 November 6-11, 2005, Ouro Preto, MG

# ON THE STABILITY PROBLEM FOR THE FLOW AROUND A 2D CYLINDER BY THE SOLENOIDAL SUBSPACE PROJECTION METHOD

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***Abstract***
*The study of wakes behind bluff bodies has received more and more attention in the fluid mechanics literature. We consider the special case of the wake behind a two-dimensional circular cylinder (of uniform cross section and infinite length) in a free stream. To perform the stability analysis of this flow field, we project the linearized discrete Navier-Stokes equations with respect to the stationary flow field, pair of vortex bubbles behind the cylinder, into the discrete divergence operator solenoidal subspace. The stability of the stationary flow field is obtained by solving the eigenvalue problem related to the projected linearized discrete Navier-Stokes equations. We use two different methods to obtain the projection matrix. The first is based on the singular value decomposition of discrete divergence operator and the second is the application of the Turnback algorithm to the discrete divergence operator. The matrices of the projected eigenvalue problem are still sparse when the projection operator obtained from the Turnback algorithm is used. The other method leads to full matrices.*
***Keywords:*** *Stability analysis, two-dimensional cylinder wake, solenoidal sub-space, Turnback algorithm, finite element method*

## 1. Introduction.

We consider the stability problem of the wake behind a two-dimensional cylinder under the action of a free stream. To perform the stability analysis of this flow problem, we need first to obtain numerically the stationary flow field, which is the pair of vortex bubble which forms behind the cylinder for $Re > 1$ (Reynolds number) and becomes unstable for $Re \sim 45$. Then, the Navier-Stokes equation governing the flow problem needs to be linearized with respect to the stationary flow and pressure fields. The spectral analysis of the linear problem contains the information about the stability of the stationary problem.

We also need to assume a discretization for the velocity field and a discretization for the scalar pressure field. We could use the finite element method. After we obtain the discrete stationary flow field, we need to linearize the discrete Navier-Stokes equation with respect to the discrete stationary flow field. To determine the stability of the stationary flow field, we need to solve the discrete eigenvalue problem associated with the discrete linearized flow problem.

At this point, we have two options. We could solve the full generalized eigenvalue problem, where the eigenvectors contain entries related to the discrete velocity field and entries related to the discrete pressure field. This problem is singular, since the mass matrix is singular (see section 2.). The other option is to project the discrete eigenvalue problem into the solenoidal sub-space. This decouples the velocity field from the pressure field, and could reduce the size of the discrete eigenvalue problem up to 1/3 of the original size. The projection operator is a matrix where each column is an element of the null vector space of the discrete divergence operator. The standard technique to obtain the projection matrix is to perform a singular value decomposition (s.v.d.) of the discrete divergence operator. The vectors related to the zero singular values form the columns of the projection matrix. This technique gives a full projection matrix, what jeopardizes the advantage of reducing the size of the original eigenvalue problem. The matrices present in the reduced eigenvalue problem are full matrices, while the matrices of the original problem are sparse.

It would be of great advantage if we could construct a projection matrix which is sparse. This could actually be done by using the Turnback algorithm (see references Berry et al., 1985 and Hall and Ye, 1992). The projection matrix which results from the application of the Turnback algorithm to the discrete divergence operator is sparse, which implies that the matrices related to the reduced eigenvalue problem are also sparse. Then, we have both advantages of the size reduction and of the sparsity of the matrices related to the eigenvalue problem.

The objective of this work is to compare both projection methods. We apply both methods to the eigenvalue problem related to the finite element discretization of the linearized flow problem mentioned above. For this numerical example, both methods gave a projection matrix of the same size (same number of columns and lines) as it should be, and we illustrate that the Trunback method provides projected matrices which are still sparse, but with a larger number of non-zero elements, while the s.v.d. based method give full projected matrices.

The problem of finding a basis for the null space of an operator is a basic problem in numerical analysis. It has application in many different areas, like sturctural optimization (Kaneko at al., 1982), linearly-constrained quadratic programming problems (Gill et al., 1982) and the dual variable method and the divergence free finite element method for solving numerically the incompressible Navier-Stokes equations (Hall and Ye, 1992). The strategy of the dual variable method and of the present work is the same, since both need to obtain a basis for the null space of the discrete divergence operator. The dual variable method solves the discrete Navier-Stokes equation by projecting it into the discrete solenoidal subspace. This decouples the discrete velocity field from discrete pressure field and reduces the size of the original problem. In the present work we seek the same advantages of decoupling the discrete velocity field from the dicrete pressure field and of reducing the size of the original problem, but for the generalized eigenvalue problem related to the stability analysis of the stationary flow mentioned above.

Numerical methods for finding a sparse basis for the null space of a discrete operator have been developed at first in the context of structural analysis. See Berry et al., (1985) and references therein. Earlier methods were based on Gauss-Jordan elimination or on an orthogonal decomposition using Givens rotations or Householder transformations. The Turnback algorithm used in this work starts with the factorization of the matrix representing a discrete operator using Gauss elimination with row pivoting or orthogonal reduction by Givens rotations. The objective of this first phase is to determine the matrix column dependency. The second phase of the Turnback algorithm involves the calculations of the basis vectors of the null space of the discrete operator which reflects the minimal column dependency of the matrix representing the discrete oerator. Details of the Turnback algorithm are given in section 5.2. It was first developed in the context of structural analysis (Berry et al., 1985) and and later applied to fluid mechanics. See Hall and Ye (1992) and references therein.

The generalized eigenvalue problem associated with the hydrodynamic stability of stationary flows satisfying the incompressible Navier-Stokes equations has been attacked before. Gervais et al. (1997), Fortin et al. (1994) and Fortin et al. (1997) used the subspace method to obtain part of the spectrum necessary for the stability analysis of different type of stationary flows. The objective was to study the localization of the bifurcations of the stationary flows considered. Only the eigenvalues with the largest real part are necessary to be computed for this type of analysis, and the subspace method is adequate, since it allows to obtain only part of the spectrum of generalized eigenvalue problems. Noak and Eckelmann (1994) developed a Galerkin type spectral method to perform the stability analysis of the two-dimensional cylinder wake and were able to compute the complete spetrum for the generalized eigenvalue problem related to this stability problem for $Re$ up to $120$, since the matrices generated by their spectral discretization of the flow field are smaller than the usual size obtained by discretizations, for example, with the finite element method. Lopez et al. (2005) is the first to use the solenoidal sub-space projection method to study the stability of the two-dimensional cylinder wake. The matrices of the genralized eigenvalue problem which resulted in their stability analysis are full, since they performed the s.v.d. of the discrete divergence operator to obtain a basis for the solenoidal subspace. The present work can be seen as an extention of Lopez et al. (2005), since we illustrate the application of the Turnback algorithm to the solenoidal subspace projection method in the stability analysis of the wake flow behind a two-dimensional cylinder.

In the next section, we obtain the generalized eigenvalue problem related to the stability analysis of the stationary flow for a two-dimensional cylinder under the action of a free-stream (pair of vortex bubbles behind the cylinder), and in section 3.we present the weak formulation for this genralized eigenvalue problem. In section 4., we define the solenoidal sub-space and project the eigenvalue problem obtained in the previous section into the solenoidal subspace. In section 5. , we discuss how to obtain the projection matrix using the s.v.d. (see section 5.1) approach and the Turnback algorithm (see section 5.2). Results are give in section 6and conclusion are discussed in section 7..

## 2. The Stability Problem

We obtained the stationary flow around a two-dimensional circular cylinder under the action of a free stream of constant speed, which comprises the flow field $U(\mathbf{x}, Re)$ and the scalar pressure field $P = P(\mathbf{x}, Re)$. For the stability analysis of the stationary flow filed, we need the governing equation for the evolution of a perturbation $u(\mathbf{x}, t, Re)$ and $p(\mathbf{x}, t, Re)$, respectively, of the velocity field $U(\mathbf{x}, Re)$ and pressure field $P(\mathbf{x}, Re)$, which is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{U}.\nabla)\mathbf{u} + (\mathbf{u}.\nabla)\mathbf{U} + (\mathbf{u}.\nabla)\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{u}, \tag{1}$$

$$\nabla.\mathbf{u} = 0, \tag{2}$$

with prescribed initial condition $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0$. Equations (1) and (2) can be written in the more compact form

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{F}(\mathbf{u}, Re) - \nabla p \tag{3}$$

$$\nabla.\mathbf{u} = 0 \tag{4}$$

where

$$\mathbf{F}(\mathbf{u}, Re) = \frac{1}{Re}\nabla^2\mathbf{u} - (\mathbf{U}.\nabla)\mathbf{u} - (\mathbf{u}.\nabla)\mathbf{U} - (\mathbf{u}.\nabla)\mathbf{u} \tag{5}$$

For very small perturbation $(\mathbf{u}(\mathbf{x}, t, Re), p(\mathbf{x}, t, Re))$, we linearize the governing Eqs. (1)-(2), which gives

$$\frac{\partial\mathbf{u}}{\partial t} = \frac{\partial\mathbf{F}}{\partial\mathbf{u}}(\mathbf{0}, Re)\mathbf{u} \tag{6}$$

$$\nabla.\mathbf{u} = 0 \tag{7}$$

where

$$\frac{\partial\mathbf{F}}{\partial\mathbf{u}}(\mathbf{0}, Re)\mathbf{v} = \frac{1}{Re}\nabla^2\mathbf{v} - (\mathbf{U}.\nabla)\mathbf{v} - (\mathbf{v}.\nabla)\mathbf{U}. \tag{8}$$

Now we write $\mathbf{u}(\mathbf{x}, t, Re) = \exp(\sigma t)\hat{\mathbf{u}}(\mathbf{x}, Re)$ and $p(\mathbf{x}, t, Re) = \exp(\sigma t)\hat{p}(\mathbf{x}, Re)$, and we substitute them into Eqs. (6) and (7). As a results, the quantities $\sigma(Re) = \zeta(Re) + i\eta(Re)$, $\hat{\mathbf{u}}$ and $\hat{p}$ satisfy the spectral problem

$$\sigma\hat{\mathbf{u}} = \frac{\partial\mathbf{F}}{\partial\mathbf{u}}(\mathbf{0}, Re)\hat{\mathbf{u}} - \nabla\hat{p} \tag{9}$$

$$\nabla.\hat{\mathbf{u}} = 0. \tag{10}$$

When the domain $\Omega$ is bounded, the spectral problem given by Eqs. (9) and (10) has an infinite enumerable and isolated set of eigenvalues $\sigma(Re)$. Most of this set lies in the part of the complex $\sigma$ plane with negative real part. The stability of the stationary flow $(\mathbf{U}(\mathbf{x}, Re), P(\mathbf{x}, Re))$ with respect to a small perturbation is reduced to the complete determination of the eigenvalues of the spectral problem given by Eqs. (9) and (10). If one or more eigenvalues $\sigma(Re)$ have real positive part, the stationary flow field is unstable with respect to small perturbations. The stationary flow field is stable with respect to small perturbations only when all eigenvalues $\sigma(Re)$ have real negative part.

## 3. Weak Formulation

We present a weak formulation for the spectral problem given by Eqs. (9) and (10). It is given by the equations

$$\sigma(\hat{\mathbf{u}}, \hat{\mathbf{v}}) + \frac{1}{Re}a(\hat{\mathbf{u}}, \hat{\mathbf{v}}) + b(\hat{\mathbf{v}}, \hat{q}) + c(\mathbf{U}, \hat{\mathbf{u}}, \hat{\mathbf{v}}) + c(\hat{\mathbf{u}}, \mathbf{U}, \hat{\mathbf{v}}) = 0 \tag{11}$$

$$b(\hat{\mathbf{u}}, \hat{q}) = 0 \tag{12}$$

where the internal product $(., .)$ and the forms above are defined by

$$(\hat{\mathbf{u}}, \hat{\mathbf{v}}) = \int_\Omega \hat{\mathbf{u}}.\hat{\mathbf{v}}d\Omega, \tag{13}$$

$$a(\hat{\mathbf{u}}, \hat{\mathbf{v}}) = -\sum_{j=1}^{2}\int_\Omega \nabla\hat{\mathbf{u}}_j.\nabla\hat{\mathbf{v}}_j d\Omega, \tag{14}$$

$$b(\hat{\mathbf{u}}, \hat{q}) = -\int_\Omega (\nabla.\hat{\mathbf{u}})\hat{q}d\Omega, \tag{15}$$

$$c(\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}) = \int_\Omega (\hat{\mathbf{u}}.\nabla)\hat{\mathbf{v}}.\hat{\mathbf{w}}d\Omega. \tag{16}$$

The term $\hat{\mathbf{u}}_j$ represents the $x$ $(y)$ component of the velocity vector $\hat{\mathbf{u}}$ when $j = 1$ $(j = 2)$. The functions $\hat{\mathbf{v}}$ and $\hat{q}$ belongs, respectively, to the spaces denoted as $\Gamma$ and $\Lambda$. The function space $\Gamma$ is a subset of the Sobolev space of functions $\hat{\mathbf{v}}$ with first derivatives square-integrable on the domain $\Omega$ such that $\hat{\mathbf{v}}|_{\partial\Omega} = 0$, and the function space $\Lambda$ is a subset of the Sobolev function space with square-integrable functions over the domain $\Omega$.

We consider a partition of the domain $\Omega$ by a finite element grid $\Xi_h$, which consists of triangles. For the discretization of the velocity field, we consider the triangular elements of $\Xi_h$ with six nodes (the three corner nodes plus one node at the middle of each face of the triangular element), and for the discretization of the pressure field, we consider the three corner nodes of each triangular elements of $\Xi_h$. The six node triangular element leads to quadratic functions to interpolate the velocity field, and three node triangular elements permit the use of linear functions to interpolate the pressure field. We assume that these linear functions are continuous across elements. The finite element method applied to Eqs. (11) and (12) leads to the system of matrix equations

$$\begin{bmatrix} \mathbf{A}+\mathbf{C}(\tilde{\mathbf{U}}) & -\mathbf{R} \\ \mathbf{R}^T & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{w} \\ \mathbf{q} \end{pmatrix} = \sigma \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{w} \\ \mathbf{q} \end{pmatrix}, \tag{17}$$

where $\mathbf{w} \in R^n$ is the vector of nodal values associated with the discretization of the velocity field $\hat{\mathbf{u}}$, $\mathbf{q} \in R^m$ is the vector of the nodal values associated with the discretization of the pressure field $p$, $\tilde{\mathbf{U}} \in R^n$ is the vector of nodal values associated with the discretization of the velocity field $\mathbf{U}$ of the stationary problem, $\mathbf{A}$ is a $(n \times n)$ sparse diffusion matrix, $\mathbf{C}(\tilde{\mathbf{U}})$ is a $(n \times n)$ non-symmetric matrix, the sum of the convective terms, $-\mathbf{R}$ is the discrete gradient $(n \times m)$ matrix of rank $m$, $\mathbf{R}^T$ is the discrete divergence $(m \times n)$ matrix operator and $\mathbf{M}$ is a mass $(n \times n)$ symmetric matrix. The eigenvalue problem represented by Eq. (17) is real and non-symmetric, therefore, the eigenvalues $\sigma$ will be either real or pairs of complex conjugate numbers. The second line of Eq. (17) implies that the eigenvectors $\mathbf{w}$ are in the kernel of the matrix operator $\mathbf{R}^T$. Therefore, we can write $\mathbf{w}$ as a linear combinations of a set of vectors $\{\mathbf{t}_1, \ldots, \mathbf{t}_s\}$, which form a base for the kernel of the matrix operator $\mathbf{R}^T$. This vector sub-space is the solenoidal sub-space for the considered finite element discretization.

## 4. The Reduced Eigenvalue Problem

The original eigenvalue problem, represented by Eq. (17), can be reduced by projecting it into the kernel of the matrix operator $\mathbf{R}^T$, the solenoidal sub-space. Here we describe how to project Eq. (17) into the solenoidal subspace of the considered finite element discretization. Assume that we have a set of vectors $\{\mathbf{t}_1, \ldots, \mathbf{t}_s\}$ which spans the kernel of the matrix operator $\mathbf{R}^T$, where $s = dim(\mathbf{R}^T)$. Since $\mathbf{w} \in ker\mathbf{R}^T$ (kernel of $\mathbf{R}^T$), we can write

$$\mathbf{w} = \sum_{l=1}^{s} \tilde{w}_l \mathbf{t}_l = \mathbf{T}\tilde{\mathbf{w}} \tag{18}$$

for some $\tilde{\mathbf{w}} \in R^s$. $\mathbf{T}$ is the $(n \times s)$ matrix with columns $\mathbf{t}_1, \ldots, \mathbf{t}_s$. Since

$$\mathbf{T}^T\mathbf{R} = (\mathbf{R}^T\mathbf{T})^T = 0, \tag{19}$$

we can multiply the first row of the matrix Eq.(17) to get rid from the pressure related term $-\mathbf{Rq}$. This results in a matrix equation in terms of $\mathbf{w}$ only. The matrices in this equation have dimensions $s \times n$. We substitute Eq. (18) into this equation and we obtain

$$\tilde{\mathbf{K}}\tilde{\mathbf{w}} = \sigma\tilde{\mathbf{M}}\tilde{\mathbf{w}}, \tag{20}$$

where $\tilde{\mathbf{K}} = \mathbf{T}^T(\mathbf{A} + \mathbf{C}(\tilde{\mathbf{U}}))\mathbf{T}$ and $\tilde{\mathbf{M}} = \mathbf{T}^T\mathbf{M}\mathbf{T}$ are $(s \times s)$ matrices embedded into the solenoidal subspace. Matrix $\mathbf{K}$ is non-symmetric and matrix $\mathbf{M}$ is symmetric.

## 5. Projection Operator

In the previous section, the eigenvalue problem for the velocity nodal values and pressure nodal values was reduced to an eigenvalue problem for the velocity field nodal values projected into the solenoidal subspace. The reduced eigenvalue problem is of dimension $(s \times s)$, smaller that the original eigenvalue problem of dimension $(n + m \times n + m)$. We described how to project the original eigenvalue problem into the solenoidal subspace, but we did not discuss how to obtain the projection matrix $\mathbf{T}$. In the two next sections, we describe two different procedures to obtain the projection matrix $\mathbf{T}$. The first procedure is based on the singular value decomposition of the matrix operator $\mathbf{R}^T$, and the second procedure is called the Turnback algorithm in the literature. The projection matrix $\mathbf{T}$ given by the first procedure is in general a full matrix, but the projection matrix $\mathbf{T}$ obtained from the second procedure is sparse, which implies that the matrices of the reduced eigenvalue problem are still sparse.

### 5.1 Singular value decomposition of the divergence operator.

The singular value decomposition of the matrix operator $\mathbf{R}^T$ is equivalent to solve the eigenvalue problem

$$\mathbf{R}\mathbf{R}^T\mathbf{t} = \eta\mathbf{t}. \tag{21}$$

We need to solve this eigenvalue problem only for the pairs $\{(\mathbf{t}_1, \eta_1 = 0), \ldots, (\mathbf{t}_s, \eta_s = 0)\}$ of eigenvectors and eigenvalues. The eigenvectors $\{\mathbf{t}_1, \ldots, \mathbf{t}_s\}$ span $ker\mathbf{R}^T = ker(\mathbf{R}\mathbf{R}^T)$, and are the columns of projection matrix operator $\mathbf{T}$.

### 5.2 The Turnback algorithm

Our main references for the Turnback algorithm are Berry et all (1985) and Hall and Ye (1992). This algorithm generates a sparse and banded projection operator matrix $\mathbf{T}$.

The version of the algorithm we implemented to obtain matrix $\mathbf{T}$ is based on the LU decomposition of matrix $\mathbf{R}^T$. We follow the steps:

1. We start the process of the LU decomposition of matrix $\mathbf{R}^T$ and continue it until we encounter the first zero pivot in column, let say, $k_1$. Then column $k_1$ is a linearly dependent of columns $1, \ldots, k_1 - 1$. Further, these first $k_1 - 1$ columns have been triangularized, and solving for the coefficients of dependence $\mathbf{t}_{1,1}, \ldots, \mathbf{t}_{1,k_1-1}$ is straightforward. The vector $\{\mathbf{t}_{1,1}, \ldots, \mathbf{t}_{1,k_1-1}, 1, 0, \ldots, 0\}^T$ is an element of $ker\mathbf{R}^T$ and the first column of the matrix $\mathbf{T}$.

2. We define the $(m \times n - 1)$ matrix $(\mathbf{R}^T)^{(1)} = \{\mathbf{R}_1^T, \ldots, \mathbf{R}_{k_1-1}^T, \mathbf{R}_{k_1+1}^T, \ldots, \mathbf{R}_n^T\}$ where $\mathbf{R}_j^T$ is the $j$-th columns of matrix $\mathbf{R}^T$.

3. We continue the LU decomposition, but of matrix $(\mathbf{R}^T)^{(1)}$ instead of matrix $\mathbf{R}^T$. Notice that the first $k_1 - 1$ columns of matrix $(\mathbf{R}^T)^{(1)}$ has already suffered the Gaussian elimination process since they are the $k_1 - 1$ columns of matrix $\mathbf{R}^T$. We proceed the LU decomposition of matrix $(\mathbf{R}^T)^{(1)}$ until another zero pivot is encountered at, let say, column $k_2$ of the original matrix $\mathbf{R}^T$ (column $k_2 - 1$ of matrix $(\mathbf{R}^T)^{(1)}$) . As before, column $k_2$ can be expressed as a linear combination of the first $k_2 - 1$ columns of matrix $\mathbf{R}^T$, with the exception of column $k_1$. This leads to the vector $\{\mathbf{t}_{2,1}, \ldots, \mathbf{t}_{2,k_1-1}, 0, \mathbf{T}_{2,k_1+1}, \ldots, \mathbf{T}_{2,k_2-1}, 0, \ldots, 0\}^T$, which is an element of $ker\mathbf{R}^T$. The process being described here so far will lead to a matrix $\mathbf{T}$ with a trapezoidal non-zero pattern. To obtain a banded sparse matrix $\mathbf{T}$, we introduce the Turnback part of the algorithm being described.

4. Turn back part of the algorithm. We define the matrix $\mathbf{E}^{(1)} = \{\mathbf{R}_{k_2}^T, \mathbf{R}_{k_2-1}^T, \ldots, \mathbf{R}_{k_1+1}^T, \mathbf{R}_{k_1-1}^T, \ldots, \mathbf{R}_1^T\}$.

5. We apply the LU decomposition to matrix $\mathbf{E}^{(1)}$ until we find a zero pivot and column, let say, $t$. Then we solve for the coefficients of dependency $c_1, \ldots, c_t$ where $t < k_2$. The elements of the second column of matrix $\mathbf{T}$ are now

$$\mathbf{t}_{2,j} = \begin{cases} 0 & \text{for } 0 \leq j < k_2 - t \\ 1 & \text{for } j = k_2 - t + 1 \\ c_{k_2-j+1} & \text{for } k_2 - t + 2 \leq j \leq k_2 \\ 0 & \text{for } n \leq j < k_2 \end{cases} \tag{22}$$

6. We repeat from step 2 up to step 5 $s - 2$ times ($s$ being the number of columns of matrix $\mathbf{T}$). To obtain the $l$-th column of matrix $\mathbf{T}$ we do the following:

   - Define matrix $(\mathbf{R}^T)^{(l-1)} = \{\mathbf{R}_1^T, \ldots, \mathbf{R}_{k_1-1}^T, \mathbf{R}_{k_1+1}^T, \ldots, \mathbf{R}_{k_2-1}^T, \mathbf{R}_{k_2+1}^T, \ldots, \mathbf{R}_{k_{l-1}-1}^T, \mathbf{R}_{k_{l-1}+1}^T, \ldots, \mathbf{R}_n^T\}$. Notice that column $k_{l-1}$ of matrix $\mathbf{R}^T$ has been eliminated.
   - Continue LU decomposition, but now of matrix $(\mathbf{R}^T)^{(l-1)}$ instead of matrix $(\mathbf{R}^T)^{l-2}$. Notice that the first $k_{l-1} - 1$ columns of matrix $(\mathbf{R}^T)^{(l-1)}$ has already suffered the Gaussian elimination process since they are the $k_{l-1} - 1$ columns of matrix $(\mathbf{R}^T)^{l-2}$. We proceed the LU decomposition of matrix $(\mathbf{R}^T)^{(1)}$ until another zero pivot is encountered at, let say, column $k_l$ of the original matrix $\mathbf{R}^T$.
   - Define matrix $\mathbf{E}^{(l-1)} = \{\mathbf{R}_{k_l}^T, \mathbf{R}_{k_l-1}^T, \ldots, \mathbf{R}_{k_{l-1}+1}^T, \mathbf{R}_{k_{l-1}-1}^T, \ldots, \mathbf{R}_1^T\}$. Notice that columns $k_{l-1}, k_{l-2}, \ldots, k_2, k_1$ of matrix $\mathbf{R}^T$ do not appear in matrix $\mathbf{E}^{(l-1)}$.
   - Apply the LU decomposition to matrix $\mathbf{E}^{(l-1)}$ until we find a zero pivot and column, let say, $t_{l-1}$. Then we solve for the coefficients of dependency $c_1, \ldots, c_{t_{l-1}}$ where $t < k_l$. The elements of the $l$-th column of matrix $\mathbf{T}$ is now

$$\mathbf{t}_{l,j} = \begin{cases} 0 & \text{for } 0 \leq j < k_l - t_{l-1} \\ 1 & \text{for } j = k_l - t_{l-1} + 1 \\ c_{k_l-j+1} & \text{for } k_l - t_{l-1} + 2 \leq j \leq k_l \\ 0 & \text{for } n \leq j < k_l \end{cases} \tag{23}$$

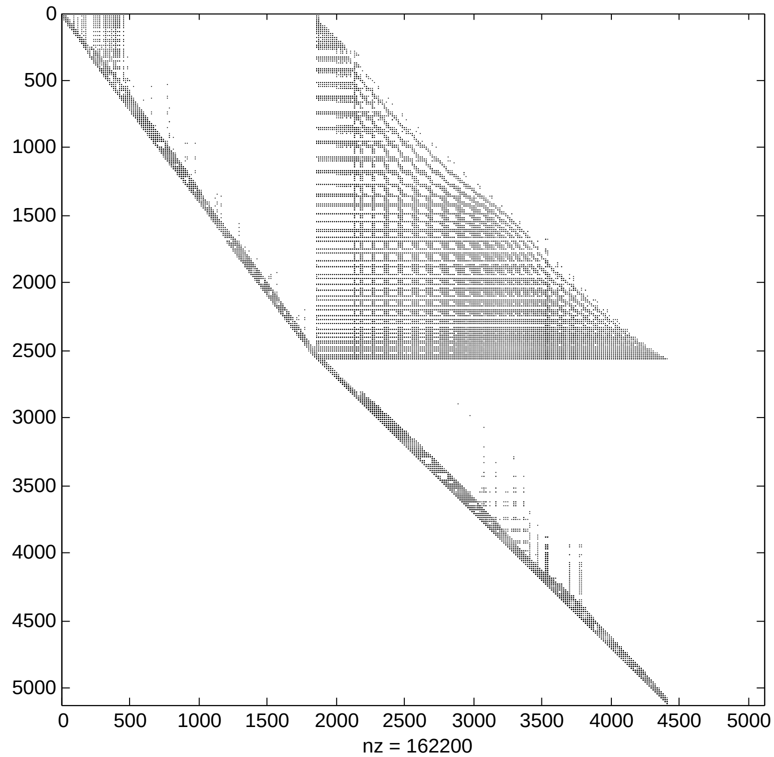


Figure 1. Matrix $\mathbf{C}$.

## 6. Results

We consider the finite element discretization $\Xi_h$ of the domain $\Omega$ described in the section 3. The velocity field was interpolated by quadratic functions and the pressure field was interpolated by linear functions which are continuous across elements of the discretization of the domain $\Omega$.

The sparcity diagram of the projection matrix $\mathbf{C}$ obtained by the turnback algorithm is given in Fig. 1 below. It is clear that the Turnback algorithm produces a sparse projection matrix. For this particular example, the band of the projection matrix $\mathbf{C}$ given by the Turnback algorithm is not small, leading to not narrow banded matrices $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{K}}$. These matrices still have a much smaller number of elements than the full matrices we would obtain if we used the projection matrix given by the s.v.d. of the discrete divergence operator.

The sparsity diagram for matrices $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{K}}$, projection of the matrices $\mathbf{M}$ and $\mathbf{A}+\mathbf{C}(\tilde{\mathbf{U}})$ into the solenoidal subspace, are displayed in Figs. 2 and 3.

## 7. Conclusion

According to sparsity diagrams displayed on Figs. 2 and 3, the projection of matrices $\mathbf{M}$ and $\mathbf{A} + \mathbf{C}(\tilde{\mathbf{U}})$ into the solenoidal subspace using the projection operator matrix $\mathbf{T}$ given by the Turnback algorithm are still sparse matrices, but with a larger number of non-zero element. Besides the increase in number of elements, these matrices are still sparse, and therefore, algorithms used to evaluate the eigenvalues/vectors of sparse matrices can still be used. This reduces the computational effort and time to perform the stability analysis of the considered flow problem.

## 8. Acknowledgements

The first author acknowledge financial support from FAPESP (fundaçao de Amparo a Pesquisa do Estado de Sao Paulo) under a pos-doctoral scholarship. grant number 02/12370-2.

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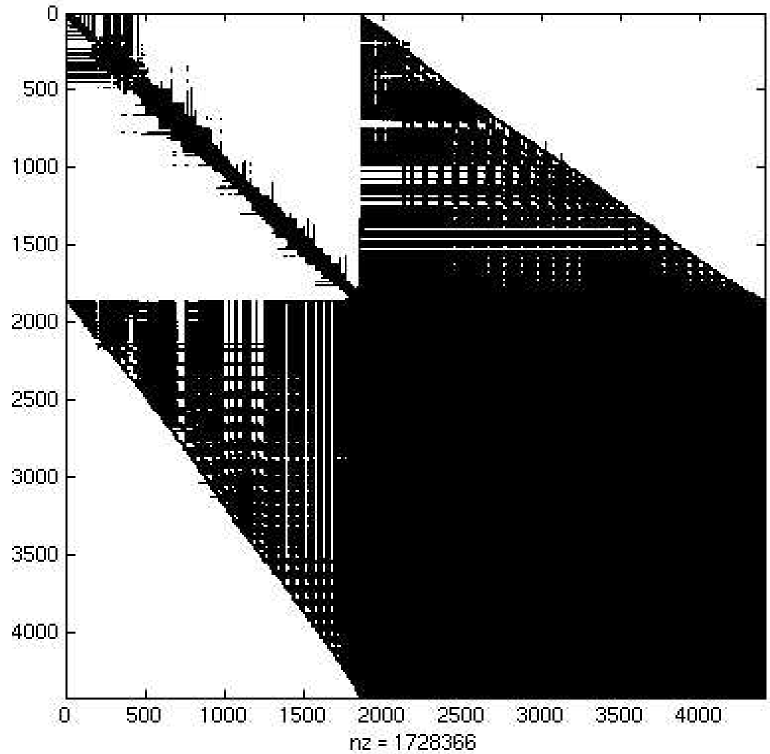


Figure 2. Matrix $\tilde{\mathbf{M}}$.

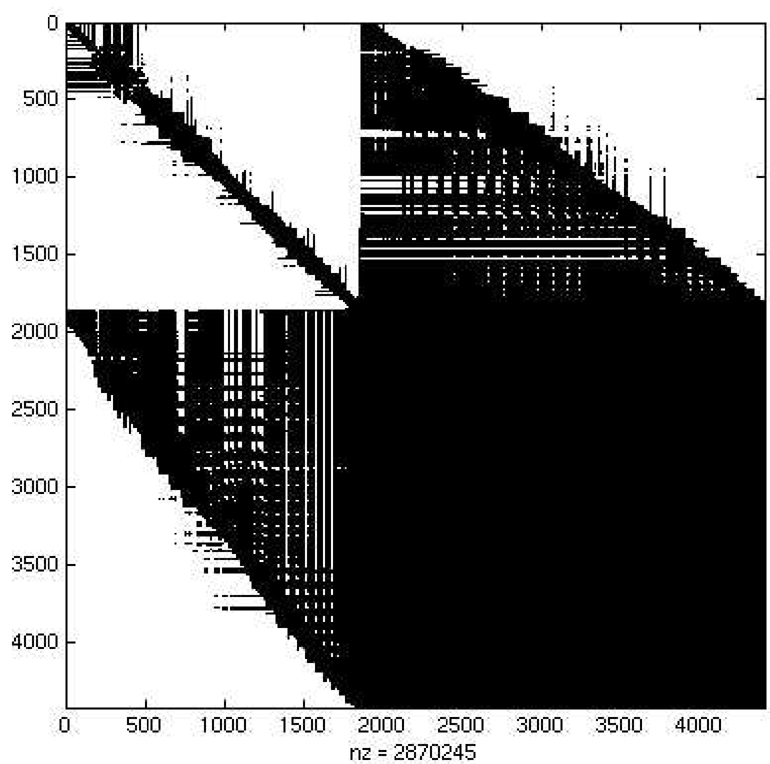


Figure 3. Matrix $\tilde{\mathbf{K}}$.

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## 10. Responsibility Notice.

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