Displacement of Semi-infinite Beams on an Elastic Foundation under Bending-Compression Loading

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Abstract. The behavior of semi-infinite uniform beams on an elastic foundation is formulated for equilibrium. A closed form solution for the displacement field, using Laplace transforms, is presented. A linear elastic brittle material, loaded in a quasi-static manner is considered. Once interface loading is deformation dependent, with dissipation occurring, determination of the deformation at different load levels is pursued. Deflections and bending moments are determined along the beam, taking into consideration the loading variables. Results show predominance of bending for contact with inclined walls.

Keywords: Beam deflections; Elastic foundation; Contact loading; Deformation dependence

1. Introduction

In this work the problem of beams of constant cross section, resting on a linear elastic foundation, loaded by in-plane as well as out-of-plane resultants, Fig. 1, is addressed. The uniform beam problem is a well studied problem in ice mechanics. Its study started with (Hetenyi, M., 1946) in the area of rail road engineering. Later on (Croasdale, K.R., 1980) obtained the solution corresponding to the failure loads in ice beams, under particular loadings. Here the solution is presented in a more elegant way, also covering other forms of boundary prescription.

2. Formulation

2.1 Equilibrium

Equilibrium of an elastic beam element, Fig. 1, constituted of a homogeneous material of rectangular uniform cross section, supported on a linear elastic foundation, being \( \kappa \) the foundation constant, under a bending compression loading, satisfies the pair of differential equations:

\[
\begin{align*}
D(N) & \equiv 0 \\
D^2 (E^i D^2 (w)) - D[N D(w)] + b \kappa w & \equiv 0; \quad D^i = \frac{d^i}{dy^i}, \; i \in Z^+ 
\end{align*}
\]  

The first equation above refers to in-plane force equilibrium, \( y \) direction, whereas the second corresponds to the lateral, out-of-plane, moment equilibrium. Combination of both equations, while disregarding the displacement of the neutral line with respect to the equal area axis, caused by the cross coupling, results in:

\[
D^4 (w) + 4 \gamma_0^4 D^2 (w) + 4 \delta_0^4 w = 0
\]

being \( w \) the lateral displacement. The coefficients are defined by:

\[
\gamma_0 = \left( \frac{n_0}{4 E I_0} \right)^{\frac{1}{4}}; \quad n_0 \geq 0; \quad \delta_0 = \left( \frac{\kappa}{4 E I_0} \right)^{\frac{1}{4}};
\]
representing \( n_0 \) the value of the normal force per unit width \( b_0 \), supposed constant. The bending proportional stiffness term is \( i_0 \) and \( E' \) is the equivalent beam Young’s modulus. Therefore:

\[
n_0 = \frac{N_0}{b_0}; \quad i_0 = \frac{h_0^3}{12}; \quad E' = \frac{E}{(1-\nu^2)}
\]

where \( N_0 \) is the normal force at the origin, supposed positive if compressive. \( E \) is the elastic modulus of the material and \( \nu \) its Poisson’s ratio. This beam model is referred to as Kirchhoff model (Mellor, M., 1983).

### 2.2 Interface Loading

Contact at the left interface supposes a loading that may be prescribed by a couple \( m_0 \) and a shear force \( v_0 \), both per unit width. Relationship between these quantities and the kinematics at the interface establishes that

\[
y = 0; \quad m_0 = -E'i_0D^2(w)_0 \\
v_0 = -E'i_0D^3(w)_0 - n_0D(w)_0
\]

what evidences dependence upon the deformation of the beam at the origin. Interface contact problems suppose the existence of local conditions that differ from a beam model. Disregarded disturbances caused by this close field, the beam model may be applied.

### 2.3 Boundary Conditions

At the other end, the far end, the boundary conditions reflect regularity, so that;

\[
y = L; \quad w_L = 0 \\
D(w)_L = 0
\]

where the subscript is employed to describe the position where the variable is to be computed. For the semi-infinite body, we need to set \( L \to \infty \). Lateral loading will add an additional term to the right hand side of Eq. (2).
3. D.E. Solution

3.1 Forward Laplace Transform Mapping

A characteristic of the differential equation of equilibrium, Eq. (2), is that, aside the central in-plane load dependent coefficient $\gamma_0$, all other coefficients are constant. Furthermore it is valid on a semi-infinite interval, what makes it ideal for a Laplace transform type of solution:

$$\mathcal{L}[w(y)] = \int_0^\infty w(y) \exp(-sy)dy$$

Applying the transform onto Eq. (2) leads to the expression

$$l(s) = \frac{s^3w_0 + s^2D(w_0) + s[D^2(w_0) + 4\gamma_0^4w_0] + [D^3(w_0) + 4\gamma_0^4D(w_0)]}{s^4 + 4\gamma_0^4s^3 + 4\delta_0^4}$$

whose pair of complex conjugated roots, $<\lambda;\bar{\lambda};-\bar{\lambda};\lambda>$ is:

$$\lambda = \alpha + i\beta; \quad \bar{\lambda} = \alpha - i\beta$$

being:

$$\alpha = \sqrt{\delta_0^2 - \gamma_0^4}; \quad \beta = \sqrt{\delta_0^2 + \gamma_0^4}$$

Partial fraction expansion of Eq. (9) may be used as a step towards the inversion mapping. Therefore if the expansion uses the roots above as a basis,

$$l(s) = \frac{A}{s + \lambda} + \frac{B}{s - \lambda} + \frac{C}{s + \bar{\lambda}} + \frac{D}{s - \bar{\lambda}}$$

with complex coefficients in the numerator.
then eight unknowns result. On equating the left and right sides of this equation, a set of relations is obtained. For the real components the system obtained is:

\[
[M][X_0] = [t_r]; \quad [M] = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
-1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
\end{bmatrix}; \quad [t_r] = \begin{bmatrix}
w_0 \\
(4\gamma_0^4 - 1)D(w) + D^3(w) \\
\frac{2\alpha}{\alpha} \\
0 \\
0 \\
\end{bmatrix}
\]

with a solution vector \( [X_0] = [A_r, B_r, C_r, D_r] \) where:

\[
A_r = \frac{1}{4} w_0 + \frac{(4\gamma_0^4 - 1)}{8\alpha} D(w) + \frac{1}{8\alpha} D^3(w); \quad C_r = A_r \\
B_r = \frac{1}{4} w_0 - \frac{(4\gamma_0^4 - 1)}{8\alpha} D(w) - \frac{1}{8\alpha} D^3(w); \quad D_r = B_r
\]

For the imaginary part, on the other hand, the same procedure will render the coefficients:

\[
A_i = \frac{1}{8\beta} D^3(w) + \frac{(4\gamma_0^4 + 1)}{8\beta} D(w) - \frac{1}{8\beta} D^3(w) - \frac{2\gamma_0^4}{8\alpha\beta} w_0; \quad C_i = -A_i \\
B_i = \frac{1}{8\beta} D^3(w) - \frac{(4\gamma_0^4 + 1)}{8\beta} D(w) - \frac{1}{8\beta} D^2(w) - \frac{2\gamma_0^2}{8\alpha\beta} w_0; \quad D_i = -B_i
\]

completing then the solution.

### 3.2 Inverse Laplace Transform Mapping

The s-space solution can be mapped back into a y-space solution, which is the solution sought, if it is observed that:

\[
\mathcal{L}^{-1}\left\{\frac{1}{s-<\lambda,-\lambda,\overline{\lambda},-\overline{\lambda}>}\right\} = \exp(<\lambda,-\lambda,\overline{\lambda},-\overline{\lambda}>)
\]

Or, on substituting Eq. (17) into Eq. (9),

\[
w(y) = \exp(-\alpha y)[A \exp(-i\beta y) + C \exp(i\beta y)] + \exp(\alpha y)[D \exp(-i\beta y) + B \exp(i\beta y)]
\]

which is a general form of lateral displacement field. Imposing the regularity conditions, Eqs. (7), and noting the relationship among real and imaginary parts of the coefficients, the positive exponential terms are dropped and the remaining term becomes

\[
w(y) = \exp(-\alpha y)[C \cos(\beta y) + S \sin(\beta y)]; \quad C = 2A_r \quad S = 2A_i
\]

Coefficients on this expression may be obtained considering the left interface loading, Eqs. (6). Resulting values are
\[ A_i = \frac{\tau_1 m_0 + \tau_2 v_0}{\sigma E' l_0} \; ; \quad \tau_1 = -\beta(\alpha^2 + \beta^2) \]
\[ \tau_2 = -2\alpha \beta \] (19)
for the first one, and:
\[ A_i = \frac{\eta_1 m_0 + \eta_2 v_0}{\sigma E' l_0} \; ; \quad \eta_1 = \alpha(\alpha^2 + \beta^2) \]
\[ \eta_2 = \beta^2 - \alpha^2 \] (20)
for the other one. The denominator term is
\[ \alpha = \beta(\alpha^2 + \beta^2)(3\alpha^2 - \beta^2) \] (21)

4. Application Problem

4.1 Quasi-static Loading

When an ice floe driven by currents and wind is loaded at contact with an inclined wall of an offshore platform, loading is essentially quasi-static, but coming from the impulse generated by the inertia change of the ice. It produces flexure, shear and compression. The rate of change of the linear momentum \( \mathbf{l} \) equals the sum of the external forces, normal and shear, per unit width at the interface, whereas the change of the angular momentum \( \alpha \) equals the rate of change of the interface bending moment. Values of normal \( n_0 \), bending moment \( m_0 \) and shear force \( v_0 \) at the interface depend on the contact between platform and beam.

They are resultants that may described by the coefficient of friction \( \mu \) between ice and the rigid wall of the platform, its slope angle \( \phi \) and the coefficient of eccentricity \( \zeta \). Solving the normal \( r_0\mathbf{e}_n \) and tangential \( \mu r_n \mathbf{e}_t \) to the inclined wall in terms of its horizontal and vertical components (Croasdale, K.R., 1980)
\[ y_0 = r_0(\cos \phi + \mu \sin \phi); \quad z_0 = r_0(\sin \phi - \mu \cos \phi) \] (22)
and denoting the rotation at the origin by \( \theta_0 \), we may write that:
\[ n_0 = -z_0 \sin \theta_0 + y_0 \cos \theta_0; \quad v_0 = z_0 \cos \theta_0 + y_0 \sin \theta_0; \quad m_0 = -n_0 e \] (23)
being \( e = h_0 \zeta; \; 0.5 \leq \zeta \leq 0.5 \) the eccentricity. The coefficient \( \mu \) depends on the existence of sliding or sticking contact conditions, for every value of \( r_0 \). A additional consideration has to be introduced here, as the shear force at the origin \( v_0 \) will also depend on the direction of the motion of the beam. For the riding-up condition, or up-slope case, under slipping and sticking stages,
\[ v_0 \leq n_0 \tan(\theta_i); \quad \theta_i < \rho \]
\[ v_0 = n_0 \tan(\theta_i - \rho); \quad \theta_i \geq \rho \] (24)
where \( \rho \) is a material parameter, related to the way the ice and wall interact, and dependent on surface roughness and temperature among other factors. \( \theta_0 \) represents the rotations at the origin and \( \theta_i = \phi + \theta_0 \).

Notice that deformation of the beam acts to create an effective value of friction angle. Table 1 presents the values considered in the present analysis.
4.2 Material Parameters

For the problem at hand we have to deal with a complex material, whose constitutive equation depends on the type of microstructure considered, time of the year, form of response sought, etc. For ice features in a brittle state, in salt water, Table 2 presents some average values of the properties of this material.

<table>
<thead>
<tr>
<th>Properties of Ice</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic modulus</td>
<td>$E = 0.50e^{10}$ Pa</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>$\nu = 0.30$</td>
</tr>
<tr>
<td>Flexural strength</td>
<td>$S_f = 0.70$ MPa</td>
</tr>
<tr>
<td>Compressive strength</td>
<td>$S_c = 5.0$ MPa</td>
</tr>
<tr>
<td>Foundation constant</td>
<td>$\gamma = 1.0045e + 4$ Pa/m</td>
</tr>
</tbody>
</table>

Table 2. Some properties of the beam material

4.3 Interface Rotation

Therefore, because the displacements do depend on the interface conditions, loading at origin depends on the rotation $\theta_0 = D(w)_0$, Eq. (20)-(24),

$$\theta_0 = -2\alpha \lambda + 2\beta \lambda$$  \hspace{1cm} (25)

or, upon substitution of the interface values, Eqs. (27),

$$a \sin \theta_0 + b \cos \theta_0 - \theta_0 = 0$$  \hspace{1cm} (26)

where,

$$a = \frac{-2\alpha \varepsilon_0 + \gamma_0}{E'(3\alpha^2 - \beta^2)}; \quad b = \frac{2\alpha \gamma_0 + \varepsilon_0}{E'(3\alpha^2 - \beta^2)}$$  \hspace{1cm} (27)

are coefficients that depend on $\theta_0$ also. Solution of Eq. (30) requires application of a Newton procedure, with expansion around the no-rotation at the zero stage to produce :

$$\theta_0 \equiv -\frac{b_0}{a_0 + \partial \theta_0(b)_0 - 1}$$  \hspace{1cm} (28)

where $a_0 = a(0)$ and $b_0 = b(0)$ depend upon

$$\alpha_0 = \sqrt{\delta_0^2 - \gamma_0^2(0)}; \quad \beta_0 = \sqrt{\delta_0^2 + \gamma_0^2(0)}$$  \hspace{1cm} (29)
being \( \gamma_0(0) = \frac{y_0}{4E' t_0} \).

5. Results
5.1 Displacement Field

Due to the nature of the problem, determination of the deformed configuration of the beam under compressive-flexural conditions has to be obtained at particular values of the in-plane load \( n_0 \). This deformed configuration will always depend upon the interface variables considered. From Eqs.(19)-(22), the lateral displacement field is:

\[
w = \frac{e^{-\alpha t}}{\sqrt{E' t_0}} \left\{ \left[ \tau_1 \cos(\beta y) + \eta_1 \sin(\beta y) \right] m_0 + \left[ \tau_2 \cos(\beta y) + \eta_2 \sin(\beta y) \right] n_0 \right\}
\]

(30)

As the interface rotations define the loading variables, once they are determined with the procedure outlined above, for every value of in-plane normal, a deflection profile of the beam may be plotted. In particular effects of the interface variables may be considered and analyzed. Results are presented in Figs. 2, 3 and 4.

In Fig. 2 the effect of the eccentricity of loading \( e = -\xi h_0 \) on the displacements \( w \) along the beam is described by:

\[
w = \hat{w}_e \left( \frac{h_0}{\lambda_0} ; E, \nu, \xi, \phi, \mu, \theta_0, \frac{n_0}{n_v} \right)
\]

(31)

for constant values of all variables but the eccentricity itself. In this plot ordinates were made non-dimensional by considering the thickness of the beam as a factor, whereas abscissas were normalized with respect to the wave-like number \( \lambda_0 = \frac{2\pi}{\delta_0} \). The crushing resistance of the beam \( n_c = h_0 S_0 \) is used as a factor as well. Observing this figure it is noticed that eccentric edge loading does not produce substantial changes in the lateral displacement profile of the beam. Only at the region closest to the loading edge, where the displacements are largest, appreciable changes occur, with an increase in the amplitude of the displacements corresponding to decreases of the eccentricity of loading.

Figure 2. Eccentricity of loading effect on displacements
The effect of the slope angle of the inclined wall produces a much more pronounced effect, as can be observed from the plot presented in Fig. 3:

\[
w = \hat{w}_0\left(\frac{\nu}{\lambda_0}; \frac{h_0}{b_0}; \frac{E, v, \xi, \phi, \mu; \theta_0}{n_0}; \frac{n_e}{n_c}\right)
\]  

Close to the loading edge, increases in the angle \( \theta \) result in large increases in the amplitude of displacements. In fact this also implies that the effect of the edge force \( v_0 \) is much more important than the bending moment.

![Figure 3. Slope angle effect on displacements](image)

Figure 3. Slope angle effect on displacements

The effect of the coefficient of friction on the displacements is shown in Fig. 4, taking into account two surfaces, steel and fine concrete, and slipping conditions,

\[
w = \hat{w}_\mu\left(\frac{\nu}{\lambda_0}; \frac{h_0}{b_0}; \frac{E, v, \xi, \phi, \mu; \theta_0}{n_0}; \frac{n_e}{n_c}\right)
\]  

Overall, it is observed a decrease of the amplitude of lateral displacements, close to the loading edge, when the coefficient of friction is augmented (Mohamed, S. and Tom Carrieres, 1999).

![Figure 4. Friction coefficient effect on displacements](image)

Figure 4. Friction coefficient effect on displacements
6. Conclusions

Solutions obtained here compare quite well with reported results for this kind of problem. In particular the solution may be completed with another form of boundary prescription. Layered forms of prescription of the constitutive equation may well be used with some changes incorporated to this mode (Smirnov, V.N. and Shushlebin, 1990)

7. References


Mellor, M., 1983, “Mechanical Behavior of Sea Ice”, Cold Regions Research and Engineering Laboratory, Monograph 83-1
