TOPOLOGICAL SENSITIVITY ANALYSIS APPLIED TO THREE-DIMENSIONAL LINEAR ELASTICITY TOPOLOGY DESIGN

Antonio André Novotny
Laboratório Nacional de Computação Científica LNCC/MCT,
Av. Getúlio Vargas 333, 25651-075 Petrópolis - RJ, Brasil
novotny@lncc.br

André Labanowski Jr. & Eduardo Alberto Fancello
Grupo de Análise e Projetos Mecânicos, GRANTE/EMC/UFSC
Campus Universitário Trindade, 88010-970 Florianópolis - SC, Brasil
andre@grante.ufsc.br, fancello@grante.ufsc.br

Abstract. *In this work we use the Topological-Shape Sensitivity Method to compute the topological derivative for three-dimensional linear elasticity problems, adopting the total potential energy as the cost function and the equilibrium equation as the constraint. This method, based on classical shape sensitivity analysis, leads to a systematic procedure to obtain the topological derivative. In particular, firstly we present the mechanical model in its varational formulation, later we compute the shape derivative for the problem under consideration and we compute the final expression for the topological derivative using classical asymptotic analysis around spherical cavities. Finally, we use the obtained sensitivity for solving some structural topology design problems.*

Keywords: *topological derivative, shape derivative, three-dimensional elasticity, topology design.*

1. Introduction

The topological derivative has been recognized as a powerful tool to solve topology optimization problems (see [1], where 425 references concerning topology optimization of continuum structures are included). See also [2, 4, 8] and references therein. Nevertheless, this concept is wider. In fact, the topological derivative may also be applied to inverse problems and to simulate physical phenomena with changes on the configuration of the domain of the problem. In addition, extension of the topological derivative in order to include arbitrary shaped holes and its applications to Elasticity, Laplace, Poisson, Helmoltz, Stokes and Navier-Stokes equations are developed by Masmoudi and his co-workers and by Sokolowsky and his co-workers. See [10] for applications of the topological derivative to the above equations, inverse problems and material properties characterization.

On the other hand, although the topological derivative is extremely general, this concept may become restrictive due to mathematical difficulties involved in its calculation. In order to overcome this problem, in [12] was proposed an alternative method to compute the topological derivative based on classical shape sensitivity analysis. This approach, called Topological-Shape Sensitivity Method, was successfully applied in several engineering problems.

Our aim in the present paper is to apply the Topological-Shape Sensitivity Method to compute the topological derivative in three-dimensional linear elasticity. This derivative can be applied in several engineering problems such as structural topology optimization. For the sake of completeness, in Section 2. we recall the Topological-Shape Sensitivity Method. In Section 3. we use this approach to compute the topological derivative for the problem under consideration. In particular, in Section 3.1 we present the mechanical model associated to three-dimensional linear elasticity. In Section 3.2 we compute the shape derivative for this problem adopting the total potential energy as the cost function and the weak form of the state equation as the constraint. In Section 3.3, we compute the final expression for the topological derivative using classical asymptotic analysis around spherical cavities. Finally, in Section 4., we use the obtained sensitivity for solving some structural topology design problems.

2. Topological-Shape Sensitivity Method

Let us consider an open bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$. If the domain Ω is perturbed by introducing a small hole at an arbitrary point $\hat{\mathbf{x}} \in \Omega$, we have a new domain $\Omega_\varepsilon = \Omega - \overline{B}_\varepsilon$, whose boundary is denoted by $\partial\Omega_\varepsilon = \partial\Omega \cup \partial B_\varepsilon$, where $\overline{B}_\varepsilon = B_\varepsilon \cup \partial B_\varepsilon$ is a ball of radius ε centered at point $\hat{\mathbf{x}} \in \Omega$. Therefore, we have the original domain without hole Ω and the new one Ω_ε with a small hole B_ε as shown in fig. (1). Thus, considering a cost function ψ, the topological derivative is defined as

$$D_T\left(\hat{\mathbf{x}}\right) = \lim_{\varepsilon \to 0} \frac{\psi\left(\Omega_\varepsilon\right) - \psi\left(\Omega\right)}{f\left(\varepsilon\right)} \ , \tag{1}$$

where $f\left(\varepsilon\right)$ is a negative function that decreases monotonically so that $f\left(\varepsilon\right) \to 0$ with $\varepsilon \to 0^+$.

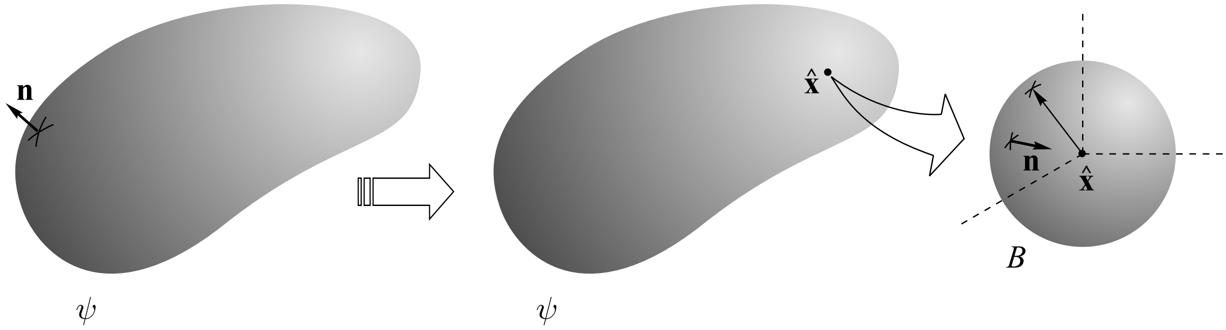


Figure 1. topological derivative concept

In [12] was proposed an alternative procedure to compute the topological derivative, called Topological-Shape Sensitivity Method, which makes use of the whole mathematical framework (and results) developed for shape sensitivity analysis (see, for instance, the pioneer work of Murat & Simon [9]). The main result obtained in [12] may be briefly summarized in the following Theorem (see also [11]):

Theorem 1 *Let $f(\varepsilon)$ be a function chosen in order to $0 < |D_T(\hat{\mathbf{x}})| < \infty$, then the topological derivative given by eq. (1) can be written as*

$$D_T(\hat{\mathbf{x}}) = \lim_{\varepsilon \to 0} \frac{1}{f'(\varepsilon)} \left. \frac{d}{d\tau} \psi(\Omega_\tau) \right|_{\tau=0} , \tag{2}$$

where $\tau \in \mathbb{R}^+$ is used to parameterize the domain. That is, for τ small enough, we have

$$\Omega_\tau := \left\{ \mathbf{x}_\tau \in \mathbb{R}^3 : \mathbf{x} \in \Omega_\varepsilon, \ \mathbf{x}_\tau = \mathbf{x} + \tau\mathbf{v}, \ \mathbf{x}_\tau|_{\tau=0} = \mathbf{x}, \ \Omega_\tau|_{\tau=0} = \Omega_\varepsilon \right\} . \tag{3}$$

In addition, the shape change velocity $\mathbf{v}$ *is a smooth vector field in Ω_ε, which assumes the following values on the boundary $\partial\Omega_\varepsilon$*

$$\begin{cases} \mathbf{v} = -\mathbf{n} & \text{on } \partial B_\varepsilon \\ \mathbf{v} = \mathbf{0} & \text{on } \partial\Omega \end{cases} \tag{4}$$

and the shape sensitivity of the cost function in relation to the domain perturbation characterized by $\mathbf{v}$ is given by

$$\left. \frac{d}{d\tau} \psi(\Omega_\tau) \right|_{\tau=0} = \lim_{\tau \to 0} \frac{\psi(\Omega_\tau) - \psi(\Omega_\varepsilon)}{\tau} . \tag{5}$$

Proof. The reader interested in the proof of this result may refer to [12] ■

This Theorem points out that the topological derivative may be obtained through the shape sensitivity analysis of the cost function (Topological-Shape Sensitivity Method). Consequently, results from shape sensitivity analysis can be used to calculate the topological derivative in simple and constructive way considering eq. (2), that we will show in the context of three-dimensional elasticity. It is also important to mention that Ω_ε and Ω_τ can be respectively seen as the material and the spatial configurations. Therefore, in order to compute the shape derivative of the cost function (see eq. 5) we can use classical results from Continuum Mechanics like the Reynolds' transport theorem and the concept of material derivatives of spatial fields [6].

3. The topological derivative in three-dimensional linear elasticity

Now, to point out the potentialities of the Topological-Shape Sensitivity Method, it will be applied to three-dimensional linear elasticity problems considering the total potential energy as the cost function and the state equation in its weak form as the constraint. Therefore, considering the above problem, firstly we introduce the mechanical model, later we perform the shape sensitivity of the cost function with respect to the shape change of the hole and finally we compute the associated topological derivative for the above problem.

3.1 Mechanical model

In this work, we consider a mechanical model restricted to smalls deformation and displacement and for the constitutive relation we adopt an isotropic linear elastic material. This assumptions leads to the classical three-dimensional linear elasticity theory [5]. In order to compute the topological derivative associated to this problem, we need to state the equilibrium equations in the original domain Ω (without hole) and in the new one Ω_ε (with hole).

3.1.1 Problem formulation in Ω (domain without hole)

The mechanical model associated to the three-dimensional linear elasticity problem can be stated in its variational formulation as following: find the displacement vector field $\mathbf{u} \in \mathcal{U}$, such that

$$\int_{\Omega} \mathbf{T}(\mathbf{u}) \cdot \mathbf{E}(\boldsymbol{\eta}) = \int_{\Gamma_N} \bar{\mathbf{q}} \cdot \boldsymbol{\eta} \qquad \forall \boldsymbol{\eta} \in \mathcal{V} \,, \tag{6}$$

where Ω represents a deformable body with boundary $\partial\Omega = \Gamma_N \cup \Gamma_D$, such that $\Gamma_N \cap \Gamma_D = \emptyset$, submitted to a set of surface forces $\bar{\mathbf{q}}$ on the Neumann boundary Γ_N and displacement constraints $\bar{\mathbf{u}}$ on the Dirichlet boundary Γ_D. Therefore, assuming that $\bar{\mathbf{q}} \in L^2(\Gamma_N)$, the admissible functions set $\mathcal{U}$ and the admissible variations space $\mathcal{V}$ are given, respectively, by

$$\mathcal{U} = \left\{ \mathbf{u} \in H^1(\Omega) : \mathbf{u} = \overline{\mathbf{u}} \text{ on } \Gamma_D \right\} \,, \tag{7}$$
$$\mathcal{V} = \left\{ \boldsymbol{\eta} \in H^1(\Omega) : \boldsymbol{\eta} = \mathbf{0} \text{ on } \Gamma_D \right\} \,. \tag{8}$$

In addition, the linearized Green deformation tensor $\mathbf{E}(\mathbf{u})$ and the Cauchy stress tensor $\mathbf{T}(\mathbf{u})$ are defined as

$$\mathbf{E}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right) := \nabla^s\mathbf{u} \qquad \text{and} \qquad \mathbf{T}(\mathbf{u}) = \mathbf{C}\mathbf{E}(\mathbf{u}) = \mathbf{C}\nabla^s\mathbf{u} \,, \tag{9}$$

where $\mathbf{C} = \mathbf{C}^T$ is the elasticity tensor, that is, since $\mathbf{I}$ and $\mathbb{I}$ respectively are the second and forth order identity tensors, E is the Young's modulus and ν is the Poisson's ratio, we have

$$\mathbf{C} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-2\nu)\,\mathbb{I} + \nu\,(\mathbf{I} \otimes \mathbf{I})\right] \quad \Rightarrow \quad \mathbf{C}^{-1} = \frac{1}{E}\left[(1+\nu)\,\mathbb{I} - \nu\,(\mathbf{I} \otimes \mathbf{I})\right] \,. \tag{10}$$

The Euler-Lagrange equation associated to the above variational problem, eq. (6), is given by the following boundary value problem:

$$\left\{ \begin{array}{l} \text{find } \mathbf{u} \text{ such that} \\ \quad \begin{array}{lll} \operatorname{div} \mathbf{T}(\mathbf{u}) = \mathbf{0} & \text{in} & \Omega \\ \mathbf{u} = \bar{\mathbf{u}} & \text{on} & \Gamma_D \\ \mathbf{T}(\mathbf{u})\mathbf{n} = \bar{\mathbf{q}} & \text{on} & \Gamma_N \end{array} \end{array} \right. . \tag{11}$$

3.1.2 Problem formulation in Ω_ε (domain with hole)

The problem stated in the original domain Ω can also be written in the domain Ω_ε with a hole B_ε. Therefore, assuming null forces on the hole, we have the following variational problem: find the displacement vector field $\mathbf{u}_\varepsilon \in \mathcal{U}_\varepsilon$, such that

$$\int_{\Omega_\varepsilon} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\boldsymbol{\eta}_\varepsilon) = \int_{\Gamma_N} \bar{\mathbf{q}} \cdot \boldsymbol{\eta}_\varepsilon \qquad \forall \boldsymbol{\eta}_\varepsilon \in \mathcal{V}_\varepsilon \,. \tag{12}$$

where the set $\mathcal{U}_\varepsilon$ and the space $\mathcal{V}_\varepsilon$ are respectively defined as

$$\mathcal{U}_\varepsilon = \left\{ \mathbf{u}_\varepsilon \in H^1(\Omega_\varepsilon) : \mathbf{u}_\varepsilon = \overline{\mathbf{u}} \text{ on } \Gamma_D \right\} \,, \tag{13}$$
$$\mathcal{V}_\varepsilon = \left\{ \boldsymbol{\eta}_\varepsilon \in H^1(\Omega_\varepsilon) : \boldsymbol{\eta}_\varepsilon = \mathbf{0} \text{ on } \Gamma_D \right\} \,. \tag{14}$$

In the same way, the tensors $\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon)$ and $\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)$ are respectively given as

$$\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) = \nabla^s\mathbf{u}_\varepsilon \qquad \text{and} \qquad \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) = \mathbf{C}\nabla^s\mathbf{u}_\varepsilon \,, \tag{15}$$

where the elasticity tensor $\mathbf{C}$ is defined in eq. (10).

Observe that according the variational problem given by eq. (12), the natural boundary condition on ∂B_ε is $\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\mathbf{n} = \mathbf{0}$ (homogeneous Neumann condition). Therefore, the Euler-Lagrange equation associated to this new variational problem is given by the following boundary value problem:

$$\left\{ \begin{array}{l} \text{find } \mathbf{u}_\varepsilon \text{ such that} \\ \quad \begin{array}{lll} \operatorname{div} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) = \mathbf{0} & \text{in} & \Omega_\varepsilon \\ \mathbf{u}_\varepsilon = \bar{\mathbf{u}} & \text{on} & \Gamma_D \\ \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\mathbf{n} = \bar{\mathbf{q}} & \text{on} & \Gamma_N \\ \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\mathbf{n} = \mathbf{0} & \text{on} & \partial B_\varepsilon \end{array} \end{array} \right. . \tag{16}$$

3.2 Shape sensitivity analysis

Considering the total potential energy as the cost function already written in the configuration Ω_τ, then $\psi(\Omega_\tau) := \mathcal{J}_\tau(\mathbf{u}_\tau)$ can be expressed by

$$\mathcal{J}_\tau(\mathbf{u}_\tau) = \frac{1}{2}\int_{\Omega_\tau} \mathbf{T}_\tau(\mathbf{u}_\tau)\cdot\mathbf{E}_\tau(\mathbf{u}_\tau) - \int_{\Gamma_N} \bar{\mathbf{q}}\cdot\mathbf{u}_\tau \ , \tag{17}$$

where the tensors $\mathbf{E}_\tau(\mathbf{u}_\tau)$ and $\mathbf{T}_\tau(\mathbf{u}_\tau)$ are respectively given by

$$\mathbf{E}_\tau(\mathbf{u}_\tau) = \nabla_\tau^s\mathbf{u}_\tau \qquad \text{and} \qquad \mathbf{T}_\tau(\mathbf{u}_\tau) = \mathbf{C}\nabla_\tau^s\mathbf{u}_\tau \ , \tag{18}$$

with $\nabla_\tau(\cdot)$ used to denote

$$\nabla_\tau(\cdot) := \frac{\partial}{\partial\mathbf{x}_\tau}(\cdot) \ . \tag{19}$$

In addition, $\mathbf{u}_\tau$ is the solution of the variational problem defined in the configuration Ω_τ, that is: find the displacement vector field $\mathbf{u}_\tau \in \mathcal{U}_\tau$ such that

$$\int_{\Omega_\tau} \mathbf{T}_\tau(\mathbf{u}_\tau)\cdot\mathbf{E}_\tau(\boldsymbol{\eta}_\tau) = \int_{\Gamma_N} \bar{\mathbf{q}}\cdot\boldsymbol{\eta}_\tau \quad \forall\,\boldsymbol{\eta}_\tau \in \mathcal{V}_\tau \ , \tag{20}$$

where the set $\mathcal{U}_\tau$ and the space $\mathcal{V}_\tau$ are defined as

$$\mathcal{U}_\tau = \left\{\mathbf{u}_\tau \in H^1(\Omega_\tau) : \mathbf{u}_\tau = \overline{\mathbf{u}} \text{ on } \Gamma_D\right\} \ , \tag{21}$$

$$\mathcal{V}_\tau = \left\{\boldsymbol{\eta}_\tau \in H^1(\Omega_\tau) : \boldsymbol{\eta}_\tau = \mathbf{0} \text{ on } \Gamma_D\right\} \ . \tag{22}$$

Observe that from the well-known terminology of Continuum Mechanics, the domains $\Omega_\tau|_{\tau=0} = \Omega_\varepsilon$ and Ω_τ can be interpreted as the material and the spatial configurations, respectively. Therefore, in order to compute the shape derivative of the cost function $\mathcal{J}_\tau(\mathbf{u}_\tau)$, at $\tau = 0$, we may use the Reynolds' transport theorem and the concept of material derivatives of spatial fields, that is [6]

$$\frac{d}{d\tau}\int_{\Omega_\tau}\varphi_\tau\bigg|_{\tau=0} = \int_{\Omega_\varepsilon}\varphi_\tau'|_{\tau=0} + \int_{\partial\Omega_\varepsilon}\varphi_\tau|_{\tau=0}(\mathbf{v}\cdot\mathbf{n}) \ , \tag{23}$$

where φ_τ is a spatial scalar field and $(\cdot)'$ is used to denote

$$(\cdot)' := \frac{\partial(\cdot)}{\partial\tau} = \frac{d(\cdot)}{d\tau}\bigg|_{\mathbf{x}_\tau \text{ fixed}} \ . \tag{24}$$

Taking into account the cost function defined through eq. (17) and assuming that the parameters E, ν, $\bar{\mathbf{u}}$, and $\bar{\mathbf{q}}$ are constants in relation to the perturbation represented by τ, we have, from eq. (23), that

$$\frac{d}{d\tau}\mathcal{J}_\tau(\mathbf{u}_\tau)\bigg|_{\tau=0} = \frac{1}{2}\int_{\partial\Omega_\varepsilon}(\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon))(\mathbf{v}\cdot\mathbf{n}) + \frac{1}{2}\int_{\Omega_\varepsilon}\frac{\partial}{\partial\tau}(\mathbf{T}_\tau(\mathbf{u}_\tau)\cdot\mathbf{E}_\tau(\mathbf{u}_\tau))\bigg|_{\tau=0} - \int_{\Gamma_N}\bar{\mathbf{q}}\cdot\dot{\mathbf{u}}_\varepsilon \ , \tag{25}$$

where $\dot{(\cdot)}$ is used to denote

$$\dot{(\cdot)} := \frac{d(\cdot)}{d\tau} \ . \tag{26}$$

In addition, $\dot{\mathbf{u}}_\varepsilon$ can be written as

$$\dot{\mathbf{u}}_\varepsilon = \mathbf{u}_\varepsilon' + (\nabla\mathbf{u}_\varepsilon)\mathbf{v} \quad \Rightarrow \quad \mathbf{u}_\varepsilon' = \dot{\mathbf{u}}_\varepsilon - (\nabla\mathbf{u}_\varepsilon)\mathbf{v} \ . \tag{27}$$

Taking into account the notation introduced through eq. (24) and from eq. (27), we have

$$\begin{aligned}\frac{\partial}{\partial\tau}(\mathbf{T}_\tau(\mathbf{u}_\tau)\cdot\mathbf{E}_\tau(\mathbf{u}_\tau))\bigg|_{\tau=0} &= 2\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon') \\ &= 2\left(\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\dot{\mathbf{u}}_\varepsilon) - \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\boldsymbol{\varphi}_\varepsilon)\right) \ ,\end{aligned} \tag{28}$$

where

$$\boldsymbol{\varphi}_\varepsilon = (\nabla\mathbf{u}_\varepsilon)\mathbf{v} \quad \Rightarrow \quad \mathbf{E}_\varepsilon(\boldsymbol{\varphi}_\varepsilon) = \nabla^s\boldsymbol{\varphi}_\varepsilon \ . \tag{29}$$

Substituting eq. (28) in eq. (25) we obtain

$$
\begin{aligned}
\left.\frac{d}{d\tau}\mathcal{J}_\tau(\mathbf{u}_\tau)\right|_{\tau=0} &= \frac{1}{2}\int_{\partial\Omega_\varepsilon}(\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon))(\mathbf{v}\cdot\mathbf{n}) - \int_{\Omega_\varepsilon}\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\boldsymbol{\varphi}_\varepsilon) \\
&\quad + \int_{\Omega_\varepsilon}\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\dot{\mathbf{u}}_\varepsilon) - \int_{\Gamma_N}\bar{\mathbf{q}}\cdot\dot{\mathbf{u}}_\varepsilon \\
&= \frac{1}{2}\int_{\partial\Omega_\varepsilon}(\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon))(\mathbf{n}\cdot\mathbf{v}) - \int_{\Omega_\varepsilon}\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\boldsymbol{\varphi}_\varepsilon)\ ,
\end{aligned} \tag{30}
$$

since $\dot{\mathbf{u}}_\varepsilon \in \mathcal{V}_\varepsilon$ and $\mathbf{u}_\varepsilon$ is the solution of eq. (12). In addition, we observe that

$$
\int_{\Omega_\varepsilon}\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\boldsymbol{\varphi}_\varepsilon) = \int_{\partial\Omega_\varepsilon}\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\boldsymbol{\varphi}_\varepsilon\cdot\mathbf{n} - \int_{\Omega_\varepsilon}\operatorname{div}(\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon))\cdot\boldsymbol{\varphi}_\varepsilon\ . \tag{31}
$$

Considering this last result (eq. 31) in eq. (30) and taking into account that $\mathbf{u}_\varepsilon$ is the solution of eq. (16), we have

$$
\begin{aligned}
\left.\frac{d}{d\tau}\mathcal{J}_\tau(\mathbf{u}_\tau)\right|_{\tau=0} &= \frac{1}{2}\int_{\partial\Omega_\varepsilon}(\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon))(\mathbf{v}\cdot\mathbf{n}) - \int_{\partial\Omega_\varepsilon}\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\boldsymbol{\varphi}_\varepsilon\cdot\mathbf{n} \\
&= \int_{\partial\Omega_\varepsilon}\left[\frac{1}{2}(\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon))\mathbf{I} - (\nabla\mathbf{u}_\varepsilon)^T\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\right]\mathbf{n}\cdot\mathbf{v}\ ,
\end{aligned} \tag{32}
$$

remembering that $\boldsymbol{\varphi}_\varepsilon$ is given by eq. (29).

Let us define $\boldsymbol{\Sigma}_\varepsilon$ as the Eshelby energy-momentum tensor (see, for instance, [3, 7]) given in this particular case by

$$
\boldsymbol{\Sigma}_\varepsilon = \frac{1}{2}(\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon))\mathbf{I} - (\nabla\mathbf{u}_\varepsilon)^T\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\ , \tag{33}
$$

then the shape derivative of the cost function $\mathcal{J}_\tau(\mathbf{u}_\tau)$ defined through eq. (17), at $\tau = 0$, can be written as

$$
\left.\frac{d}{d\tau}\mathcal{J}_\tau(\mathbf{u}_\tau)\right|_{\tau=0} = \int_{\partial\Omega_\varepsilon}\boldsymbol{\Sigma}_\varepsilon\mathbf{n}\cdot\mathbf{v}\ , \tag{34}
$$

which becomes an integral defined on the boundary $\partial\Omega_\varepsilon$.

3.3 Topological sensitivity analysis

In order to compute the topological derivative using the Topological-Shape Sensitivity Method, we need to substitute eq. (34) in the result of Theorem 1 (eq. 2). Therefore, from the definition of the velocity field (eq. 4) and considering the shape derivative of the cost function (eq. 34), we have that

$$
\left.\frac{d}{d\tau}\mathcal{J}_\tau(\mathbf{u}_\tau)\right|_{\tau=0} = -\int_{\partial B_\varepsilon}\boldsymbol{\Sigma}_\varepsilon\mathbf{n}\cdot\mathbf{n}\ , \tag{35}
$$

where

$$
\boldsymbol{\Sigma}_\varepsilon\mathbf{n}\cdot\mathbf{n} = \frac{1}{2}\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) - \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\mathbf{n}\cdot(\nabla\mathbf{u}_\varepsilon)\mathbf{n}\ . \tag{36}
$$

In addition, taking into account homogeneous Neumann boundary condition on the hole, we have, from eq. (16), that $\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\mathbf{n} = \mathbf{0}$ on ∂B_ε, therefore

$$
\left.\frac{d}{d\tau}\mathcal{J}_\tau(\mathbf{u}_\tau)\right|_{\tau=0} = -\frac{1}{2}\int_{\partial B_\varepsilon}\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon)\ . \tag{37}
$$

Finally, substituting eq. (37) in the result of the Theorem 1 (eq. 2), the topological derivative becomes

$$
D_T(\hat{\mathbf{x}}) = -\frac{1}{2}\lim_{\varepsilon\to 0}\frac{1}{f'(\varepsilon)}\int_{\partial B_\varepsilon}\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon)\ . \tag{38}
$$

Considering the inverse of the constitutive relation $\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) = \mathbf{C}^{-1}\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)$ (see eq. 10), then the integrand of eq. (38) may be expressed as a function of the stress tensor as following

$$
\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) = \frac{1}{E}\left[(1+\nu)\,\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) - \nu\,(\operatorname{tr}\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon))^2\right]\ . \tag{39}
$$

Let us introduce a spherical coordinate system (r, θ, φ) centered in $\hat{\mathbf{x}}$, then the stress tensor $\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) = (\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon))^T$, when defined on the boundary ∂B_ε, can be decomposed as

$$\begin{aligned} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)|_{\partial B_\varepsilon} &= T_\varepsilon^{rr}\,(\mathbf{e}_r \otimes \mathbf{e}_r) + T_\varepsilon^{r\theta}\,(\mathbf{e}_r \otimes \mathbf{e}_\theta) + T_\varepsilon^{r\varphi}\,(\mathbf{e}_r \otimes \mathbf{e}_\varphi) \\ &+ T_\varepsilon^{r\theta}\,(\mathbf{e}_\theta \otimes \mathbf{e}_r) + T_\varepsilon^{\theta\theta}\,(\mathbf{e}_\theta \otimes \mathbf{e}_\theta) + T_\varepsilon^{\theta\varphi}\,(\mathbf{e}_\theta \otimes \mathbf{e}_\varphi) \\ &+ T_\varepsilon^{r\varphi}\,(\mathbf{e}_\varphi \otimes \mathbf{e}_r) + T_\varepsilon^{\theta\varphi}\,(\mathbf{e}_\varphi \otimes \mathbf{e}_\theta) + T_\varepsilon^{\varphi\varphi}\,(\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi) \ , \end{aligned} \tag{40}$$

where $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$ are the basis of the spherical coordinate system such that

$$\mathbf{e}_r \cdot \mathbf{e}_r = \mathbf{e}_\theta \cdot \mathbf{e}_\theta = \mathbf{e}_\varphi \cdot \mathbf{e}_\varphi = 1 \qquad \text{and} \qquad \mathbf{e}_r \cdot \mathbf{e}_\theta = \mathbf{e}_r \cdot \mathbf{e}_\varphi = \mathbf{e}_\theta \cdot \mathbf{e}_\varphi = 0 \ . \tag{41}$$

Since we have homogeneous Neumann boundary condition on ∂B_ε, then

$$\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\mathbf{n} = \mathbf{0} \qquad \Rightarrow \qquad \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\mathbf{e}_r = \mathbf{0} \qquad \text{on} \qquad \partial B_\varepsilon \ . \tag{42}$$

From the decomposition of the stress tensor shown in eq. (40) and taking into account eqs. (41,42), we observe that

$$\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\mathbf{e}_r = T_\varepsilon^{rr}\mathbf{e}_r + T_\varepsilon^{r\theta}\mathbf{e}_\theta + T_\varepsilon^{r\varphi}\mathbf{e}_\varphi = \mathbf{0} \qquad \Rightarrow \qquad T_\varepsilon^{rr} = T_\varepsilon^{r\theta} = T_\varepsilon^{r\varphi} = 0 \ . \tag{43}$$

Substituting eqs. (40,43) into eq. (39), the topological derivative given by eq. (38) may be written in terms of the components of the stress tensor in spherical coordinate, as following

$$\begin{aligned} D_T\,(\hat{\mathbf{x}}) &= -\frac{1}{2E}\lim_{\varepsilon\to 0}\frac{1}{f'\,(\varepsilon)}\int_{\partial B_\varepsilon} d_T(T_\varepsilon^{\theta\theta}, T_\varepsilon^{\theta\varphi}, T_\varepsilon^{\varphi\varphi}) \\ &= -\frac{1}{2E}\lim_{\varepsilon\to 0}\frac{1}{f'\,(\varepsilon)}\int_0^{2\pi}\left(\int_0^{\pi} d_T(T_\varepsilon^{\theta\theta}, T_\varepsilon^{\theta\varphi}, T_\varepsilon^{\varphi\varphi})\varepsilon^2 \sin\theta d\theta\right) d\varphi \ , \end{aligned} \tag{44}$$

where

$$d_T(T_\varepsilon^{\theta\theta}, T_\varepsilon^{\theta\varphi}, T_\varepsilon^{\varphi\varphi}) = (T_\varepsilon^{\theta\theta})^2 + (T_\varepsilon^{\varphi\varphi})^2 - 2\nu T_\varepsilon^{\theta\theta}T_\varepsilon^{\varphi\varphi} + 2(1+\nu)(T_\varepsilon^{\theta\varphi})^2 \ . \tag{45}$$

Now, it is enough to calculate the limit $\varepsilon \to 0$ in the eq. (44) to obtain the final expression of the topological derivative. Thus, an asymptotic analysis shall be performed in order to know the behavior of the solution $T_\varepsilon^{\theta\theta}$, $T_\varepsilon^{\theta\varphi}$ and $T_\varepsilon^{\varphi\varphi}$ when $\varepsilon \to 0$. This behavior may be obtained from the analytical solution for a stress distribution around a spherical void in a three-dimensional elastic body [13], which is given, for any $\delta > 0$ and at $r = \varepsilon$, by

$$\begin{aligned} T_\varepsilon^{\theta\theta}|_{\partial B_\varepsilon} &= \frac{3}{4(7-5\nu)}\left\{\sigma_1\,(\mathbf{u})\left[3 - 5(1-2\nu)\cos 2\varphi + 10\cos 2\theta \sin^2\varphi\right]\right. \\ &\quad +\sigma_2\,(\mathbf{u})\left[3 + 5(1-2\nu)\cos 2\varphi + 10\cos 2\theta\cos^2\varphi\right] \\ &\quad \left.+\sigma_3\,(\mathbf{u})\left[2(4-5\nu) - 10\cos 2\theta\right]\right\} + \mathcal{O}(\varepsilon^{1-\delta}) \ , \end{aligned} \tag{46}$$

$$T_\varepsilon^{\theta\varphi}|_{\partial B_\varepsilon} = \frac{15(1-\nu)}{2(7-5\nu)}(\sigma_1\,(\mathbf{u}) - \sigma_2\,(\mathbf{u}))\cos\theta\sin 2\varphi + \mathcal{O}(\varepsilon^{1-\delta}) \ , \tag{47}$$

$$\begin{aligned} T_\varepsilon^{\varphi\varphi}|_{\partial B_\varepsilon} &= \frac{3}{4(7-5\nu)}\left\{\sigma_1\,(\mathbf{u})\left[8 - 5\nu + 5(2-\nu)\cos 2\varphi + 10\nu\cos 2\theta\sin^2\varphi\right]\right. \\ &\quad +\sigma_2\,(\mathbf{u})\left[8 - 5\nu - 5(2-\nu)\cos 2\varphi + 10\nu\cos 2\theta\cos^2\varphi\right] \\ &\quad \left.-2\sigma_3\,(\mathbf{u})\,(1 + 5\nu\cos 2\theta)\right\} + \mathcal{O}(\varepsilon^{1-\delta}) \ , \end{aligned} \tag{48}$$

where $\sigma_1\,(\mathbf{u})$, $\sigma_2\,(\mathbf{u})$ and $\sigma_3\,(\mathbf{u})$ are the principal stress values of the tensor $\mathbf{T}\,(\mathbf{u})$, associated to the original domain without hole Ω (see eq. 6), evaluated in the point $\hat{\mathbf{x}} \in \Omega$, that is $\mathbf{T}\,(\mathbf{u})|_{\hat{\mathbf{x}}}$.

Substituting the asymptotic expansion given by eqs. (46,47,48) in eq. (44) we observe that function $f\,(\varepsilon)$ must be chosen such that

$$f'\,(\varepsilon) = -\,|\partial B_\varepsilon| = -4\pi\varepsilon^2 \quad \Rightarrow \quad f\,(\varepsilon) = -\,|B_\varepsilon| = -\frac{4}{3}\pi\varepsilon^3 \tag{49}$$

in order to take the limit $\varepsilon \to 0$ in eq. (44), where $|B_\varepsilon|$ is used to denote the Lebesgue measure of the ball B_ε.

Therefore, from this choice of function $f\,(\varepsilon)$ shown in eq. (49), the final expression for the topological derivative becomes a scalar function that depends on the solution $\mathbf{u}$ associated to the original domain Ω (without hole), that is (see also [4, 8]):

- in terms of the principal stress values $\sigma_1(\mathbf{u})$, $\sigma_2(\mathbf{u})$ and $\sigma_3(\mathbf{u})$ of tensor $\mathbf{T}(\mathbf{u})$

$$D_T(\hat{\mathbf{x}}) = \frac{3(1-\nu)}{4(7-5\nu)E}\left[10(1+\nu)\left(\sigma_1(\mathbf{u})^2+\sigma_2(\mathbf{u})^2+\sigma_3(\mathbf{u})^2\right)-(1+5\nu)\left(\sigma_1(\mathbf{u})+\sigma_2(\mathbf{u})+\sigma_3(\mathbf{u})\right)^2\right] \; ; \tag{50}$$

- in terms of the stress tensor $\mathbf{T}(\mathbf{u})$

$$D_T(\hat{\mathbf{x}}) = \frac{3(1-\nu)}{4(7-5\nu)E}\left[10(1+\nu)\mathbf{T}(\mathbf{u})\cdot\mathbf{T}(\mathbf{u})-(1+5\nu)(\mathrm{tr}\mathbf{T}(\mathbf{u}))^2\right] \; ; \tag{51}$$

- in terms of the stress $\mathbf{T}(\mathbf{u})$ and strain $\mathbf{E}(\mathbf{u})$ tensors

$$D_T(\hat{\mathbf{x}}) = \frac{3(1-\nu)}{4(7-5\nu)}\left[10\mathbf{T}(\mathbf{u})\cdot\mathbf{E}(\mathbf{u})-\frac{1-5\nu}{1-2\nu}\mathrm{tr}\mathbf{T}(\mathbf{u})\;\mathrm{tr}\mathbf{E}(\mathbf{u})\right] \; , \tag{52}$$

  which was obtained from a simple manipulation considering the constitutive relation given by eq. (9). See also eq. (10).

Remark 2 *It is interesting to observe that if we take $\nu = 1/5$ in eq. (52), the final expression for the topological derivative in terms of $\mathbf{T}(\mathbf{u})$ and $\mathbf{E}(\mathbf{u})$ becomes*

$$D_T(\hat{\mathbf{x}}) = \mathbf{T}(\mathbf{u})\cdot\mathbf{E}(\mathbf{u}) \; . \tag{53}$$

4. Numerical Results

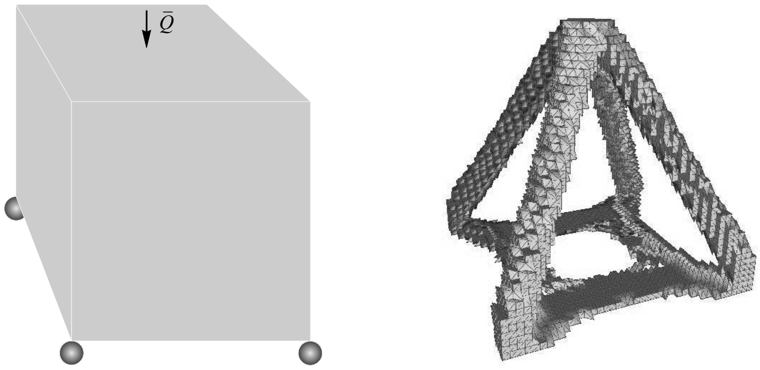
As already mentioned in this paper, the topological derivative allow us to quantify the sensitivity of a given cost function when the domain under consideration is perturbed by introducing a hole. Thus, let us write eq. (1) like a Taylor series expansion, then

$$\psi(\Omega_\varepsilon) = \psi(\Omega) + f(\varepsilon)D_T(\hat{\mathbf{x}}) + \mathcal{O}(f(\varepsilon)) \; , \tag{54}$$

where $\mathcal{O}(f(\varepsilon))$ contains all higher order terms than $f(\varepsilon)$. From analysis of eq. (54), $D_T(\hat{\mathbf{x}})$ may be seen as a first order correction of $\psi(\Omega)$ to obtain $\psi(\Omega_\varepsilon)$, which allow us to naturally use this derivative as a descent direction in topology design algorithm (for a survey on topology optimization methods, the reader is referred to the review paper [1], where 425 references concerning topology optimization of continuum structures are included). In other words, the topological sensitivity gives the information on the opportunity to create holes. Therefore, let us present the following topology design algorithm based on the topological derivative given by eqs. (50,51 or 52): considering the sequence $\left\{\Omega^i : \left|\Omega^i\right| \geq |\Omega^*|\right\}$, where i is the i-th iteration and $|\Omega^*|$ corresponds to the required final volume, then,

1. **Provide** the initial domain Ω and the stop criterion $|\Omega^*|$.
2. **While** $\left|\Omega^i\right| \geq |\Omega^*|$ **do**:
   (a) compute $D_T^i(\hat{\mathbf{x}})$;
   (b) create holes at the points $\hat{\mathbf{x}}$ satisfying $\xi^i_{\inf} \leq D^i_T(\hat{\mathbf{x}}) \leq \xi^i_{\sup}$, where $\xi^i_{\inf}$ and $\xi^i_{\sup}$ are specified depending on the volume of material to be removed at each iteration;
   (c) define the new domain Ω^{i+1};
   (d) make $i \leftarrow i+1$.
3. **Ensure** the desired final topology Ω^*.

In order to show the potentiality of the topological derivative, let us apply the above algorithm to perform the topology design of a benchmark example. In fig. 2a is shown the initial domain given by a simply supported cube of $5\times5\times5m^3$, submitted to a concentrated load $\bar{Q} = 10^6N$ on the top. The material properties are given by $E = 210\times10^3MPa$ and $\nu = 1/4$. Due to the symmetry of the problem, only a quarter part of the cube is discretized with 96×10^3 tetrahedric finite elements. Taking $|\Omega^*| = 0.07\,|\Omega|$, the final topology is obtained at iteration $i = 78$, as shown in fig. 2b, where we have a classical result.



(a) model (b) topology at $i = 78$

Figure 2. a benchmark example.

5. Conclusions

In this work, we have computed the topological derivative in three-dimensional linear elasticity taking the total potential energy as the cost function and the state equation in its weak form as the constraint. The relationship between shape and topological derivatives was formally established in Theorem 1, leading to the Topological-Shape Sensitivity Method. Therefore, results from classical shape sensitivity analysis could be used to compute the topological derivative in a simple and constructive way. In particular, we have obtained the explicit formula for the topological derivative for the problem under consideration given by eqs. (50,51,52), whose result can be applied in several engineering problems such as topology optimization of three-dimensional linear elastic structures.

6. Acknowledgments

This research was partly supported by the brazilian agencies CNPq/FAPERJ-PRONEX (E-26/171.199/2003) and FAPERJ (E-26/150.712/2003). The support from these agencies is greatly appreciated.

7. References

H.A. Eschenauer & N. Olhoff. Topology Optimization of Continuum Structures: A Review. *Applied Mechanics Review*, **54**:331-390, 2001.

H.A. Eschenauer, V.V. Kobelev & A. Schumacher. Bubble Method for Topology and Shape Optimization of Structures. *Structural Optimization*, **8**:42-51, 1994.

J.D. Eshelby. The Elastic Energy-Momentum Tensor. *Journal of Elasticity*, **5**:321-335, 1975.

S. Garreau, Ph. Guillaume & M. Masmoudi. The Topological Asymptotic for PDE Systems: The Elasticity Case. *SIAM Journal on Control and Optimization*, **39**:1756-1778, 2001.

P. Germain & P. Muller. *Introduction à la Mécanique des Milieux Continus*. Masson, 1994.

M.E. Gurtin. *An Introduction to Continuum Mechanics*. Mathematics in Science and Engineering vol. 158. Academic Press, 1981.

M.E. Gurtin. *Configurational Forces as Basic Concept of Continuum Physics.* Applied Mathematical Sciences vol. 137. Springer-Verlag, 2000.

T. Lewiński & J. Sokolowski. Energy change due to the appearance of cavities in elastic solids. *International Journal of Solids and Structures*, **40**:1765-1803, 2003.

F. Murat & J. Simon. *Sur le Contrôle par un Domaine Géométrique*. Thesis, Université Pierre et Marie Curie, Paris VI, França, 1976.

P. Neittaanmäki *et alli* (eds.) *Mini-symposium on Topological Sensitivity Analysis: Theory and Applications, European Congress on Computational Methods in Applied Sciences and Engineering*, Jyväskylä, Finland, 2004.

A.A. Novotny. *Análise de Sensibilidade Topológica*. Ph. D. Thesis, LNCC/MCT, Petrópolis - RJ, Brasil, 2003.

A.A. Novotny, R.A. Feijóo, C. Padra & E. Taroco. Topological Sensitivity Analysis. *Computer Methods in Applied Mechanics and Engineering*, **192**:803-829, 2003.

M.A. Sadowsky & E. Sternberg. Stress Concentration Around a Triaxial Ellipsoidal Cavity. *Journal of Applied Mechanics*, 149-157, June 1949.

8. Responsibility notice

The authors are the only responsible for the printed material included in this paper