\bar{\dot{q}}'''_{lim}}{\bar{\dot{q}}'''_{C\,lim}}\,, \qquad \bar{\dot{q}}'''_{C\,lim}=\frac{\left(T^*_{lim}-T^*_P\right)k_0}{\theta_{CO}\;R^2}\,, \qquad \theta_{CO}=\frac{1}{4}\,. \tag{63a, b, c}$$

where $\bar{\dot{q}}'''_{C\,lim}$ corresponds to the limiting thermal generation rate for a cell with circular cross-section, R is the circular cell radius and θ_{CO} is the dimensionless temperature computed at the cell center. Admitting, for comparison purposes, that the temperature T_P prescribed on the boundary is the same for both cells, the non dimensional generation rate can be determined by:

$$\zeta=\frac{\bar{\dot{q}}'''_{lim}}{\bar{\dot{q}}'''_{C\,lim}}=\frac{\theta_{CO}\,R^2}{\theta_{max}\,l^2}=\frac{1}{4\theta_{max}}\left(\frac{R}{l}\right)^2. \tag{64}$$

It can be observed that for $\zeta\ge 1$ the cell with circular cross-section minimizes the area for heat transfer and, consequently, minimizes the thermal generation rate.

The area A_S of a cell cross-section can be written as $A_S=l^2\,A^*_S$, where A^*_S corresponds to the sectional area in its dimensionless format. Therefore, from the condition that the cross-section areas must be equal ($\pi R^2=l^2\,A^*_S$), the non dimensional generation rate (ζ) can be written as:

$$\zeta=\frac{1}{4\theta_{max}}\frac{A^*_S}{\pi}=\frac{1}{\theta^*_{max}}\,, \qquad \theta^*_{max}=max\left\{\theta^*(u,v)\right\}=\frac{4\pi\,\theta_{max}}{A^*_S}\,, \qquad \theta^*(u,v)=\frac{4\pi\,\theta(u,v)}{A^*}\,. \tag{65a, b, c}$$

where $\theta^*(u,v)$ is now called cell characteristic temperature. It can be observed that as consequence of the ζ definition, the characteristic potential θ^*_{max} will be limited in the interval (0,1) and can be interpreted as a parameter for measument of the degree of deformation for the cylindrical cell cross-section, since that the limit $\theta^*_{max} \to 1$ is achieved for cells with circular cross-section.

4. Results and Discussion

For determination of the transformed potential $\tilde{\tilde{\theta}}_{ij}$ it was necessary the determination of B_{ij}, B_{ijmn} and D_{ij} coefficients related to each one proposed problem. The integration involved in the computation of these coefficients was done by using the Gauss quadrature rule. Consequently, all eigenfunctions and the Jacobian of the coordinate change must have to be calculated at quadrature points.

The transformed potential $\tilde{\tilde{\theta}}_{ij}$ was calculated for each cell shape, after truncating the series expansion for a given order M and N. For cells with rectangular cross-section, it was observed that the series convergence to compute temperature distribution becomes slower when the aspect ratio l/L is small $(l/L < 0.1)$, being necessary a high number of terms to get stable results over four or five decimal numeric places $(M = N > 35)$. For cells with elliptical cross-section this fact occurs when the focal distance (α) tends to zero, when an ellipse tends toward a circular shape $(l/L \to 1)$. For all cases, it is verified that the series converge to 4 or 5 decimal places when it is truncated to an order of approximately $M = N = 20$. Anyway, even considering a high number of terms in the series, the computer processing time is small.

The results obtained for the maximum characteristic temperature (θ^*_{max}) and for the non dimensional generation rate (ζ) are presented in Table 1 and Table 2 for cells with rectangular and elliptical cross-section, respectively, for several aspect ratios (l / L) and diffusion length (L_d).

As expected, the characteristic temperature increases when the aspect ratio increases. Consequently, the non dimensional generation rate for these cases decrease. It is presented in Table 1, a comparison of results obtained by using the classical technique known as variable splitting for the cells with rectangular cross-section Özisik (1993). It has been observed an excellent agreement between the results obtained by using those two techniques.

In Figures 2 and 3 are presented the non dimensional generation rate as function of the aspect ratio for several diffusion lengths for rectangular and elliptical cross-section, respectively.

Finally, the three-dimensional temperature distributions with aspect ratio $l / L = 0.5$, for $L_d = \infty$ and for $L_d = 0.5$, respectively, are presented in the Figs. 4 and 5 (for rectangular cells) and in the Figs. 6 and 7 (for elliptical cells). From these results can be observed the influence of the diffusion length in the maximum temperature reached.

Table 1. Maximum characteristic temperature and non dimensional generation rate in cells with rectangular cross-section.

b/a	$L_d^2 = \infty$		$L_d^2 = \infty$		$L_d^2 = 2.0$		$L_d^2 = 1.0$		$L_d^2 = 0.5$	
	θ^*_{max}	ζ	θ^*_{max} (1)	ζ (1)	θ^*_{max}	ζ	θ^*_{max}	ζ	θ^*_{max}	ζ
0.05	0.07851	12.7375	0.07854	12.7324	0.07470	13.386	0.07135	14.0144	0.06569	15.224
0.10	0.15706	6.3668	0.15708	6.36620	0.14797	6.7580	0.14022	7.1315	0.12755	7.8402
0.15	0.23559	4.2446	0.23561	4.24438	0.21954	4.5551	0.20629	4.8475	0.18533	5.3957
0.20	0.31392	3.1855	0.31391	3.18565	0.28917	3.4582	0.26937	3.7124	0.23901	4.1839
0.25	0.39120	2.5563	0.39118	2.55633	0.35633	2.8064	0.32910	3.0386	0.28846	3.4667
0.30	0.46605	2.1457	0.46606	2.14563	0.42022	2.3797	0.38504	2.5971	0.33364	2.9972
0.35	0.53701	1.8622	0.53702	1.86212	0.48002	2.0832	0.43678	2.2895	0.37458	2.6696
0.40	0.60277	1.6590	0.60278	1.65899	0.53492	1.8694	0.48383	2.0668	0.41121	2.4319
0.45	0.66242	1.5096	0.66243	1.50960	0.58439	1.7112	0.52594	1.9014	0.44355	2.2545
0.50	0.71547	1.3977	0.71548	1.39767	0.62815	1.5920	0.56299	1.7762	0.47173	2.1199
0.55	0.76174	1.3128	0.76175	1.31276	0.66617	1.5011	0.59506	1.6805	0.49591	2.0165
0.60	0.80134	1.2479	0.80134	1.24790	0.69860	1.4314	0.62231	1.6069	0.51635	1.9367
0.65	0.83451	1.1983	0.83452	1.19829	0.72570	1.3780	0.64503	1.5503	0.53330	1.8751
0.70	0.86167	1.1605	0.86168	1.16053	0.74783	1.3372	0.66355	1.5070	0.54706	1.8279
0.75	0.88328	1.1321	0.88329	1.13214	0.76542	1.3065	0.67824	1.4744	0.55795	1.7923
0.80	0.89984	1.1113	0.89985	1.11130	0.77888	1.2839	0.68948	1.4504	0.56625	1.7660
0.85	0.91188	1.0966	0.91189	1.09663	0.78865	1.2680	0.69762	1.4334	0.57227	1.7474
0.90	0.91989	1.0871	0.91990	1.08707	0.79515	1.2576	0.70304	1.4224	0.57627	1.7353
0.95	0.92438	1.0818	0.92438	1.08180	0.79879	1.2519	0.70607	1.4163	0.57850	1.7286
1.00	0.92577	1.0802	0.92578	1.08017	0.79992	1.2501	0.70702	1.4144	0.57919	1.7265

(1): Calculate through separation of variables (Özisik, 1993).

Table 2. Maximum characteristic temperature and non dimensional generation rate in cells with elliptical cross-section.

b/a	$L_d^2=\infty$		$L_d^2=2.0$		$L_d^2=1.0$		$L_d^2=0.5$	
	θ^*_{max}	ζ	θ^*_{max}	ζ	θ^*_{max}	ζ	θ^*_{max}	ζ
0.05	0.09975	10.0250	0.09973	10.0270	0.09971	10.0291	0.09967	10.0332
0.10	0.19802	5.0500	0.19786	5.0540	0.19771	5.0580	0.19739	5.0660
0.15	0.29340	3.4083	0.29290	3.4141	0.29241	3.4199	0.29142	3.4314
0.20	0.38462	2.6000	0.38353	2.6073	0.38246	2.6147	0.38034	2.6292
0.25	0.47059	2.1250	0.46868	2.1337	0.46679	2.1423	0.46307	2.1595
0.30	0.55046	1.8167	0.54751	1.8265	0.54459	1.8362	0.53888	1.8557
0.35	0.62361	1.6036	0.61944	1.6143	0.61535	1.6251	0.60735	1.6465
0.40	0.68966	1.4500	0.68416	1.4616	0.67877	1.4733	0.66828	1.4964
0.45	0.74845	1.3361	0.74154	1.3485	0.73479	1.3609	0.72171	1.3856
0.50	0.80000	1.2500	0.79167	1.2632	0.78353	1.2763	0.76783	1.3024
0.55	0.84454	1.1841	0.83479	1.1979	0.82530	1.2117	0.80704	1.2391
0.60	0.88236	1.1333	0.87127	1.1478	0.86049	1.1621	0.83984	1.1907
0.65	0.91389	1.0942	0.90159	1.1092	0.88966	1.1240	0.86686	1.1536
0.70	0.93961	1.0643	0.92629	1.0796	0.91340	1.0948	0.88884	1.1251
0.75	0.96001	1.0417	0.94598	1.0571	0.93242	1.0725	0.90664	1.1030
0.80	0.97562	1.0250	0.96131	1.0402	0.94750	1.0554	0.92127	1.0855
0.85	0.98695	1.0132	0.97299	1.0278	0.95951	1.0422	0.93392	1.0708
0.90	0.99449	1.0055	0.98179	1.0185	0.96951	1.0314	0.94615	1.0569
0.95	0.99871	1.0013	0.98870	1.0114	0.97898	1.0215	0.96035	1.0413
1.00	1.00000	1.0000	0.99468	1.0053	0.98943	1.0107	0.97927	1.0212

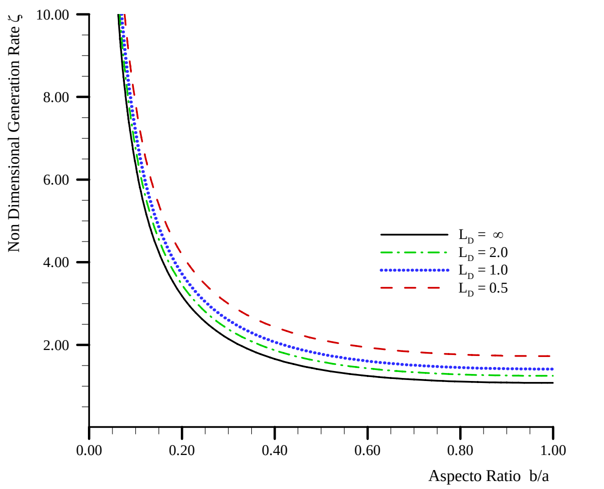

Figure 2. Non dimensional generation rate for rectangular cylinders as a function of the aspect ratio for several diffusion lengths.

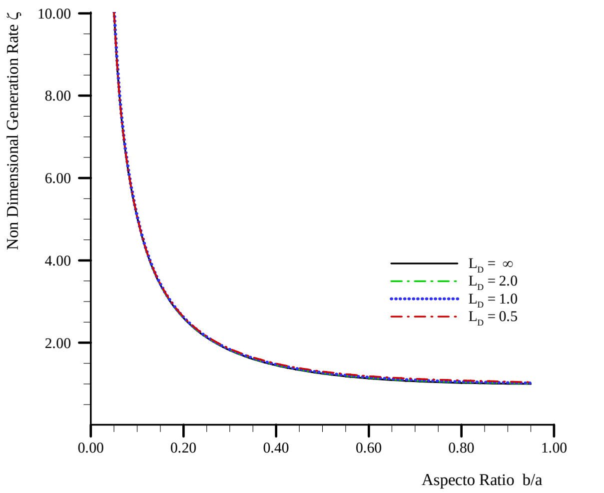

Figure 3. Non dimensional generation rate for elliptical cylinders as a function of the aspect ratio for several diffusion lengths.

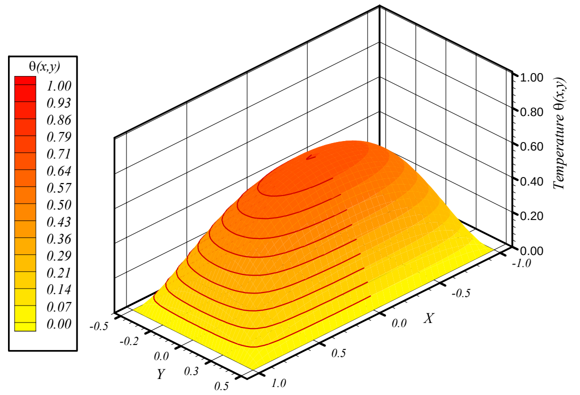

Figure 4. Rectangular cell temperature distribution for $b/a = 0.5$ and $L_D = \infty$.

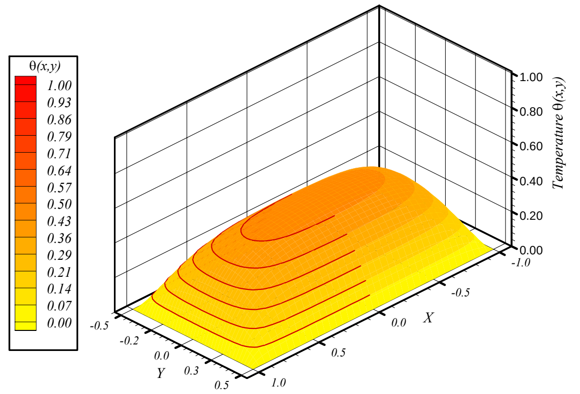

Figure 5. Rectangular cell temperature distribution for $b/a = 0.5$ and $L_D = 0.5$.

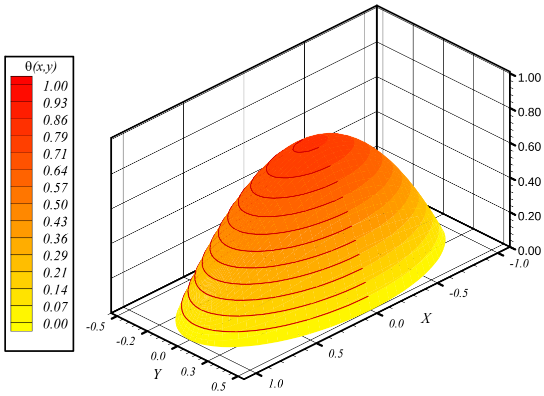

Figure 6. Elliptical cell temperature distribution for $b/a = 0.5$ and $L_D = \infty$.

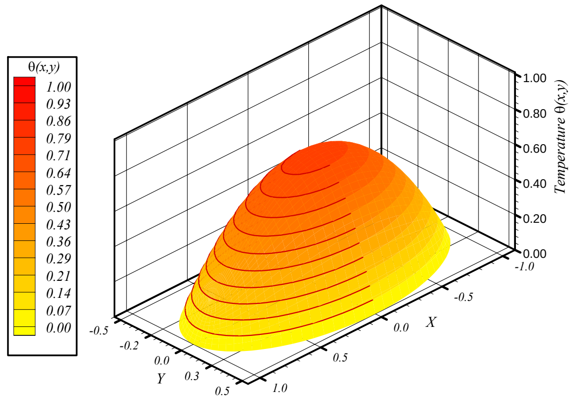

Figure 7. Elliptical cell temperature distribution for $b/a = 0.5$ and $L_D = 0.5$.

5. Conclusion

In this work it was analyzed a class of diffusion problems that characterizes cylindrical fuel cells with rectangular and elliptical cross-section. The diffusive problem studied presenting variables sources in its domain that is proportional to the thermal neutrons flux within the fuel cell. Assuming a general thermal dependency on physical properties, the diffusion equation was linearized through the use of the Kirchhoff's transformation, and to facilitate the application of the boundary conditions, the coordinate system was changed from cartesian to elliptical, according to the case considered.

Analytical solutions were obtained, by applying the Generalized Integral Transform Technique to the diffusion equation, resulting in a decoupled system of linear equations for the transformed potential. The expansion that determines the temperature distribution presented a slow convergence for rectangular cells with aspect ratio tending to zero ($l/L \to 0$) and for elliptical cells presenting focal distance tending to zero ($\alpha \to 0$), being necessary to consider a higher quantity of terms to achieve accurate results. But, computationally, numerical results were computed quickly since the transformed potentials are obtained explicitly from simple algebraic expressions.

Finally, the results presented are interesting since that was possible to demonstrate the efficiency of GITT to obtain analytical solution for diffusive complex problems, which does not have solution through classical techniques, such as separation of variables, as is the case for the problem of fuel cells with elliptical cross-section.

6. References

Alves, T.A., Pelegrini, M.F., Woiski, E.R. and Ramos, R.A.V., 2004, "Limit Power of Nuclear Fuel Cells With Biconcave Cross Sections", Proceedings of the 10th Brazilian Congress of Thermal Sciences and Engineering, Rio de Janeiro, RJ, Brazil, in CD-ROM (*in portuguese*).

Aparecido, J.B, Cotta, R.M. and Özisik, M.N., 1989, "Analytical Solutions to Two-Dimensional Diffusion Type Problems in Irregular Geometries", Journal of the Franklin Institute, Vol. 326, pp. 421-434.

Cotta, R.M., 1998, "The Integral Transform Method in Thermal and Fluids Science and Engineering", Begell House Inc., New York, USA.

Diniz, A.J., Aparecido, J.B. and Cotta, R.M., 1990, "Heat Conduction with Ablation in a Finite Slab", International Journal of Heat and Technology, Vol. 8, pp. 30-43.

Glasstone, S. and Sesonske, A., 1994, "Nuclear Reactor Engineering: Reactor Systems Engineering", 4th ed., Chapman & Hall, London, UK.

IMSL Library, 1979, Edition 7, GNB Building, 7500 Ballaire Blod, Houston, Texas 77036.

Maia, C.R.M., Aparecido, J.B. and Milanez, L.F., 2002, "Critical Specific Power in Fuel Rods with Biconcave Cross-Section", Proceedings of the 12th International Heat Transfer Conference, Grenoble, France, in CD-ROM, code 0868.

Maia, C.R.M., Aparecido, J.B. and Milanez, L.F., 2003, "Limiting Thermal Energy Generation Rate", Proceedings of the 24th Iberian Latin American Congress on Computational Methods, Ouro Preto, MG, Brazil, in CD-ROM.

Özisik, M.N., 1993, "Heat Conduction, 2nd ed., John Wiley & Sons, New York, USA.

Pelegrini, M.F., Alves, T.A., Maia, C.R.M. and Ramos, R.A.V., 2004, "Solution of Unstead Difusives Problems with Sources by Generalized Integral Transform Technique", Proceedings of the 25th Iberian Latin American Congress on Computational Methods, Recife, PE, Brazil (*in portuguese*).

7. Responsibility notice

The autors are the only responsible for the printed material included in this paper.